universität wien

DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

„Agents in superposition“

verfasst von / submitted by

Veronika Baumann BSc MSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktorin der Naturwissenschaften (Dr.rer.nat.)

Wien, 2021 / Vienna, 2021

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 796 605 411
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Physik
Betreut von / Supervisor:	Univ.- Prof. Mag. Dr. Časlav Brukner
Betreut von / Supervisor:	Prof. Dr. Stefan Wolf

Agents in superposition

Doctoral Dissertation submitted to the
Faculty of Informatics of the Università della Svizzera Italiana
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

presented by
Veronika Baumann

under the supervision of
Prof. Časlav Brukner and Prof. Stefan Wolf

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Dissertation accepted on May 11, 2021

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I certify that except where due acknowledgement has been given, the work presented in this thesis is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; and the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program.

Veronika Baumann
Vienna & Lugano, May 11, 2021

Abstract

The idea of agents in superposition is central to both encapsulated observers, i.e. Wigner's-friend-type experiments, and correlations without definite causal order. An agent might be regarded as a thinking or computing entity that is able to perform operations on (other) quantum systems. In case of Wigner's-friend setups, an agent's ability to make predictions about other agents potentially leads to testable contradictions, if said agent is in superposition. Indefinite (quantum) causal order can be understood as superposition of the order of operations different agents perform. In this case, agents with access to non-casual processes can outperform agents with only causal processes as a resource in certain computational tasks.

We started working on the topic of indefinite causal order contributing to the notion of effectively definite causal order in the bipartite case. The main part of the work presented in this thesis was done on scenarios containing observations of observers, i.e. encapsulated observers or Wigner's-friend-type experiments. We made a clear distinction between the formalism and the interpretations of quantum theory and showed that there are actually two inequivalent quantum formalisms. If different agents in a Wigner's-friend setup use different formal descriptions for a quantum measurement, they arrive at contradicting statements. This, however, becomes manifest only if there are classical records of these statements that can be compared at some point. In order to arrive at consistent probability assignments for setups comprising encapsulated observers, we analyzed such a setup within the Page-Wootters formalism, arriving at three possible probability rules, all of which exclude any observable contradictions for Wigner's-friend experiments. Finally, we adapted the Page-Wootters formalism to enable it to capture certain, well understood, processes with indefinite causal order: Those where the order of event is coherently controlled by a quantum system.

Zusammenfassung

Die Idee von Beobachtern in Superposition ist sowohl für verschachtelte Beobachter, i.e. Wigners-Freund Experimente, als auch für unbestimmte kausale Strukturen von zentraler Bedeutung. Ein Beobachter ist in diesem Fall eine denkende oder rechnende Einheit, die Operationen an (anderen) Quantensystemen durchführen kann. Im Fall von Wigners-Freund Experimenten kann die Fähigkeit eines Beobachters Vorhersagen über andere Beobachter zu machen zu testbaren Widersprüchen führen, wenn sich ersterer in Superposition befindet. Unbestimmte Quanten-Kausalstrukturen können als Superpositionen der Abfolge von Operationen verschiedener Beobachter verstanden werden. In diesem Fall können Beobachter, die Zugang zu nicht-kausalen Prozessen haben, bestimmte Aufgabenstellungen besser erfüllen als solche Beobachter, die lediglich über kausale Ressourcen verfügen.

Zuerst haben wir an unbestimmten kausalen Strukturen gearbeitet und zum Verständnis des Konzepts einer effektiven Kausalität für zwei Beobachter beigetragen. Der Hauptteil dieser Arbeit befasst sich mit Beobachtungen von Beobachtern, i.e. verschachtelte Beobachter oder Wigners-Freund Experimente. Hierbei haben wir klar zwischen dem Formalismus und den Interpretationen der Quantentheorie unterschieden und gezeigt, dass es zwei nicht äquivalente Quantenformalismen gibt. Wenn verschiedene Beobachter in einem Wigners-Freund Szenario verschiedene formale Beschreibungen eines Messvorgangs verwenden, führt dies zu widersprüchlichen Aussagen, was allerdings nur dann offenkundig wird, wenn diese Aussagen als klassische Aufzeichnungen vorliegen, die miteinander verglichen werden können. Um den Messergebnissen verschiedener Beobachter in Wigners-Freund Szenarien konsistent Wahrscheinlichkeiten zuzuordnen, haben wir solch ein Gedankenexperiment mit Hilfe des Page-Wootters Formalismus analysiert. Dabei fanden wir drei mögliche Wahrscheinlichkeitsformeln, die allesamt Widersprüche ausschließen. Zuletzt haben wir den Page-Wootters Formalismus angewendet um bestimmte nicht-kausale Prozesse zu beschreiben, nämlich solche bei denen die Abfolge von Operationen durch den Quantenzustand eines Kontrollsystems bestimmt wird.

Contents

Chapter 1

Introduction

1.1 Cotutelle de Thèse

The cotutelle de thèse enables PhD students to conduct research and study at two universities in different countries – in my case these are the Università della Svizzera italiana in Lugano and the University of Vienna. For a cotutelle de thèse, the doctoral candidate has to be enrolled in and fulfill the necessary requirements of both universities involved as well as work at both universities for at least one year. The supervision of the PhD thesis is shared between Prof.Stefan Wolf at the Università della Svizzera italiana, where I spent the first half of my PhD studies, and Prof. Časlav Brukner at the University of Vienna, where I spent the second half of my degree and will defend this thesis. The thesis reviewer committee was also chosen such that the respective requirements of both the Università della Svizzera italiana and the University of Vienna are satisfied.

1.2 Wigner's-friend experiments

The quantum measurement problem, despite lacking an unambiguous, unique definition, is roughly the question of how, when and under what circumstances definite values of physical variables are obtained in quantum theory, see Busch et al. [1996]. In the literature it has often been subdivided into conceptually different sub-problems that can be addressed separately Brukner [2017]; Bub and Pitowsky [2010]; Maudlin [1995]. The importance of the measurement problem is due to the fact that (standard) quantum theory comprises two different dynamics, namely the collapse of the wave function, which without any ontological commitment means the application of the state-update rule, and unitary evolution. Collapse dynamics describe the evolution of a quantum system upon measurement, while in the absence of measurements these systems evolve unitarily. Answering the above questions, that together constitute the measurement problem, would give an unambiguous prescription for when to use which dynamics. Note, however, that quantum theory in its current form lacks such a prescription altogether.

The Wigner's-friend thought experiment was first proposed by Eugene Wigner Wigner [1963] and considered the observation of an observer in order to illustrate the measurement problem. It comprises an observer – historically called Wigner's friend–, who measures a quantum system, as well as a so-called superobserver – Wigner– who performs a joint measurement on the quantum system, the friend and potentially other relevant degrees of freedom. The latter are sometimes called the friend's laboratory. Wigner's friend uses the state-update rule after obtaining a definite outcome for her measurement. However, provided that the friend's laboratory is sufficiently isolated, Wigner assigns an entangled state to the composite laboratory system and describes the friend's measurement via unitary dynamics. Hence, the two observers disagree on the dynamics during the friend's measurement. Such setups allowing for quantum measurements on observers are also referred to as encapsulated observers and can, in principle, contain more than just two levels of observation, i.e. beyond observers and superobservers, see Figure 1.1.

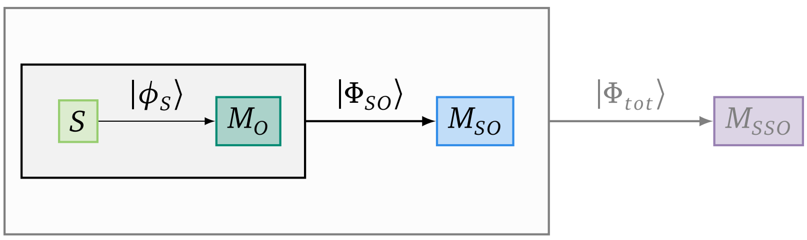


Figure 1.1. Encapsulated observers: The source S emits a system in quantum state $|\phi_S\rangle$, which is measured by an observer (the friend), who performs measurement M_O and applies the state-update rule. To the superobserver (Wigner) the joint system of S and F evolves unitarily to the overall state $|\Phi_{SO}\rangle$, on which he then performs measurement M_{SO}. This can in principle be continued indefinitely and gives rise to an ever higher order of observation: observers, superobservers, super-superobservers etc. To each order of observer, all lower-oder observations constitute one big joint quantum system described by unitary evolution and state $|\Phi_{tot}\rangle$.

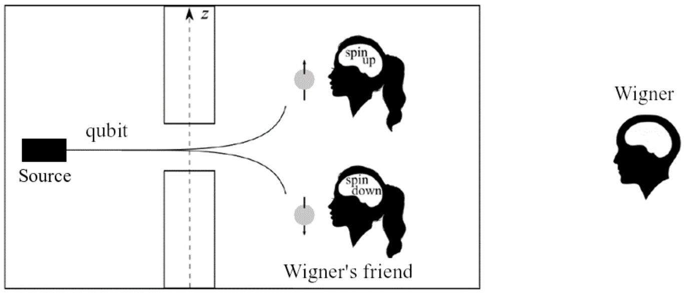
Despite the measurement problem being one of the oldest problems of quantum theory, there have in recent years been new proposals combining Wigner's-friend-type scenarios with other known setups, where quantum theory was known to give puzzling predictions, for example Brukner [2018]; Frauchiger and Renner [2018]. These setups, which will be introduced in detail in Section 1.2.2, have been used to devise arguments against the consistency of quantum theory and the existence of objective, observer-independent facts, which sparked renewed debate in the quantum-foundations community Baumann et al. [2016]; Bub [2018]; Cavalcanti [2021]; Sudbery [2017]; Losada et al. [2019].

1.2.1 The simplest Wigner's-friend experiment

The simplest version of the Wigner's-friend experiment, as depicted in Figure 1.2, features a source S which emits a qubit state $|\phi\rangle_S \in \mathscr{H}_S$. Wigner's friend F measures this qubit in the σ_z-basis observing outcome "up" or "down" corresponding to states $|\uparrow\rangle_S$ or $|\downarrow\rangle_S$. Assuming collapse dynamics for F's measurement gives states $|\uparrow\rangle_S|u\rangle_F$ or $|\downarrow\rangle_S|d\rangle_F$ for the joint system $S+F$ associated with Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_F$, where $|u\rangle_F$ and $|d\rangle_F$ are the states describing the friend having seen outcome "up" and "down" respectively. In general, $\mathscr{H}_F$ can be regarded as the space of those degrees of freedom that store the measurement result the friend observed. Wigner W measures the joint system $S+F$ in some basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, where the first two basis elements are spanned by $|\uparrow\rangle_S|u\rangle_F$ and $|\downarrow\rangle_S|d\rangle_F$ while the two basis vectors $|3\rangle$ and $|4\rangle$ are linear combinations of $|\uparrow\rangle_S|d\rangle_F$ and $|\downarrow\rangle_S|u\rangle_F$. Assuming that the friend's measurement satisfies the rules of standard quantum mechanics, the result she sees is perfectly correlated with the corresponding eigenstate of the system. Hence, the terms $|\uparrow\rangle_S|d\rangle_F$ and $|\downarrow\rangle_S|u\rangle_F$ will not appear in the overall state of $S+F$ regardless of whether one describes F's measurement via collapse or unitary dynamics. However, if the friend's laboratory is sufficiently isolated, Wigner describes F's measurement as an entangling unitary resulting in a joint state $|\Phi\rangle_{SF}$, which in general is neither $|\uparrow\rangle_S|d\rangle_F$ nor $|\downarrow\rangle_S|u\rangle_F$. Assuming collapse dynamics for W's measurement, then, means that the post-measurement state of $S+F+W$ is either $|1\rangle_{SF}|w_1\rangle_W$ or $|2\rangle_{SF}|w_2\rangle_W$ where states $|w_j\rangle_W \in \mathscr{H}_W$ correspond to Wigner having seen outcome j.

The fact that Wigner and his friend use different dynamical descriptions for the same process, i.e. F's measurement, leads to them assigning different probabilities to W's measurement outcomes. In general, the state emitted by the source

a.



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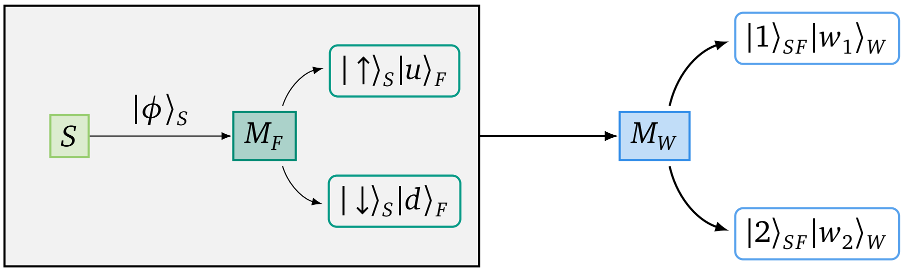


Figure 1.2. The simplest Wigner's-friend experiment: a. The setup consist of a qubit source and an observer – Wigner's friend –inside an isolated laboratory as well as superobserver –Wigner– who is situated outside said laboratory. b. The source S emits a qubit $|\phi\rangle_S$, which is measured by the friend F in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Wigner's friend obtains one of the results "up" or "down", which corresponds to states $|u\rangle_F$ and $|d\rangle_F$ respectively. Wigner W then measures the joint system in a basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, where $|1\rangle_{SF} = \alpha\ |\uparrow\rangle_S|u\rangle_F + \beta e^{i\phi_{SF}}|\downarrow\rangle_S|d\rangle_F$ and $|2\rangle_{SF} = \beta e^{-i\phi_{SF}}|\uparrow\rangle_S|u\rangle_F - \alpha|\downarrow\rangle_S|d\rangle_F$ with $\alpha, \beta, \phi_{SF} \in \mathbb{R}$. If F applies the state-update rule after her measurement, she will predict probabilities for W's measurement on her laboratory, which differ from those predicted by W, if he describes F's measurement unitarily.

is

$$|\phi\rangle_S = a|\uparrow\rangle_S + be^{i\phi_S}|\downarrow\rangle_S\,, \tag{1.1}$$

and the states $|1\rangle$ and $|2\rangle$ of W's measurement basis are given by

$$|1\rangle_{SF} = \alpha|\uparrow\rangle_S|u\rangle_F + \beta e^{i\phi_{SF}}|\downarrow\rangle_S|d\rangle_F \tag{1.2}$$

$$|2\rangle_{SF} = \beta e^{-i\phi_{SF}}|\uparrow\rangle_S|u\rangle_F - \alpha|\downarrow\rangle_S|d\rangle_F\,, \tag{1.3}$$

with $a, b, \phi_S, \alpha, \beta, \phi_{SF} \in \mathbb{R}$. For now, assume for simplicity that the source emits the state $|\phi\rangle_S = 1/\sqrt{2}(|\uparrow\rangle_S + |\downarrow\rangle_S)$ and that Wigner's measurement results 1 and 2 correspond to states (1.2) and (1.3) with $\phi_{SF} = 0$. The friend employing collapse dynamics uses the product states $|z\rangle_S|f_z\rangle_F$, where $z \in \{\uparrow, \downarrow\}$, $f_\uparrow = u$ and $f_\downarrow = d$, for calculating the the probabilities for any subsequent measurement and hence assigns probabilities

$$P_F(w) = |\langle z|\langle f_z|w\rangle_{SF}|^2 \tag{1.4}$$

to W's measurement results with $w \in \{1,2,3,4\}$. Wigner, however, assigns the post-measurement state $|\Phi\rangle_{SF} = 1/\sqrt{2}(|\uparrow\rangle_S|u\rangle_F + e^{i\nu}|\downarrow\rangle_S|d\rangle_F)$ to the friend's laboratory, where $\nu \in \mathbb{R}$ depends on the specifics of the interaction Hamiltonian between F and the system S. Therefore, W predicts probabilities

$$P_W(w) = |\langle \Phi|w\rangle_{SF}|^2 \tag{1.5}$$

for the outcomes of his measurement. Plugging in the corresponding states gives the following, clearly different, probability distributions according to F and W:

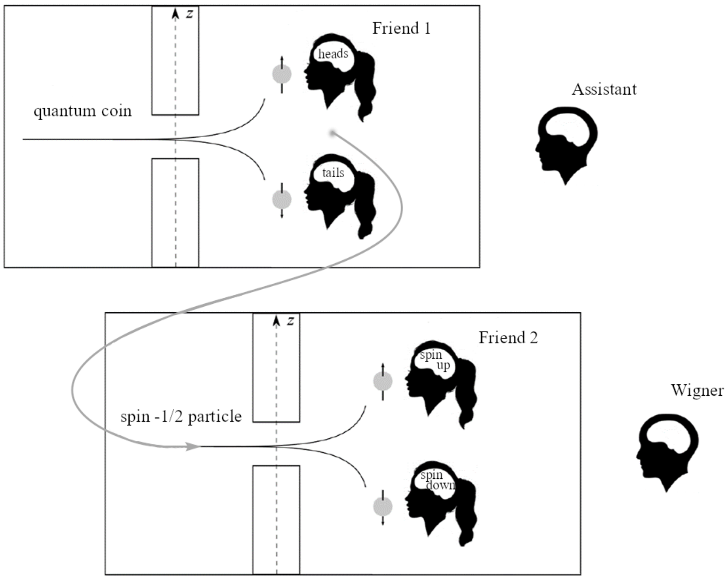
$$\begin{array}{c|c|c} P_F(w): & 1 & 2 \\ \hline f=u & \alpha^2 & \beta^2 \\ f=d & \beta^2 & \alpha^2 \end{array} \qquad \begin{array}{c|c|c} P_W(w): & 1 & 2 \\ \hline f=u & \frac{1}{2}(\alpha+\beta)^2 & \frac{1}{2}(\beta-\alpha)^2 \\ f=d & \frac{1}{2}(\alpha+\beta)^2 & \frac{1}{2}(\beta-\alpha)^2. \end{array} \tag{1.6}$$

These differing probability assignments constitute the core of the Wigner's friend paradox and can, in principle, be resolved in three different ways. The two straight forward solutions to the paradox are that one agent instead of using the dynamical description above adopts the description of the other agent. In these cases, Wigner and his friend will both agree either on probabilities P_F or on probabilities P_W. Another approach to the Wigner's friend paradox is accepting the disagreement in the probability assignments and abandoning the requirement that two agents, even when describing the same experiment, must agree under all circumstances. These different approaches are discussed in more detail in Chapter 2.

1.2.2 Extended Wigner's-friend setups

Different proposals combining two or more simple Wigner's-friend setups drew renewed attention to the thought experiment and provided new kinds of paradoxes for encapsulated observers. The most famous of these proposals is the one by Daniela Frauchiger and Renato Renner Frauchiger and Renner [2018] and is

a.



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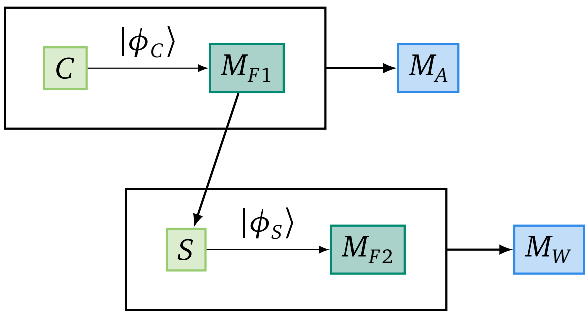


Figure 1.3. The Frauchiger-Renner setup: There are two friends – F_1 and F_2 – who perform measurement on two qubits observing results "heads" or "tails" in case of F_1, and "up" or "down" in case of F_2. The two superobservers – Wigner W and his assistant A – perform measurements on the laboratories of F_2 and F_1 respectively. b. The source in the first laboratory emits a coin state $|\phi_C\rangle$ which is measured by F_1. Depending on the result, F_1 sends state $|\phi_S\rangle$, which is measured by F_2. Assistant A and Wigner W perform measurements in analogous superposition bases on the first and second laboratory respectively.

depicted in Figure 1.3. The four agents in this setup are two observers – F_1 and F_2 – and two superobservers – Wigner W and his assistant A – who perform the following protocol:

0 The source in F_1's laboratory emits a quantum coin state $|\phi_C\rangle = \sqrt{\frac{1}{3}}|h\rangle_{S_1} + \sqrt{\frac{2}{3}}|t\rangle_{S_1}$ which is an unequal superposition of "head" and "tail".

1 Friend F_1 measures the quantum coin in the basis $\{|h\rangle, |t\rangle\}$ and prepares a spin state $|\phi_S\rangle = |\downarrow\rangle_{S_2}$, if the result of the measurement was "head", and a spin state $|\phi_S\rangle = \sqrt{\frac{1}{2}}(|\downarrow\rangle_{S_2} + |\uparrow\rangle_{S_2})$, if the result was "tail". The spin state $|\phi_S\rangle$ is sent to F_2's laboratory.

2 Friend F_2 measures the spin state in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ obtaining result "up" or 'down".

3 The assistant A measures the joint system of the quantum coin C and friend F_1 in a basis containing states

$$|o\rangle_{S_1F_1} = \sqrt{1/2}(|h\rangle_{S_1}|H\rangle_{F_1} - |t\rangle_{S_1}|T\rangle_{F_1}),$$
$$|f\rangle_{S_1F_1} = \sqrt{1/2}(|h\rangle_{S_1}|H\rangle_{F_1} + |t\rangle_{S_1}|T\rangle_{F_1}),$$

obtaining results "ok" or "fail" respectively. The states $|H\rangle_{F_1}$ and $|T\rangle_{F_1}$ correspond to F_1 having seen outcome "head" or"tail" during her measurement.

4 Wigner W performs a joint measurement on the spin state S and friend F_2 in a basis containing

$$|O\rangle_{S_2F_2} = \sqrt{1/2}(|\downarrow\rangle_{S_2}|d\rangle_{F_2} - |\uparrow\rangle_{S_2}|u\rangle_{F_2}),$$
$$|F\rangle_{S_2F_2} = \sqrt{1/2}(|\downarrow\rangle_{S_2}|d\rangle_{F_2} + |\uparrow\rangle_{S_2}|u\rangle_{F_2}),$$

where $|u\rangle_{F_2}$ and $|d\rangle_{F_2}$ are the states of F_2 having seen "up" or "down". Also Wigner's results are labeled "ok" and "fail".

5 Wigner and his assistant compare their results. If they both obtain the result "ok" corresponding to $|O\rangle$ and $|o\rangle$ respectively, they stop the protocol, otherwise they repeat steps **0** - **5**.

Similar to the famous Hardy paradox Hardy [1993], they derive a contradiction for the measurement results of all agents in the round of the protocol, where both superobservers measure "ok", see Figure 1.4. This contradiction is particularly striking since it is phrased in terms of deterministic predictions of results in one particular run of the experiment. As in the simple Wigner's-friend setup, the contradiction can be understood as caused by one of the observers, namely F_1, describing her own measurement via collapse dynamics, while to the superobservers the state of both laboratories evolves unitarily.

Another kind of extended Wigner's-friend setups Brukner [2018]; Healey [2018]; Leegwater [2018] combines encapsulated observers with Bell- and GHZ-

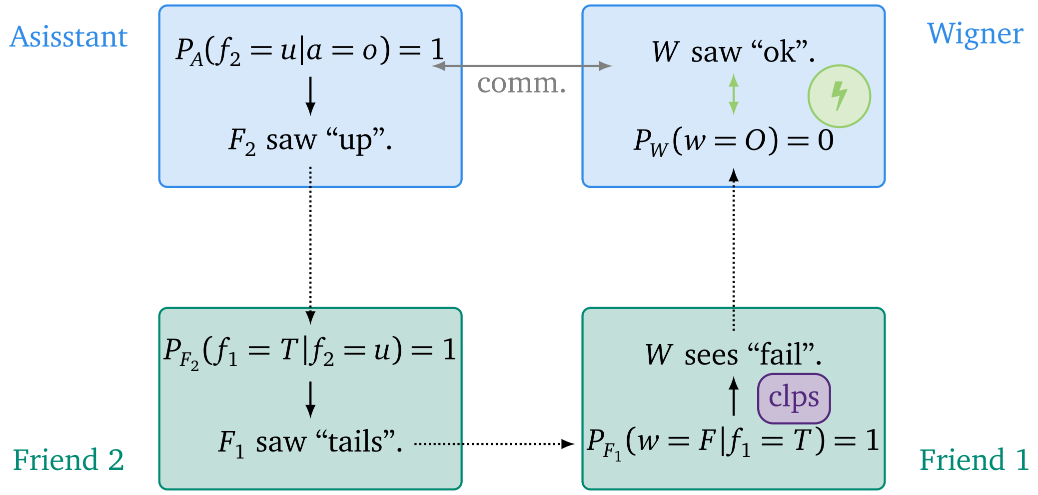


Figure 1.4. The contradiction for the setup in Figure 1.3: Starting with the assistant A, each agent deduces, based on their own result, the measurement outcome of another with certainty (black arrows). Requiring that such predictions match the actual observation of the other agent (dotted arrows) gives a circle of reasoning that leads to a contradiction for Wigner W. The protocol ensures that during the final round both superobservers measure "ok". Wigner can confirm this by communicating with his assistant (gray double arrow). Due to A having seen "ok", F_2 should have seen "up". If F_2 saw up, F_1 should have seen "tails". However, if F_1 saw "tails", W should have seen "fail" which is in contradiction with his actual observation. Note, that F_1's prediction of W's result is due to her using the state-update rule after her measurement.

type setups Bell [1964]; Greenberger et al. [1989] in order to construct arguments against the existence of objective, observer-independent facts. According to Bell's theorem, the violation of Bell-like inequalities precludes the possibility of combining all the outcomes under consideration in one joint probability distribution such the observed statistics are the respective marginal distributions. The setup in Brukner [2018] is depicted in Figure 1.5 and combines two Wigner's-friend experiments and a CHSH-type Bell setup Clauser et al. [1969]. Each Winger's friend setup receives one part of an entangled two-qubit state

$$\begin{aligned}|\Psi\rangle &= \cos\theta|\Phi^+\rangle_{S_1S_2} - \sin\theta|\Psi^-\rangle_{S_1S_2} \\ &= \frac{\cos\theta}{\sqrt{2}}(|\uparrow\rangle_{S_1}|\uparrow\rangle_{S_2} + |\downarrow\rangle_{S_1}|\downarrow\rangle_{S_2}) - \frac{\sin\theta}{\sqrt{2}}(|\uparrow\rangle_{S_1}|\downarrow\rangle_{S_2} - |\downarrow\rangle_{S_1}|\uparrow\rangle_{S_2}),\end{aligned} \tag{1.7}$$

which is measured by the observers – Charlie C and Debbie D– in the σ_z-basis. Then the two superobservers – Alice A and Bob B – can each choose between either performing the measurement that reveals the respective observer's result or the complementary measurement that confirms the superposition state of the respective laboratory. More concretely, Charlie measures qubit 1 and his laboratory is measured by Alice, while Debbie measures qubit 2 and her laboratory is measured by Bob. This means $M_F := |\uparrow\rangle\langle\uparrow|_{S_f} - |\downarrow\rangle\langle\downarrow|_{S_f}$ with $(S_f, F) \in \{(S_1, C), (S_2, D)\}$, while the measurements M_W are either $W_z := |u\rangle|\uparrow\rangle\langle\uparrow|\langle u|_{S_f F} - |d\rangle|\downarrow\rangle\langle\downarrow|\langle d|_{S_f F}$ or $W_x := |u\rangle|\uparrow\rangle\langle\downarrow|\langle d|_{S_f F} - |d\rangle|\downarrow\rangle\langle\uparrow|\langle u|_{S_f F}$ where $(F, W) \in \{(C, A), (D, B)\}$, compare Figure 1.5.

a.

b.

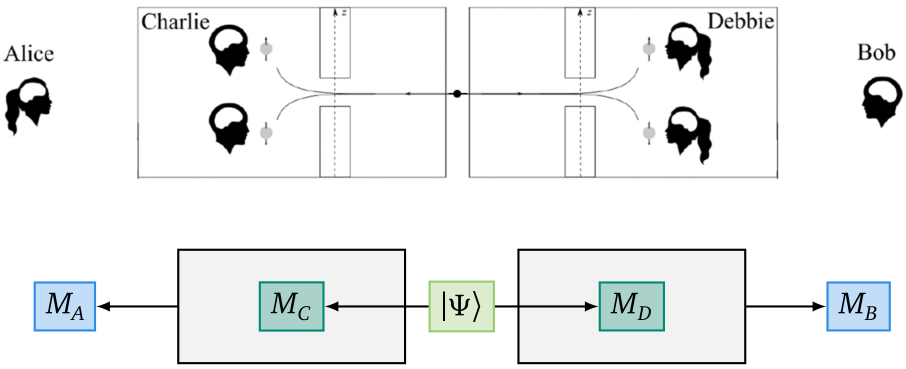


Figure 1.5. A Wigner-Bell setup: a. Two spacelike separated observers – Debbie and Charlie– become entangled due to their measurements on the two halves of an entangled pair. The two superobservers – Alice and Bob – can choose between two different measurements on the observers' laboratories. b. The state $|\Psi\rangle = \cos\theta|\Phi^+\rangle_{S_1S_2} - \sin\theta|\Psi^-\rangle_{S_1S_2}$, where $|\Phi^+\rangle$ and $|\Psi^-\rangle$ are the maximally entangled Bell states, is distributed to the two Wigner's-friend setups. The observers Charlie and Debbie each measure the subsystem they receive in the σ_z-basis. The superobservers Alice and Bob choose between two measurements $M_A \in \{A_x, A_z\}$ and $M_B \in \{B_x, B_z\}$ on the joint system of qubit and respective observer. The z-measurements of the superobservers reveal the measurement results of the observers, while the x-measurements confirm a superposition state of the whole laboratory after the respective observer's measurement.

Like for any Bell setup, one assumes 'locality' and 'freedom of choice' and in addition to that 'universal validity of quantum theory' and the 'existence of observer-independent facts'. The first of these two additional assumptions ensures that the unitary descriptions of the two superobservers of the observers' laboratories is correct, i.e. their probability assignments match the frequencies of their observed results. Hence, starting from the state in Equation (1.7), the overall state of the two laboratories after the measurements of C and D is given by

$$\begin{aligned} |\Psi_{tot}\rangle = \ & \frac{\cos\theta}{\sqrt{2}}\big(|\uparrow,u\rangle_{S_1C}|\uparrow,u\rangle_{S_2D} + |\downarrow,d\rangle_{S_1C}|\downarrow,d\rangle_{S_2D}\big) \\ & -\frac{\sin\theta}{\sqrt{2}}\big(|\uparrow,u\rangle_{S_1C}|\downarrow,d\rangle_{S_2D} - |\downarrow,d\rangle_{S_1C}|\uparrow,u\rangle_{S_2D}\big), \end{aligned} \quad (1.8)$$

where $|z,f_z\rangle_{S_fF}$ stands for $|z\rangle_{S_f}|f_z\rangle_F$ and, again, $|u\rangle_F$ and $|d\rangle_F$ are the states of the respective observer $F \in \{C,D\}$ having seen outcomes "up" and "down". The assumption of 'existence of observer-independent facts' means that one can jointly assign values to the observed outcomes of all four agents, which would have to be determined by a joint probability distribution. Moreover, in those runs of the experiment where both superobservers measure in the superposition basis, the observers' outcomes correspond to those of the other two observables of Alice and Bob, namely those that would reveal Charlie's and Debbie's observed results. Hence, these four assumptions together imply that the correlation functions for the superobservers' measurements satisfy the CHSH-type Wigner-Bell inequality

$$S = |E(A_x,B_x) + E(A_x,B_z) + E(A_z,B_x) - E(A_z,B_z)| \leq 2. \quad (1.9)$$

For $E(A_i,B_j) = \langle\Psi_{tot}|A_i \otimes B_j|\Psi_{tot}\rangle$, with $i,j \in \{x,z\}$ and $|\Psi_{tot}\rangle$ according to Equation (1.8), however, one obtains $S = 2\sqrt{2} \geq 2$, which constitutes a violation of the Wigner-Bell inequality in Equation (1.9). This, in turn, implies that no joint probability distribution $P(A_x,A_z,B_x,B_z)$ exists, such that its marginals give the respective correlation functions. Brukner argues that this precludes an assignment of definite values to the observers' measurement outcomes and, therefore, excludes the existence of observer-independent facts, if one maintains the other three assumptions. Note that the empirical content of the above argument, i.e. experimental violation of the Wigner-Bell inequality in Equation (1.9), has been confirmed Bong et al. [2020]; Proietti et al. [2019], although with a very minimal observer model that allowed for considering photons as observers.

1.3 Indefinite causal order

The notion of a global causal order is deeply rooted in all areas of physics, where physical processes are happening in the causal structure of spacetime. However, in any theory unifying quantum physics and general relativity the casual structure is expected to be both dynamical, as in general relativity, as well as indefinite, due to quantum theory Butterfield and Isham [2001]; Hardy [2005, 2007, 2009];

Kiefer [2012]; Rovelli [2004]; Zych et al. [2019]. The process-matrix framework Oreshkov et al. [2012] allows for scenarios with no well-defined global order of the operations by different agents, for whom locally the order of events is well-defined. These non-causal processes are a consequence of the formalism and in many cases still lack a clear physical interpretation.

In Baumann and Brukner [2016] we showed that for two agents a certain class of operations makes any process appear effectively causal. Moreover, we investigated non-causal processes in the context of the Page-Wootters formalism, which is introduced in Section 1.4, see Baumann et al. [2021], to provide further arguments which types of processes might be physically meaningful. These results are summarized in Chapter 3.

1.3.1 The process-matrix framework

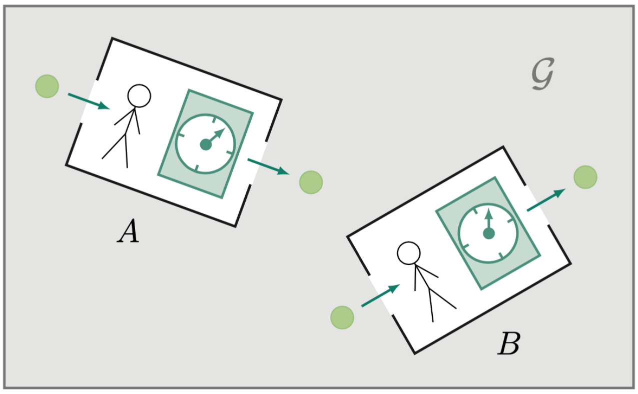


Figure 1.6. The idea behind the process-matrix framework for the bipartite example: Two agents, A and B, are situated inside their laboratories and each obtains some input system from the environment, acts on it with a quantum operation, and then send it out again. While inside the laboratories the order of events is well-defined, signified by the two clocks, there need not be a well-defined global ordering of the agents' operations. The outcome statistics of the operations performed by A and B is described by a process matrix $\mathcal{G}$, which can be such that it is not possible to define a global order parameter (i.e. time) for these operations.

The process-matrix framework was introduced to capture the most general correlations between agents, labeled $X^k \in \{A \ldots N\}$, without assuming a well-defined causal order between them. The agents are associated with their laboratories inside of which the order of events is well-defined, i.e. there is a well-defined local time for each agents. Each laboratory is assumed to receive some input quantum system with Hilbert space $\mathscr{H}^{X_1^k}$ and send away an output quantum system described by Hilbert space $\mathscr{H}^{X_2^k}$. Between these two events, agent X^k is thought to perform a quantum operation or quantum instrument, i.e. a probabilistic map from quantum states to quantum states, which potentially allows him or her to obtain some classical outcome i_k, see Figure 1.6. Hence, agent X^k is described by a set of quantum operations $\{\mathscr{M}_{i_k}^{X_k}\}$. The proposal of Oreshkov et al. [2012] is that the joint probability of maps of different agents is given by a generalized Born rule

$$P(\mathscr{M}_{i_1}^{X^1} \dots \mathscr{M}_{i_n}^{X^n}) = \mathrm{Tr}\left(W \left(\bigotimes_{k=1}^{n} M_{i_k}^{X_1^k X_2^k} \right) \right), \tag{1.10}$$

where $M_{i_k}^{X_1^k X_2^k} \in \mathscr{L}(\mathscr{H}^{X_1^k} \otimes \mathscr{H}^{X_2^k})$ are the Choi-Jamiołkowski (CJ) matrices Choi [1975]; Jamiołkowski [1972] of the agents' operations, which are completely positive and trace non-increasing. Moreover, summing over the outcomes i_k of an agent should give a completely positive, trace preserving map, which means the CJ matrices satisfy

$$\forall k : \mathrm{Tr}_{X_2^k} \left(\sum_{i_k} M_{i_k}^{X_1^k X_2^k} \right) = \mathbb{1}_{X_1^k} \,. \tag{1.11}$$

The mathematical object relating the agents' operation is the so-called process matrix $W \in \mathscr{L}\left(\bigotimes_{k=1} \mathscr{H}^{X_1^k} \otimes \mathscr{H}^{X_2^k}\right)$. Requiring that Equation (1.10) gives proper probabilities, numbers in $[0,1]$ which sum to 1, excludes so-called *causal loops*, which would allow some agent to signal to his or her own past creating logical problems like the famous grandfather paradox. In general, however, process matrices allow for scenarios with indefinite causal order, where no global order, i.e. a well-defined global time, exists. Such non-causal process could, for example, be superpositions of causally ordered space-time structures.

Alternatively, process matrices can be identified with generalized higher order quantum maps $\mathscr{G}^W(\mathscr{M}^A \dots \mathscr{M}^N)$, which, if the global order of the operations performed is well-defined, are equivalent to quantum combs Chiribella et al. [2008]; Perinotti [2017]; Bisio et al. [2011]; Chiribella et al. [2009]; Yokojima et al. [2021].

Causal inequalities are bounds on the success probabilities of guessing games between agents, see Baumeler et al. [2014]; Branciard et al. [2016]; Brukner [2015], which hold if there is a definite global causal order but can be violated with non-causal processes and certain local strategies. Processes, which violate causal inequalities, can be understood in terms of post selected closed timeline curves Baumeler et al. [2019]; Lloyd et al. [2011]. Analogous to separable quantum states one can define causally separable processes Oreshkov and Giarmatzi [2016]. They correspond to definite global causal orders or convex mixtures of them and can never violate a causal inequality. However, not all processes encoding indefinite causal order can violate causal inequalities. Such processes can still be shown to exhibit indefinite causal order by causal witnesses Araújo et al. [2015]. An example for this kind of processes, which has also been implemented experimentally, is the *quantum-switch*, where the order of operations is determined by a control quantum system in superposition Chiribella et al. [2013]; Goswami et al. [2018]; Guo et al. [2020]; Procopio et al. [2015]; Rubino et al. [2017]; Taddei et al. [2021]; Wei et al. [2019].

1.3.2 Pure processes and causal reference frames

The fact that many non-causal processes lack a clear interpretation motivated looking for additional principles which might be relevant for identifying physically meaningful processes. In Araújo et al. [2017], the authors suggest that only processes that can be obtained form *pure processes* are physical. A pure process $\mathcal{G}$ is a a multi-linear unitary-preserving map, i.e. $\mathcal{G}(U_A, U_B \ldots U_N)$ is unitary for any unitary operations $U_A, U_B \ldots U_N$. Note that, by introducing suitable ancillary systems $A', B' \ldots N'$, one can always represent the agents' quantum operations as unitary operations acting on the respective ancilla and the system parts that the agents obtain as inputs, see Figure 1.7. Pure processes can be understood as reversible transformations from a well-defined casual past to a well-defined casual future, with potentially indefinite causal order in between.

Note that the quantum switch mentioned in the previous section is an example of a pure process. In fact it has been shown in Barrett et al. [2021] that for two agents any pure process is either causally ordered or some variant of coherent control of causal order, of which the switch is the best known example. This in turn means that pure bipartite processes cannot violate causal inequalities. For more than two agents, however, restricting oneself to purifiable processes does not exclude violations of causal inequalities. An example of a pure tripartite process with indefinite causal order, often called the *Lugano process*, is known to violate causal inequalities, see Araújo et al. [2017]; Baumeler and Wolf [2016].

In Allard Guérin and Brukner [2018] the authors introduced the notion of

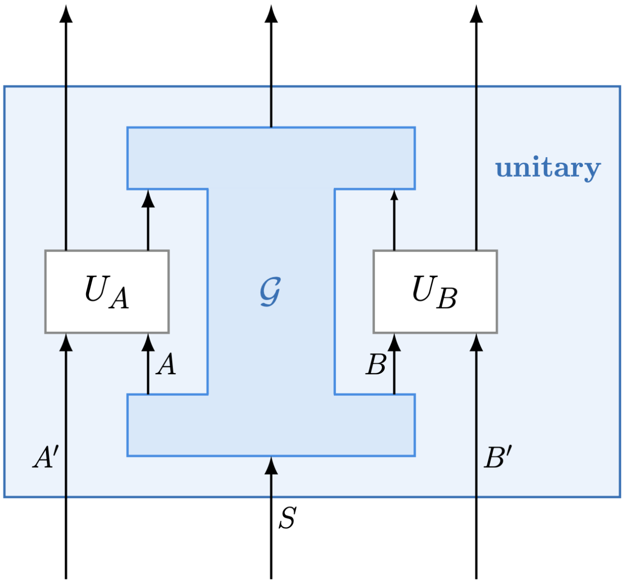


Figure 1.7. Pure processes for the bipartite scenario: The quantum operations of agents A and B can be represented as unitaries U_A, U_B by introducing ancillary systems A', B'. These ancillas are not acted upon by anything other than the quantum operations of the respective agents. A pure process $\mathscr{G}$ is a multilinear supermap that gives an induced unitary transformation on $S \otimes A' \otimes B'$ when the agents are applying unitary operations U_A, U_B.

causal reference frames and showed it to be equivalent to the process-matrix framework for pure processes. More concretely, every pure process has a causal reference frame decomposition and any such decomposition corresponds to some pure process. The casual reference of an agent describes that agent's perspective inside such a process. Consider, for example, agent A who from his or her perspective applies quantum operation U_A at a certain time as measured inside his or her laboratory. The evolution from the well-defined global past, which is common to all agents, up to that point is called the causal past of A and given by a unitary $\Pi_A(U_B, \ldots U_N)$, which can depend on the operations of all other agents. While A applies unitary U_A to the input to his or her laboratory and ancilla A', all other degrees of freedom evolve in an uncorrelated way. The evolution following this application up to the well-defined common global future is given by another unitary $\Phi_A(U_B, \ldots U_N)$, which is called the causal future of A and can again depend on the other agents' operations. Any pure process $\mathscr{G}$ can then be written as

$$\mathscr{G}(U_A, U_B, \ldots U_N) = \Phi_A(U_B, \ldots U_N)\,(U_A \otimes \mathbb{1})\,\Pi_A(U_B, \ldots U_N), \tag{1.12}$$

which corresponds to A's point of view. Similar decompositions exit for all agents whose operations are related by process $\mathcal{G}$. The casual reference frame decomposition for both the bipartite quantum switch and the time-reversed Lugano process are depicted in Figure 1.8.

1.4 The Page-Wootters formalism

The Page-Wooters formalism (PWF) was introduced by Don Page and William Wootters Page and Wootters [1983]; Wootters [1984] to address the problem of time in quantum theory Achuthan and Venkatesan [1958]; Isham [1993]; Kuchař [2011]. It associates a Hilbert-space structure with time similar to how standard quantum theory does with spatial position. By introducing a quantum clock system, the PWF allows for describing time-evolution of quantum systems via correlations between the clock and said systems. Consider the clock and systems being associated with Hilbert spaces $\mathcal{H}_c \simeq L^2(\mathbb{R})$ and $\mathcal{H}_S$ respectively. Choosing $\mathcal{H}_c$ to be spanned by square integrable functions on the real line allows to, in analogy to the spatial momentum operator, define "temporal momentum" operator $\hat{p}_t := -i\frac{\partial}{\partial t}$. The so-called physical state or history state $|\Psi\rangle\rangle$ of the clock and the systems under consideration is a solution to the Wheeler-DeWitt-like equation DeWitt [1967]

$$\hat{C}|\Psi\rangle\rangle = \left(\hat{p}_t + \hat{H}_S\right)|\Psi\rangle\rangle = 0, \tag{1.13}$$

which can in some cases be interpreted as arising from the canonical quantization of a gauge theory with a Hamiltonian constraint. One defines the time observable $\hat{T}_c$ as the one that is covariant with respect to the clock Hamiltonian $\hat{p}_t$ Holevo [1982]; Busch et al. [1995], which means that

$$\hat{T}_c = \int_{\mathbb{R}} dt\, t|t\rangle\langle t|, \tag{1.14}$$

where $|t\rangle$ are improper eigenvectors of $\hat{T}_c$ associated with the eigenvalue $t \in \mathbb{R}$, which are connected to one another by the unitary generated by the clock Hamiltonian, i.e. $|t_2\rangle = e^{-i\hat{p}_c(t_2-t_1)}|t_1\rangle$. Enforcing the covariance condition then implies that $\hat{T}_c$ is canonically conjugate to the clock Hamiltonian, $[\hat{T}_c, \hat{p}_c] = i$. If one now defines the conditional state of the systems given that the clock reads the time t as

$$|\psi(t)\rangle_S := \left(\langle t| \otimes \mathbb{1}_S\right)|\Psi\rangle\rangle\,, \tag{1.15}$$

a.

b.

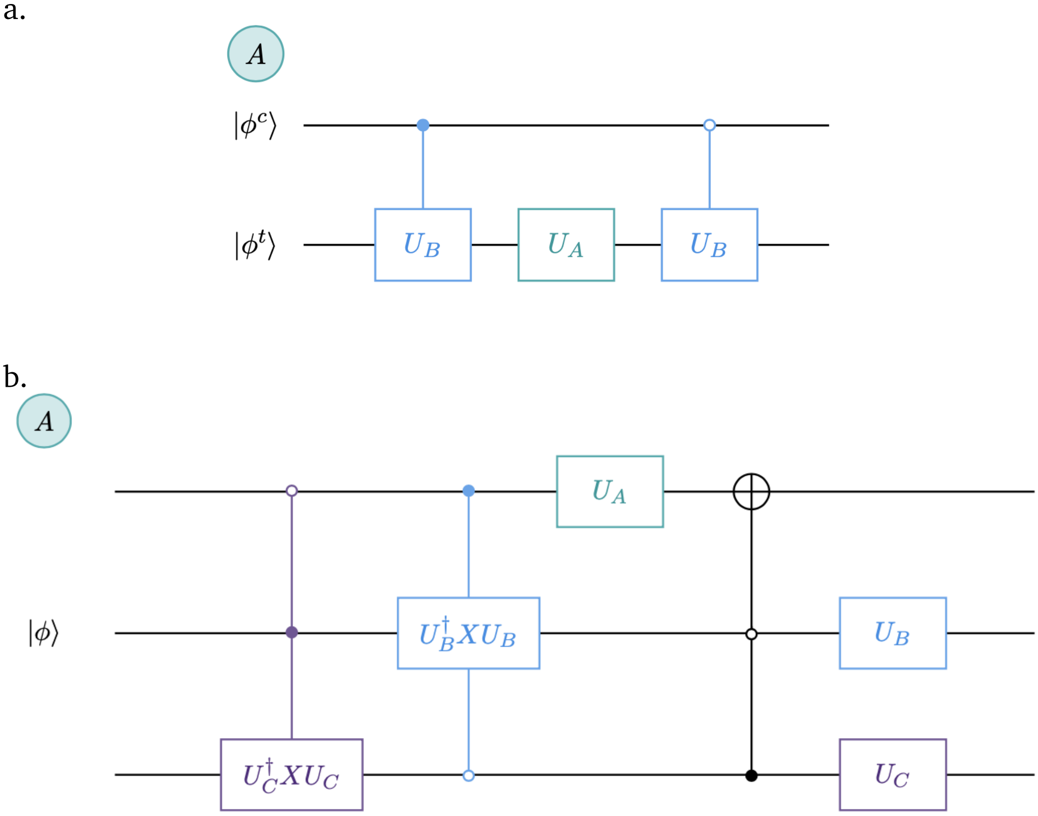


Figure 1.8. The causal reference frame of agent A for the quantum switch (a) and for the time reversed Lugano process (b) both according to Allard Guérin and Brukner [2018]. a.: In case of the quantum switch, both agents apply their operations to the target system $|\phi_t\rangle$. According to agent A's perspective his or her unitary U_A is applied in the middle of the process, while agent B applies their unitary either before or after that, depending on the value of the control $|\phi_c\rangle$. b.: In case of the Lugano process, both the casual past and casual future of A depend on the operations of B and C in a non-trivial manner. Note that, other than for the switch, there is no separate control system in case of the time reversed Lugano process. All agents act on different subsystems of input $|\phi\rangle$. Moreover, the agent's action is partially controlled by the subsystems the other agents act on.

one can formally expand the physical states $|\Psi\rangle\rangle$ as

$$|\Psi\rangle\rangle = \int \mathrm{d}t \; |t\rangle \otimes |\psi(t)\rangle_S. \tag{1.16}$$

In this expansion it is clear that an ordered time sequence $t_0 < t_1 < t_2$ corresponds to a history of the system given by states $|\psi(t_0)\rangle, |\psi(t_1)\rangle$ and $|\psi(t_2)\rangle$. Hence, the physical states $|\Psi\rangle\rangle$ are also called history states. The conditional system state $|\psi(t)\rangle_S$ can be shown to satisfy the standard Schrödinger equation,

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S, \tag{1.17}$$

see for example Wootters [1984]. The probability that a system observable $\hat{M} = \sum_m m|m\rangle\langle m|_S$ takes the value m when the clock reads time t is given by

$$P\left(\hat{M} = m \text{ when } \hat{T}_C = t\right) = \frac{\langle\langle\Psi|\left(|t\rangle\langle t| \otimes \Pi_m\right)|\Psi\rangle\rangle}{\langle\langle\Psi|\left(|t\rangle\langle t| \otimes \mathbb{1}_S\right)|\Psi\rangle\rangle}, \tag{1.18}$$

which using Equation (1.15) gives the Born rule of standard quantum theory

$$P(m \text{ when } t) = \langle\psi_S(t)|m\rangle\langle m|\psi_S(t)\rangle. \tag{1.19}$$

From here on we will use the abbreviated expression for the probabilities, in which we omit the operators, i.e., $P(\hat{M} = m \text{ when } \hat{T}_C = t) = P(m \text{ when } t)$. Together Equations (1.17) and (1.19) mean that the Page-Wootters formalism recovers the two main features of standard quantum theory, i.e., the Schrödinger equation and the standard Born rule.

In general, solutions to Equation (1.13) can be obtained from arbitrary states $|\phi\rangle\rangle$ in the kinematical Hilbert space $\mathscr{K} \simeq \mathscr{H}_c \otimes \mathscr{H}_S$ with the operator

$$\hat{P} := \int_{\mathbb{R}} ds\, e^{-is\hat{C}}. \tag{1.20}$$

That is, supposing $|\phi\rangle \in \mathscr{K}$ is some arbitrary state in the kinematical space, then $|\Psi\rangle\rangle = \hat{P}|\phi\rangle$ is a solution to the constraint equation, i.e. $\hat{C}(\hat{P}|\phi\rangle) = 0$. For this reason the operator $\hat{P}$ is sometimes called the physical projector, see Dolby [2004], although it is not a projector in the strict mathematical sense. Note that, for the choice of clock Hamiltonian above, which we refer to as an *ideal quantum clock*, the solutions to Equation (1.13) are often referred to as the physical space $\mathscr{H}_{\text{ph}}$, but do not form a proper subspace of the kinematical Hilbert space $\mathscr{K}$. Since the the spectrum of $\hat{C}$ is continuous around zero, the physical states $|\Psi\rangle\rangle$ are not normalizable in the kinematical inner product Kiefer [2012]; Rovelli [2004]. One can define a new inner product to normalize the physical states, which then defines the physical Hilbert space $\mathscr{H}_{\text{ph}}$,

$$\langle\langle\Psi|\Phi\rangle\rangle_{\text{ph}} := \langle\langle\Psi|\hat{P}|\Phi\rangle\rangle, \tag{1.21}$$

where the right hand-side signifies the inner product on the kinematical Hilbert space $\mathscr{K}$. Requiring that the physical states are normalized with respect to the physical inner product in Equation (1.21) then implies that the conditional states $|\psi(t)\rangle_S$ are normalized in $\mathscr{H}_S$. Moreover, $\langle t_2|\hat{P}|t_1\rangle = \mathscr{U}(t_2, t_1)$ is a unitary operator on $\mathscr{H}_S$ and in case of $\hat{C} = \hat{p}_c + \hat{H}_S$ is given by

$$\langle t_2|\hat{P}|t_1\rangle = e^{-i(t_2-t_1)H_S}, \tag{1.22}$$

which is the time evolution according to the Schrödinger equation.

As pointed out by Karel Kuchař Kuchař [2011] the PWF does not allow to calculate conditional probabilities for two consecutive measurements in accordance with standard quantum theory, i.e.

$$P(b|a) = |\langle b|e^{-i(t_2-t_1)H_S}|a\rangle|^2. \tag{1.23}$$

Moreover, Kuchař claimed that the Page-Wootters formalism is unable to reproduce the correct propagator of the system between two measurements at times t_1 and t_2 respectively. The problem with attempting to determine the probability of the first observable $\hat{A}$ giving outcome a at time t_1 and another observable $\hat{B}$ giving outcome b at t_2, is that after applying the operator for the first measurement one ends up with a state that is in general no longer part of the physical space, i.e. $\hat{C}(|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|\,|\Psi\rangle\rangle) \neq 0$. Hence, applying the second operator, i.e. $|t_2\rangle\langle t_2| \otimes |b\rangle\langle b|$, does *not* give the probabilities in Equation (1.23). Carl Dolby solved this problem by exclusively considering objects on the physical Hilbert space, see Dolby [2004]. More concretely, he applies the physical projector form Equation (1.20) to the measurement operators, threfore considering $\hat{P}|t_1\rangle\langle t_1|\otimes|a\rangle\langle a|\hat{P}$ and $\hat{P}|t_2\rangle\langle t_2|\otimes|b\rangle\langle b|\hat{P}$. This means that the sate after the first measurement is $\hat{P}(|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|)\,|\Psi\rangle\rangle$, which *is* a solution of the constraint equation by construction and indeed Dolby was able to recover the standard quantum probabilities in Equation (1.23)

$$\begin{aligned} &P(b \text{ when } t_2|a \text{ when } t_1) \qquad (1.24)\\ &:= \frac{\langle\langle\Psi\,|\,\hat{P}|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|\hat{P}|t_2\rangle\langle t_2| \otimes |b\rangle\langle b|\hat{P}|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|\hat{P}\,|\,\Psi\rangle\rangle}{\langle\langle\Psi\,|\,\hat{P}|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|\hat{P}\,|\,\Psi\rangle\rangle}\\ &= |\langle b|e^{-i(t_2-t_1)H_S}|a\rangle|^2. \end{aligned}$$

Following arguments in Hellmann et al. [2007] concerning the lack of an operational meaning for certain constructions in Dolby's proposal, Giovannetti *et al.* Giovannetti et al. [2015], proposed another solution to the problem pointed out by Kuchař. They used purified (i.e. von Neumann-) measurement model, incorporating the measurement devices into the Page-Wooters formalism. The measurements are then entangling unitaries between the measured system and apparata. The corresponding interaction Hamiltonian is an additive term in the constraint operator, which allows the authors to equate the two measurements on the system at different times t_1 and t_2 with one joint measurement of the two apparata at some final time $t > t_2 > t_1$

$$\begin{aligned} &P(b \text{ when } t_2 | a \text{ when } t_1) \\ &:= \frac{\langle\langle\Psi|\,|t\rangle\langle t| \otimes |A_a\rangle\langle A_a| \otimes |B_b\rangle\langle B_b|\,|\Psi\rangle\rangle}{\langle\langle\Psi|\,|t_1\rangle\langle t_1| \otimes |a\rangle\langle a|\,|\Psi\rangle\rangle} = |\langle b|e^{-i(t_2-t_1)H_S}|a\rangle|^2, \end{aligned} \tag{1.25}$$

where $|A_a\rangle$ and $|B_b\rangle$ are the states of the first measurement apparatus having registered outcome a and the second measurement apparatus having registered outcome b respectively. Just like Carl Dolby's approach, Equation (1.25) recovers standard quantum probabilities for two consecutive measurements, but avoids those constructions possible by the approach in Dolby [2004], which were criticized in Hellmann et al. [2007].

1.4.1 Wigner's friend in terms of the Page-Wootters formalism

In order to model the Wigner's-friend experiment described in Section 1.2.1 in terms of the PWF introduced in the previous section, we employ a circuit-like description as shown in Figure 1.9. The clock C associated with Hilbert space $\mathscr{H}_C \simeq L^2(\mathbb{R})$ keeps track of time and in particular indicates the times of the friend's and Wigner's measurements t_F and respectively t_W. The qubit system S is initially in state

$$|\psi\rangle_S = a|\uparrow\rangle_S + be^{i\phi_S}|\downarrow\rangle_S, \tag{1.26}$$

where $a, b, \phi_S \in \mathbb{R}$, and together with the friend and Wigner, both in "ready" states $|r\rangle_F \in \mathscr{H}_F$ and $|r\rangle_W \in \mathscr{H}_W$, constitutes the input to the circuit. The friend's measurement in the σ_z-basis yielding outcomes "up" or "down" is described by the set of projectors $\{\Pi_\uparrow = |\uparrow\rangle\langle\uparrow|_S, \Pi_\downarrow = |\downarrow\rangle\langle\downarrow|_S\}$, where $\Pi_\uparrow + \Pi_\downarrow = \mathbb{1}_S$. Wigner's measurement is also binary giving outcomes "yes" and "no" where the first outcome corresponds to the friend's laboratory being in state

$$|\text{yes}\rangle_{SF} = \alpha|\uparrow\rangle_S|u\rangle_F + \beta e^{i\phi_{SF}}|\downarrow\rangle_S|d\rangle_F, \tag{1.27}$$

where $\alpha, \beta, \phi_{SF} \in \mathbb{R}$. Again we can associate the measurement with projectors $\Pi_{\text{yes}} = |\text{yes}\rangle\langle\text{yes}|_{\text{SF}}$ and $\Pi_{\text{no}} = \mathbb{1}_{SF} - |\text{yes}\rangle\langle\text{yes}|_{\text{SF}}$. The measurements of F and W

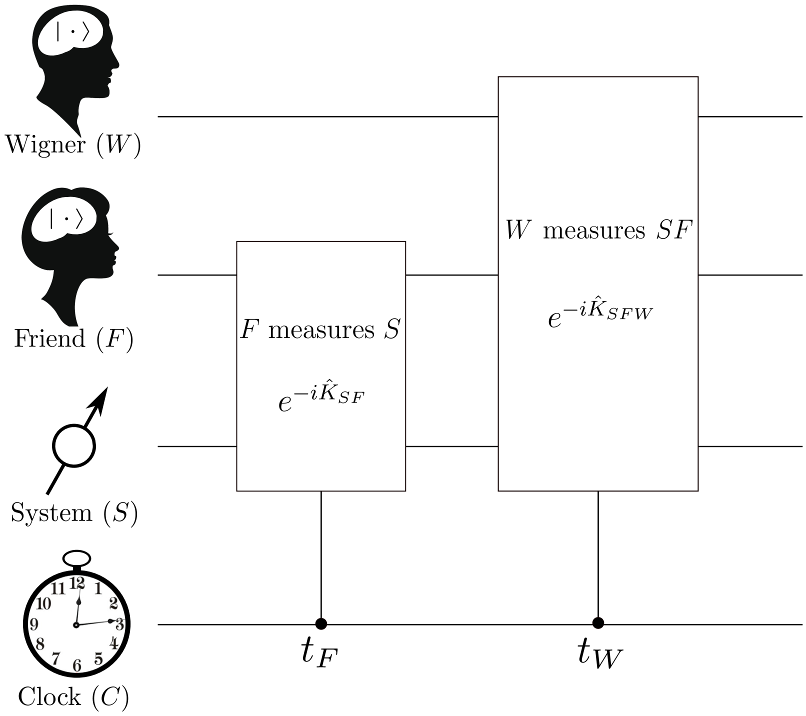


Figure 1.9. The circuit representation of the Wigner's friend experiment as encoded in the physical state $|\Psi\rangle\rangle$ in Equation (1.29). At time t_F indicated by the clock C the Friend performs a measurement entangling her with the system S. At another time $t_W > t_F$ Wigner performs a measurement on both S and F which entangles him with the joint system $S+F$. (picture taken from Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz and Brukner [2019])

are described by interaction Hamiltonians $\hat{K}_{SF}$ and $\hat{K}_{SFW}$, that couple respectively S and F during F's measurement, and S, F, and W during W's measurement as follows

$$e^{-i\hat{K}_{SF}}|\psi\rangle_S|r\rangle_F = \sum_{f\in\{u,d\}} \Pi_f|\psi\rangle_S|f\rangle_F,$$
$$e^{-i\hat{K}_{SFW}}|\phi\rangle_{SF}|r\rangle_W = \sum_{w\in\{\text{yes},\,\text{no}\}} \Pi_w|\phi\rangle_{SF}|w\rangle_W,$$

where $|\psi\rangle_S$ is the state of the system for clock readings $t < t_F$ and $|\phi\rangle_{SF}$ the joint state of $S+F$ at times $t_F < t < t_W$. We assume for simplicity that there are no free dynamics of S, F, and W, which means that the constraint equation is given by

$$\left(\hat{H}_c + \delta(\hat{T} - t_F)\hat{K}_{SF} + \delta(\hat{T} - t_W)\hat{K}_{SFW}\right)|\Psi\rangle\rangle = 0, \tag{1.28}$$

where the clock Hamiltonian is $\hat{H}_c = \hat{p}_t$ and the delta couplings mean that the measurements are instantaneous. The solutions to Equation (1.28) take the form

$$\begin{aligned}|\Psi\rangle\rangle = &\int_{-\infty}^{t_F} dt\,|t\rangle|\psi\rangle_S|r\rangle_F|r\rangle_W \\ &+\int_{t_F}^{t_W} dt\,|t\rangle \sum_{f\in\{u,d\}} \Pi_f|\psi\rangle_S|f\rangle_F|r\rangle_W \\ &+\int_{t_W}^{\infty} dt\,|t\rangle \sum_{\substack{f\in\{u,d\}\\ w\in\{\text{yes},\text{no}\}}} \Pi_w\Pi_f|\psi\rangle_S|f\rangle_F|w\rangle_W,\end{aligned} \tag{1.29}$$

where state $|\psi\rangle_S$ is given by Equation (1.26).

In Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz and Brukner [2019] we used the Page-Wootters formulation of the Winger's friend experiment to remove the ambiguity of the different dynamical descriptions. Using the Page-Wooters description above, both Wigner and his friend, provided that they have the same prior information about the experiment, would agree on the global state, i.e. the history state in Equation (1.29). Different probability assignments can then only arise if they use different types of Born-rules. The details are presented in Section 2.3.

1.4.2 A Page Wootters formulation of quantum circuits

The Page-Wootters formalism can be adapted to quantum circuits with a discrete quantum clock, see for example Breuckmann and Terhal [2014]; Caha et al. [2018]; Feynman [1985]; Kitaev et al. [2002]. Form a fundamental point of view, one could argue that, since the information acquired via measurements is finite, also physics should be discrete and indeed finite on a fundamental level, see for example Gisin [2019]. Regardless of that, a discrete, finite dimensional clock Hilbert space $\mathscr{H}_c$ in naturally suited for describing quantum circuits within the Page-Wootters formalism, see Figure 1.10. The constraint equation looks as follows

$$\hat{C}|\Psi\rangle\rangle = \sum_t \hat{H}_t|\Psi\rangle\rangle = 0, \tag{1.30}$$

where the Hamiltonians

$$\hat{H}_t = -\frac{1}{2}\left(|t\rangle\langle t-1| \otimes U_t + |t-1\rangle\langle t| \otimes U_t^\dagger - |t-1\rangle\langle t-1| - |t\rangle\langle t|\right), \tag{1.31}$$

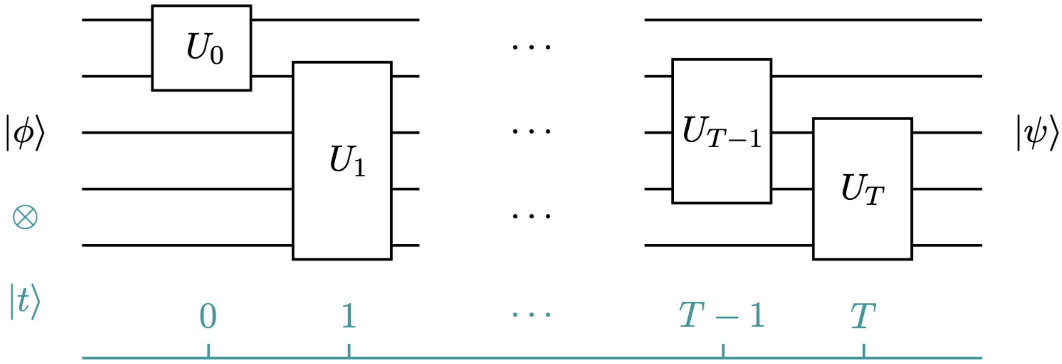


Figure 1.10. The Page-Wootters formulation of quantum circuits: One considers a quantum clock that keeps track of the number of computational steps that have happened so far. At computational step t, the circuit applies the gate U_t. The input to the quantum circuit is $|\phi\rangle$ and the output of the circuit is $|\psi\rangle = U_T \cdots U_0 |\phi\rangle$.

can be understood as making the clock tick once and applying the unitary U_t. Solutions to Equation (1.13) are history states of a quantum circuit where at time t unitary U_t is applied to the target systems described by Hilbers space $\mathscr{H}_S$:

$$|\Psi\rangle\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle_c \otimes U_t \ldots U_0 |\phi\rangle_S = \sum_{t=0}^{T} |t\rangle_c \otimes |\psi(t)\rangle_S, \tag{1.32}$$

with $|\phi\rangle \in \mathscr{H}_S$ being the circuit's input. The conditional system state at the final time T corresponds to the output of the circuit, $|\psi(T)\rangle_S = |\psi\rangle = U_T \ldots U_1 |\phi\rangle$.

Note that, for finite-dimensional clock Hilbert spaces, the physical Hilbert space is a proper subspace of the kinematical Hilbert space. As discussed in Section 1.4.1, this is not the case in the infinite dimensional case Marolf [2000]; Hoehn et al. [2019] and constitutes a technical simplification. We can again define a physical projector

$$\hat{P} := \sum_i |\Psi_i\rangle\rangle\langle\langle\Psi_i|, \tag{1.33}$$

which is now a proper projector onto the space of physical states. The states $|\Psi_i\rangle\rangle$ are given according to Equation (1.32) with initial states $|\phi_i\rangle$, which form an orthonormal basis of $\mathscr{H}_S$. Similar to the continuous case, the physical projector can be related to the unitary evolution of the circuit between two times t_2 and t_1 by

$$\langle t_2 | \hat{P} | t_1 \rangle = \frac{1}{T+1} U_{t_2} \cdots U_{t_1+1}. \tag{1.34}$$

In Baumann et al. [2021] we generalize the Page-Wooters description of quantum circuits to multiple discrete clocks, which allows us to implement processes with indefinite casual order, compare Section 1.3, and impose some constraints on the physicality of such processes. This is presented in detail in Sections 3.2 and 3.3.

Chapter 2

Wigner's friend

This Chapter contains the results on the topic of encapsulated observers obtained during my PhD, which are published in Allard Guérin et al. [2020]; Baumann et al. [2016]; Baumann and Wolf [2018]; Baumann, Del Santo and Brukner [2019]; Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz and Brukner [2019] and Baumann and Brukner [2020]. In order the address the Wigner's friend paradox and Wigner's-friend-type setups in general, we first make a clear distinction between the *formalism* and the *interpretations* of scientific theories and quantum theory in particular. Encapsulated observers allow for deciding between two inequivalent quantum formalisms usually employed when describing measurements – the relative-state formalism and the standard formalism with the Born- and state-update rules, see Section 2.1. We then consider different paradoxical situations in Wigner's-friend setups in terms of their observable consequences analyzing which setups can give rise to actual contradictions, i.e. two contradicting pieces of information that can be compared by some agent, in Section 2.2. We further address the problem of different probability assignments for encapsulated observers within a Page-Wootters formulation of the Wigner's-friend experiment in Section 2.3. Finally we formulate a no-go theorem for the persistence of the friend's perception in Wigner's-friend experiments, see Section 2.4.

2.1 Formalism and interpretation

A physical theory consists of a *mathematical formalism*, which allows for predicting the outcomes of experiments, together with an *ontological interpretation*, which establishes a correspondence between the mathematical objects of the theory and elements of some notion of physical reality. In most physical theories the identification of the entities in the formalism with an ontological description is unambiguous and generally agreed upon. In quantum theory, however, there exist multiple interpretations that differ vastly in the ontological meaning they associate with, in particular, the quantum wave function Cabello [2017]; Leifer [2014]. That an empirically adequate scientific theory should be able to make predictions, in order to have testable empirical content, means that one can associate a measure of likelihood to an event y to happen, given that another event x has already occurred. In the words of Eugene Wigner:

> "One realizes that *all* the information which the laws of physics provide consists of probability connections between subsequent impressions that a system makes on one if one interacts with it repeatedly, i.e., if one makes repeated measurements on it." Wigner [1995]

Mathematically speaking, this corresponds to conditional probability distributions $P(y|x)$, which are, in general, conditioned on the specifics of the experiment. If a user of the theory lacks the information about these specifics, he or she can subjectively assign their best guess for a probability distribution over these unknown variables, which then allows for computing an estimate of $P(y|x)$. The theory should then provide a list of the variables that would have to be known in order to determine $P(y|x)$. Moreover, if a theory prescribes how to assign probabilities to single events[1], standard probability theory allows for the definition of a joint probability distribution through the identity $P(x,y) = P(x|y)P(y)$.

Since physical theories are in general not "complete" in the sense that their formalisms give a unique and exhaustive description of some part of an objective ontological reality, different mathematical formalisms can describe the same physical phenomena. We, therefore define *empirical equivalence* of two mathematical formalisms as follows.

Definition 1. *Two mathematical formalisms are* empirically equivalent *if they yield the same predictions for the outcomes of all possible experiments.*

A well known example of two empirically equivalent formalism are the Lagrange and Hamilton formulation of classical mechanics. Moreover, a mathematical formalism might have different ontological interpretations. A well known example thereof are the Bayesian and frequentist interpretation of probability theory. We will extend this idea not only to the very same mathematical formalism but also those formalisms that are empirically equivalent to it and, hence, define the following.

[1]It is worth noting that also single event probabilities are fundamentally conditional probabilities, namely a priori conditioned on all past events that can possibly influence the probability of that one event the probability is assigned to.

Definition 2. *If two* physical theories *are empirically equivalent, they are called different* interpretations *of the same* formalism.

In line with Karl PopperPopper [1935] a physical theory needs to be *falsifyable*, meaning that the predictions of the theory are such that experiments can in principle contradict them. If such a contradiction appears with statistical significance the theory is said to be falsified and should be dismissed. According to our Definitions 1 and 2 there then must exist at least one *decisive experiment* where the two theories make different predictions about the outcome. This experiment is then able to falsify only one of the two competing theories. If no such experiment exists the two are not competing theories but competing interpretations of the same theory, which cannot be differentiated based on empirical evidence but rather based on metaphysical arguments.

2.1.1 Two quantum formalisms and some of their interpretations

In relation to the quantum measurement problem discussed in Section 1.2 we now examine two possible descriptions of a quantum measurement, namely via non-unitary collapse dynamics or via unitary dynamics entangling the system and its observer. As we will discuss these two descriptions amount to two different quantum formalisms, that are empirically equivalent as long as one considers the same level of observation. For encapsulated observers, however, these two formalisms predict different probabilities for the measurement results. We will describe only pure states and projective measurements in the main text for the sake of readability. The statements derived, however, equally hold for mixed states and POVMs as shown in Appendix A.1.

First there are the standard Born and state-update rule Born [1954], which give the following evolution of quantum state $|\phi\rangle \in \mathscr{H}_S$ upon measurement of observable $A = \sum_a a|a\rangle\langle a| \in \mathscr{L}(\mathscr{H}_S)$

$$|\phi\rangle_S \xmapsto[\text{result: } a]{A} |a\rangle_S, \tag{2.1}$$

and the probability for observing outcome a in said measurement as

$$P_\phi^{clps}(a) = \mathrm{Tr}(|a\rangle\langle a||\phi\rangle\langle\phi|) = |\langle a|\phi\rangle_S|^2. \tag{2.2}$$

Equations (2.1) and (2.2) are assumed to be correct in most interpretations of quantum theory albeit with different interpretations accompanying their meaning. While some regard Equation (2.1) as a physical process of the system, for example Bassi et al. [2013]; Bohm [1952a,b]; DeWitt and Graham [2015]; Lombardi and Dieks [2017]; Wallace [2012], others consider it to describe a change of knowledge of the observer, who performed the measurement, for example Brukner and Zeilinger [2003]; Bub and Pitowsky [2010]; Fuchs [2010]; Healey [2012]; Rovelli [1996]. Another sort of approach, compatible with Equations (2.1) and (2.2) are objective collapse models like Diósi [2014]; Ghirardi et al. [1986]; Penrose [2000]. While they modify the standard quantum formalism with non-linear terms, which account for a "real" collapse, they do this in a way that ensures empirical equivalence to standard quantum theory for all experiments conducted so far. As an alternative to Equations (2.1) and (2.2) Hugh Everett proposed the relative-state formalism Everett III [1957], which incorporates the measurement apparatus or the observer into a unitary description of the measurement

$$\begin{aligned} U_O : \quad & \mathscr{H}_S \otimes \mathscr{H}_O \to \mathscr{H}_S \otimes \mathscr{H}_O \\ & |a\rangle_S |r\rangle_O \mapsto |a\rangle_S |A_a\rangle_O \quad \forall a, \end{aligned} \tag{2.3}$$

where $|r\rangle_O$ and $|A_a\rangle_O$ correspond to the state of the observer (or apparatus) being ready to measure and having measured outcome a respectively. Hence, initial states $|\phi\rangle_S$ of the system and $|r\rangle_O \in \mathscr{H}_O$ of the observer evolve as follows:

$$|\phi\rangle_S |r\rangle_O = \sum_a \langle a|\phi\rangle |a\rangle_S |r\rangle_O \mapsto \sum_a \langle a|\phi\rangle |a\rangle_S |A_a\rangle_O = |\Phi_{tot}\rangle. \tag{2.4}$$

Note that, a priori the relative-state formalism, meaning the description of a measurement according to Equation (2.4), constitutes a different quantum formalism compared to Equation (2.2). It is not necessarily a different interpretation of quantum theory, unless the two formalisms can be shown to be empirically equivalent. However, Equation (2.4) alone does not allow for predicting the outcomes of experiments. In order to do that, the use of standard Born rule in Equation (2.1) has been motivated by a many-worlds interpretation Wheeler [1957] and decision-theoretical arguments Deutsch [1999]; Saunders [2004]. In this case the two formalism are trivially empirically equivalent, since they use exactly the same probability rule. According to Definition 2 we can regard generalized Bohmian mechanics Sudbery [1986] as a different interpretation of the relative-state formalism, since it also features unitary evolution of the global wave function regardless of a measurement happening or not.

According to the ideas of Grete Hermann Hermann [1935] the state of the observing system is the one that represents the result of the measurement and that the systems sate is only defined relative to the state of the observer. Based on that we propose a different use of the Born-rule for the relative state formalism, namely

$$P_{\phi}^{rels}(a) = \mathrm{Tr}(\mathbb{1}_S \otimes |A_a\rangle\langle A_a|_O \cdot U_O|\phi\rangle\langle\phi|_S \otimes |r\rangle\langle r|_O U_O^{\dagger}) \,, \tag{2.5}$$

which means that the probability of outcome a is given by the projection onto the state $|A_a\rangle$ of the observer having seen a acting on the overall state according to Equation (2.4).

For one observer measuring a quantum system it is obvious that the two formalisms – namely Equations (2.1) and (2.2) on the one hand and Equations (2.4) and (2.5) on the other hand – give the same probabilities

$$P_{\phi}^{rels}(a) = |\langle a|\phi\rangle_S|^2 = P_{\phi}^{clps}(a) \,. \tag{2.6}$$

In Shrapnel et al. [2018] the authors motivate the Born and state-update rules from the process-matrix formalism by considering separate and consecutive measurements. More concretely, they obtain the state-update rule from the Born rule by considering consecutive measurements on one quantum system and by requiring that the Born rule provides the correct probabilities for the results.

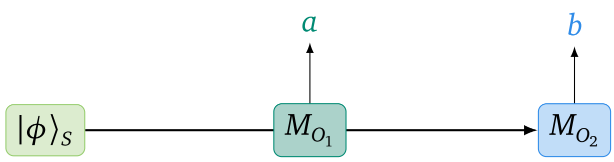


Figure 2.1. The same level of observation: Two observers O_1 and O_2 are performing consecutive measurements on a quantum system, initially in state $|\phi\rangle \in \mathscr{H}_S$, obtaining results a and b respectively. For such setups the state-update rule and the relative-state formalism give the same conditional probabilities for the results of the two agents.

Similarly we now consider two observers O_1 and O_2 successively measuring the same quantum system at times t_1 and t_2 respectively, see Figure 2.1. Their measurements are given by $M_{O_1} = \sum_a a|a\rangle\langle a|$ and $M_{O_2} = \sum_b b|b\rangle\langle b|$, where $\{|a\rangle\}$ and $\{|b\rangle\}$ are two arbitrary bases of $\mathscr{H}_S$. The system is in state $|\phi\rangle_S$ before the first measurement and its free evolution between times t_1 and t_2 is given by $U_S(t_2, t_1) = e^{-i(t_2-t_1)H_S}$, where H_S is the system Hamiltonian. According to

Equations (2.1) and (2.2), the conditional probability of result b given a is

$$\begin{aligned} P_\phi^{clps}(b|a) &= \frac{P_\phi^{clps}(a,b)}{\sum_b P_\phi^{clps}(a,b)} \\ &= \frac{\mathrm{Tr}\left(|b\rangle\langle b|U_S(t_2,t_1)|a\rangle\langle a|\phi\rangle\langle\phi|a\rangle\langle a|U_S(t_1,t_2)|b\rangle\langle b|\right)}{\mathrm{Tr}\left(|a\rangle\langle a|\phi\rangle\langle\phi|a\rangle\langle a|\right)} \\ &= |\langle b|U_S(t_2,t_1)|a\rangle|^2, \end{aligned} \tag{2.7}$$

where $P_\phi^{clps}(a,b)$ is the joint probability of O_1 measuring result a and O_2 obtaining result b. In analogy to Equation (2.5) we define this joint probability for the relative-state formalism as

$$P_\phi^{rels}(a,b) := \mathrm{Tr}\left(\mathbb{1}_S \otimes |A_a\rangle\langle A_a| \otimes |B_b\rangle\langle B_b| \cdot |\Phi_{tot}\rangle\langle\Phi_{tot}|\right), \tag{2.8}$$

where $|\Phi_{tot}\rangle = U_{O_2}U_S(t_2,t_1)U_{O_1}|\phi\rangle_S|r\rangle_{O_1}|r\rangle_{O_2}$. Hence, the conditional probability of result b given a is

$$\begin{aligned} P_\phi^{rels}(b|a) &= \frac{P_\phi^{rels}(a,b)}{\sum_b P_\phi^{rels}(a,b)} = \frac{\mathrm{Tr}\left(\mathbb{1}_S \otimes |A_a\rangle\langle A_a| \otimes |B_b\rangle\langle B_b| \cdot |\Phi_{tot}\rangle\langle\Phi_{tot}|\right)}{\sum_b \mathrm{Tr}\left(\mathbb{1}_S \otimes |A_a\rangle\langle A_a| \otimes |B_b\rangle\langle B_b| \cdot |\Phi_{tot}\rangle\langle\Phi_{tot}|\right)} \\ &= \frac{\langle b|U_S(t_2,t_1)|a\rangle\langle a|\phi\rangle\langle\phi|a\rangle\langle a|U_S(t_1,t_2)|b\rangle}{\langle a|\phi\rangle\langle\phi|a\rangle} \\ &= |\langle b|U_S(t_2,t_1)|a\rangle|^2 = P_\phi^{clps}(b|a), \end{aligned} \tag{2.9}$$

and once more the two formalisms give the same probabilities. This means that for the same level of observation the two formalisms are empirically equivalent and one can regard Equation (2.9) as justifying the use the standard Born and state-update rules within the relative-state formalism. This agrees with former motivations thereof but is independent of the interpretation of the relative-state formalism.

2.1.2 Inequivalence for encapsulated observers

We will now consider a Wigner's-friend-type setup and see that in this case the two quantum formalisms, introduced in Section 2.1.1, are no longer empirically equivalent, meaning they give different predictions for the outcomes of some measurements. We can, therefore, regard the Wigner's-friend experiment as the decisive experiment between the standard Born and state-update rules one the one hand and the relative state formalism with the use of the Born rule given by Equations (2.5) and (2.8) on the other hand.

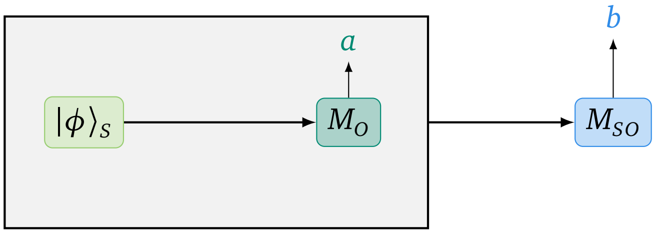


Figure 2.2. Different levels of observation: Observer O performs a measurement on a quantum system S in state $|\phi\rangle$. A superobserver SO then performs a measurement on the joint system including the observer O. For this type of consecutive measurements, the standard Born and state-update rule yield conditional probabilities for the measurement results that are different from those predicted by the the relative-state formalism. Hence, the two formalisms are empirically inequivalent and can be tested against one another in a Wigner's-friend setup.

Consider again a quantum system in state $|\phi\rangle_S$ that is measured by observer O, who performs measurement $M_O = \sum_a a|a\rangle\langle a|_S$. Afterwards superobserver SO performs the measurement $M_{SO} = \sum_b b|b\rangle\langle b|_{S,O}$ on the joint system, where $\{|b\rangle_{S,O}\}$ is some orthonormal basis of $\mathscr{H}_S \otimes \mathscr{H}_O$. The standard Born and state-update rule, i.e. Equations (2.1) and (2.2), correspond to the evolution

$$|\phi\rangle_S|r\rangle_O \xrightarrow[\text{result: } a]{M_O} |a\rangle_S|A_a\rangle_O = |a,A_a\rangle_{S,O}\,, \tag{2.10}$$

of the joint system $S+O$ during the observer's measurement. Equally, the superobserver's measurement is described by

$$|\Phi\rangle_{S,O}|r\rangle_{SO} \xrightarrow[\text{result: } b]{M_{SO}} |b\rangle_{S,O}|B_b\rangle_{SO}\,, \tag{2.11}$$

where the state $|\Phi\rangle_{S,O}$ is the state of the joint system $S+O$ right before the superobserver's measurement. We assume that nothing happens to the observer between her measurement and that of the superobserver, while the system may undergo some free evolution, which means that $|\Phi\rangle_{S,O} = \mathbb{1}_O \otimes U_S(t_2,t_1)|a,A_a\rangle_{S,O}$. Hence, analogous to Equation (2.7), we find the conditional probabilities for the results b measured by the superobserver given the result a of the observer to be

$$P_\phi^{clps}(b|a) = |{}_{S,O}\langle b|\mathbb{1}_O \otimes U_S(t_2,t_1)|a,A_a\rangle_{S,O}|^2. \tag{2.12}$$

The relative state formalism treats both measurements unitarily leading to the following overall state

$$
\begin{aligned}
|\phi\rangle_S|r\rangle_O|r\rangle_{SO} &\xrightarrow{U_O} \sum_a \langle a|\phi\rangle|a\rangle_S|A_a\rangle_O|r\rangle_{SO} \\
&\longrightarrow \sum_a \langle a|\phi\rangle \mathbb{1}_O \otimes U_S(t_2,t_1)|a,A_a\rangle_{S,O}|r\rangle_{SO} \\
&\xrightarrow{U_{SO}} \sum_{ab} \langle a|\phi\rangle\langle b|\mathbb{1}_O \otimes U_S(t_2,t_1)|a,A_a\rangle|b\rangle_{S,O}|B_b\rangle_{SO} = |\Phi_{tot}\rangle
\end{aligned}
\tag{2.13}
$$

for the case of encapsulated observers. Now, using Equation (2.9), the conditional probabilities for the results b measured by the superobserver given the result a of the observer are

$$
\begin{aligned}
P_\phi^{rels}(b|a) &= \frac{P_\phi^{rels}(a,b)}{\sum_b P_\phi^{rels}(a,b)} \qquad (2.14)\\
&= \frac{\langle b|\left(|A_a\rangle\langle A_a| \otimes \mathbb{1}_S\right)|b\rangle \sum_{a'a''} \langle\phi|a''\rangle\langle a'|\phi\rangle\langle b|\left(|A_{a'}\rangle\langle A_{a''}| \otimes U_S|a'\rangle\langle a''|U_S^\dagger\right)|b\rangle}{\sum_b P_\phi^{rels}(a,b)}
\end{aligned}
$$

which are equal to those in Equation (2.12) if and only if $\forall b : \exists x, x' : |b\rangle_{S,O} = |x\rangle_S|A_{x'}\rangle_O$. In that case both formalisms once more give the same conditional probabilities

$$
P_\phi^{clps}(b|a) = \delta_{a,x'}|\langle x|U_S(t_2,t_1)|a\rangle|^2 = P_\phi^{rels}(b|a). \tag{2.15}
$$

In genreal, however, the probabilities predicted by the standard Born and measurement-update rule are different from those predicted by the relative state formalism when one considers encapsulated observers, which renders the two formalisms *empirically inequivalent*.

In light of the considerations above one can rephrase and potentially resolve the Wigner's friend paradox in the following way, compare also Table 2.1. If all agents in a Wigner's-friend-type experiment apply the state-update rule for every measurement and, hence, calculate the conditional probabilities for their results according to Equation (2.12) they will all agree in their probability assignments, which are those traditionally attributed the the friend, compare $P_F(w)$ in Equation (1.4). This case corresponds to an *objective collapse* during a measurement. Alternatively, all agents can use the relative-state formalism and calculate the conditional probabilities using Equation (2.14). They will again all agree in their probability assignments, which are however those traditionally attributed to Wigner, compare $P_W(w)$ in Equation (1.5). We call this the *no-collapse* description. The description were F uses the state-update rule after her measurement and, therefore, Equation (2.12) but W, to whom the joint quantum system evolves unitarily, uses the relative-state formalism and Equation (2.14) instead, is paradoxical. We will call this the *subjective-collapse* model and it is most likely to be endorsed by relational Bub and Pitowsky [2010]; Pitowsky [2006]; Rovelli [1996] and subjectivistic interpretations of quantum theory Fuchs [2010, 2017]; Mermin [2013]. As we will show in the next section, however, this subjective application of inequivalent formalisms is highly problematic since it allow for an observable contradiction for encapsulated observers.

model	Friend	Wigner	agreement
subjective collapse	$P_F(w)= P^{clps}(w)$	$P_W(w)= P^{rels}(w)$	no
no collapse	$P_F(w)= P^{rels}(w)$	$P_W(w)= P^{rels}(w)$	yes
objective collapse	$P_F(w)= P^{clps}(w)$	$P_W(w)= P^{clps}(w)$	yes

Table 2.1. Probability assignments for Wigner's friend setups: The classical Wigner's-friend paradox arises for the subjective-collapse model, where the friend and Wigner use different dynamical descriptions for the friend's measurement, and hence, disagree in their probability assignments for Wigner's measurement. This is equivalent to them using different formalism to calculate these probabilities. Both the no-collapse model and the objective-collapse model don't lead to a disagreement between observer and superobserver, since Wigner and his friend use the same formalism. Note, however, that the probabilities for Wigner's result are different for these two models. This means that the outcome statistics observed by Wigner will confirm either one model or the other.

2.2 Observability and contradictions

Different probability assignments of different agents in Wigner's-friend type setups are potentially contradictory, if they can be compared to either one another or experimental data. Only then can the agents' predictions result in a *contradiction* that is potentially able to falsify the particular version of quantum theory used to make those predictions. We want to emphasize that both results of scientific experiments and statements regarding these experiments correspond to pieces of classical information, i.e., bit strings, and that only such pieces of classical information can falsify a theory. In Baumann and Wolf [2018] we formalize this as follows:

Definition 3. *Pieces of information are* classical *if and only if they satisfy the requirements of* interoperability — i.e.*, they can be copied — and* distinguishability — i.e.*, different information can be told apart perfectly.*

This definition of classicality focuses on qualitative notions regarding information and not on the concrete physical realization. Moreover, we define the following:

Definition 4. *A* scientific contradiction *is given by two pieces of contradictory classical information in one point in space and time.*

If a scientific theory gives rise to a scientific contradiction, it should be dismissed. More concretely, a scientific contradiction arises, if the theory's predictions are unambiguous but contradict the actual outcomes of experiments as well as in the case when a theory makes contradictory predictions about such outcomes, at least one of which is then necessarily falsified.

Descriptions of Wigner's-friend-type setups based on the relative-state formalism do not give rise to potentially contradicory predictions, neither do objective-collapse descriptions, or any other version in which there is consensus about the application of the state-update rule.

David Deutsch was the first one to consider a Wigner's-friend experiment where the friend and Wigner communicate Deutsch [1985] by exchanging classical information. There, the friend signals to Wigner that she has performed her measurement and observed a definite outcome without revealing which outcome. Note that, the friend's measurement outcome in a Wigner's-friend experiment does not represent classical information according to Definition 3 , since it does not satisfy *interoperability*. To see this consider a classical record of the observer's result, which is not affected by Wigner's measurement. The notion of classicality in Definition 3 allows for modeling a classical record by an orthonormal basis $\{|i\rangle\}$, if and only if all *accessible observables* are diagonal in that basis. We introduce a register system $\{|s_i\rangle \in \mathscr{H}_R\}$ which can encode classical statements of an agent. Since $\{|s_i\rangle\}$ forms a basis, the statements they encode should be informationally complete, by which we mean that if one of the basis states encodes a certain statement, the negation of that statement should also have a corresponding state in that basis, i.e. $\forall \mathrm{s} : |\mathrm{s}_\mathrm{j}\rangle \cong \mathrm{s}\ \exists |\mathrm{s}_{\mathrm{j}'}\rangle \in \{|\mathrm{s}_\mathrm{i}\rangle\} : |\mathrm{s}_{\mathrm{j}'}\rangle \cong \neg \mathrm{s}$. In the relative-state formalism, the formation of such a classical record corresponds to yet another unitary

$$\begin{aligned} U_R : \quad & \mathcal{H}_S \otimes \mathcal{H}_O \otimes \mathcal{H}_R \to \mathcal{H}_S \otimes \mathcal{H}_O \otimes \mathcal{H}_R \\ & |a\rangle_S |A_a\rangle_O |s_0\rangle_R \mapsto |a\rangle_S |A_a\rangle_O |s_a\rangle_R \quad \forall a, \end{aligned} \tag{2.16}$$

where the record system undergoes no further evolution after the measurement. If the states $|s_a\rangle \in \mathcal{H}_R$ are classical records of the friend's result, different a correspond to different basis states of the register space. We now define the joint probability of measurement results a and b in the presence of classical records as

$$P^{rels}_{class}(a,b) := \mathrm{Tr}\left(\mathbb{1}_S \otimes |A_a\rangle\langle A_a| \otimes |s_a\rangle\langle s_a| \otimes |B_b\rangle\langle B_b| \cdot |\Phi_{rec}\rangle\langle\Phi_{rec}|\right),$$

where $|\Phi_{rec}\rangle$ is the overall state according to the relative-state formalism which now includes the classical records as described above. For the simple Wigner's-friend experiment presented in Section 1.2.1 we have that

$$\begin{aligned} |\Phi_{rec}\rangle = \frac{1}{\sqrt{2}}(&\alpha|1\rangle_{SF}|s_u\rangle_R|1\rangle_W + \beta|1\rangle_{SF}|s_d\rangle_R|1\rangle_W \\ &+ \beta|2\rangle_{SF}|s_u\rangle_R|2\rangle_W - \alpha|2\rangle_{SF}|s_d\rangle_R|2\rangle_W), \end{aligned}$$

which gives conditional probabilities

	$P^{rels}_{class}(1 \mid f)$	$P^{rels}_{class}(2 \mid f)$
$f = u$	α^2	β^2
$f = d$	β^2	α^2 .

Compare this to the conditional probabilities without the classical records of the friend's result

	$P^{clps}(1 \mid f)$	$P^{clps}(2 \mid f)$
$f = u$	α^2	β^2
$f = d$	β^2	α^2

	$P^{rels}(1 \mid f)$	$P^{rels}(2 \mid f)$
$f = u$	$\frac{1}{N_u}\alpha^2(1+2\alpha\beta)$	$\frac{1}{N_u}\beta^2(1-2\alpha\beta)$
$f = d$	$\frac{1}{N_d}\beta^2(1+2\alpha\beta)$	$\frac{1}{N_d}\alpha^2(1-2\alpha\beta)$,

where $N_u = 1 + 2\alpha\beta(\alpha^2 - \beta^2)$ and $N_d = 1 + 2\alpha\beta(\beta^2 - \alpha^2)$. If there is a classical copy of the friend's observed outcome, the conditional probabilities for Wigner's result given his friend's result are the same for collapse dynamics and a unitary description of F's measurement. This means a Wigner's-friend setup with said classical records is no longer the paradoxical gedankenexperiment with differing probability assignments for W and F.

In Deutsch's version of the gedankenexperiment the classical record exchanged between Wigner and his friend is the same regardless of the result F observes. That means that in U_R in Equation (2.16) the state $|s_a\rangle$ is the same for all a and orthogonal to $|s_0\rangle$. In this case the message is not correlated to the specific post-measurement state of the observer and does not affect the probabilities obtained in the relative-state formalism, i.e. $P^{rels}_{class}(w \mid f) = P^{rels}(w \mid f)$.

2.2.1 Scientific contradictions for Wigner's-friend experiments

The classical records introduced above can, in principle, model any type of message exchanged between observers and superobservers in Wigner's-friend-type setups. In the context of the Wigner's friend paradox the main question is whether the conflicting probabilistic predictions, which arise in the subjective collapse model, can be stored in such classical records. If that is the case, these records can be compared to one another. Collected data at the end of (multiple runs) of the experiment would then falsify one of the two predictions and hence constitute a scientific contradiction, which would exclude the subjective collapse model, compare Section 2.1.

Consider again the simplest Wigner's-friend experiment, see Section 1.2.1 where the initial state of the qubit is $|\phi\rangle_S = 1/\sqrt{2}(|\uparrow\rangle_S + |\downarrow\rangle_S)$. If the friend applies the state-update rule after her measurement her prediction for Wigner's measurement, if she sees "up", will differ from her prediction, if she sees "down", compare $P_F(w)$ in (1.6). The only exception is the case where W measures in a basis containing states $|\Phi^\pm\rangle = 1/\sqrt{2}(|\uparrow, u\rangle_{SF} \pm |\downarrow, d\rangle_{SF})$, meaning $\alpha = \beta = 1/\sqrt{2}$. For this particular setting we label the two outcomes of W's measurement as "+" corresponding to $|\Phi^+\rangle$ and "−" corresponding to $|\Phi^-\rangle$ and the probabilities Wigner and his friend predict for Wigner's measurement are

$$
\begin{array}{c|c|c}
P_F(w): & + & - \\
\hline
f = u & \frac{1}{2} & \frac{1}{2} \\
f = d & \frac{1}{2} & \frac{1}{2}
\end{array}
\qquad\qquad
\begin{array}{c|c|c}
P_W(w): & + & - \\
\hline
f = u & 1 & 0 \\
f = d & 1 & 0.
\end{array}
\tag{2.17}
$$

Since the prediction "s_F: $P_F(+) = P_F(-) = \frac{1}{2}$" is the same for both $|\uparrow\rangle_S |u\rangle_F$ and $|\downarrow\rangle_S \, |d\rangle_F$, the state of F's laboratory before Wigner's measurement is $|\Phi^+\rangle_{SF} |s_F\rangle_R$.

In this case, the classical record of the friend's prediction does not affect the conditional probabilities predicted by W, who uses the relative state formalism

Simple Wigner's-friend experiment

$$\begin{array}{c|c|c} & P^{rels}_{class}(+\mid f) = P_W(w) & P^{rels}_{class}(-\mid f) = P_W(w) \\ \hline f = u & 1 & 0 \\ f = d & 1 & 0, \end{array} \tag{2.18}$$

and, hence, his prediction differs from the one made by his friend. This fact allows for a scientific contradiction in the simple Wigner's-friend experiment, see Figure 2.3. Consider the following protocol:

0 The source emits the quantum state $|\phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S)$.

1 The friend F measures the system in the σ_z-basis and then applies the state-update rule according to the outcome she observed. Based on that she predicts $P_F(+) = P_F(-) = \frac{1}{2}$ for Wigner's measurement result, which she encodes in a classical record system.

2 F sends that classical record out of her laboratory in a way that keeps all other degrees of freedom fully isolated and hence preserves the coherence of the state $|\Phi\rangle_{SF}$ assigned by Wigner.

3 Wigner W encodes his prediction $P_W(+) = 1, P_W(-) = 0$ in another classical register.

4 W performs his Bell-basis measurement on the system $S + F$, recording the observed result in yet another classical register.

5 Wigner and his friend repeat steps **0-4** for a previously agreed upon number N times.

At the very end of the protocol F leaves the laboratory, which is equivalent to W performing a final measurement in the product basis of $S + F$. Irrespective of the specific state to which F's laboratory is reduced, she and W can now compare the three kinds of messages, i.e. F's predictions, W's predictions and W's observed results, and convince themselves that there is a contradiction, since the actually observed statistics of Wigner' measurement can only agree with one of the two predictions if N is chosen large enough. Alternatively to the friend leaving the laboratory, she can also keep her own prediction while Wigner sends his prediction and observed statistics into the laboratory. In that that case, however, only F can observe the contradiction.

This contradiction can be made effectively deterministic in the following limit. To see this, now consider the source emitting a higher dimensional quantum system in state

$$|\phi_d\rangle_S = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle_S \tag{2.19}$$

and the friend measures in the respective basis, $M_F : \{|j\rangle\langle j|_S\}$ with $j = 1 \dots d$. Wigner, again, models F's measurement as a unitary process that results in state

$$|\phi_d\rangle_S |0\rangle_F \mapsto \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle_S |J_j\rangle_F =: |\Phi_d^+\rangle_{SF}, \tag{2.20}$$

where $|J_j\rangle_F$ is, once more, the state corresponding to the friend having observed outcome j.

For Wigner's measurement M_W we now choose $\{|\Phi_d^+\rangle\langle\Phi_d^+|_{SF}, \mathbb{1} - |\Phi_d^+\rangle\langle\Phi_d^+|_{SF}\}$, which, like before, just confirms the entangled state of the laboratory, if Wigner's description is correct. In that case W records the "+" result with unit probability, i.e. $P_W(+) = 1$ and $P_W(-) = 0$. Using the state-update rule the friend, however, now predicts

$$\begin{aligned} P_F(+|j) &= |\langle j|\langle J_j|\Phi_d^+\rangle_{SF}|^2 = \frac{1}{d} \quad \xrightarrow[d\to\infty]{} \quad 0 \\ P_F(-|j) &= 1 - \frac{1}{d} \xrightarrow[d\to\infty]{} \quad 1\,, \end{aligned} \tag{2.21}$$

independently of the actual outcome she registers in her measurement. For large enough dimension d the friend's prediction becomes the exact opposite of Wigner's. They can again repeat the protocol described above and will arrive at an observable contradiction at the end.

It is traditionally assumed that Wigner's prediction is correct and he will measure the corresponding statistics. In that case the friend can convince herself that she made the wrong predictions using what she would consider standard quantum theory. If the actual statistics confirmed the friend's prediction, Wigner would reach the same conclusion regarding his prediction.

Another type of contradiction can be obtained if the friend purposefully ignores her obtained outcome and makes predictions about her own perception after Wigner's measurement. Consider again the simplest Wigner's-friend experiment as presented in Section 1.2.1, namely the initial state given by Equation (1.1) and Wigner's measurement by (1.2) and (1.3). The friend ignoring

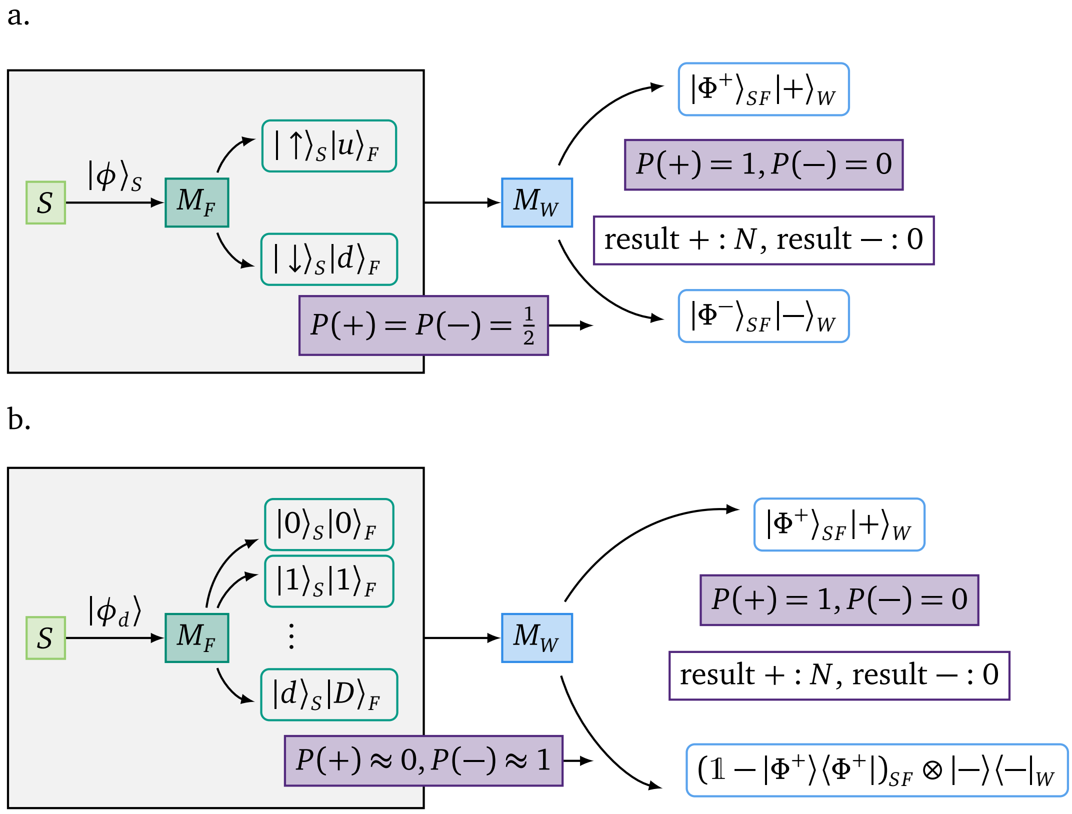


Figure 2.3. Observable contradictions in the simple Wigner's-friend experiment: The source emits a state, which is an equal superposition of the eigenstates of the friend's measurement M_F. If Wigner performs a measurement M_W that confirms the overall entangled state of the friend's laboratory their probability assignments will disagree. According to F's collapse description the conditional probabilities of W's outcomes "+" and "−" is the same independent of the outcome she observed, i.e. $|\langle j|\langle J_j|\Phi_d^+\rangle_{SF}|^2 = \frac{1}{d}$ where d is the dimension of the system the source emitted. Wigner, however, expects to see result "+" with certainty. a. : If the source emits a qubit the friends predicts a 50:50 distribution for Wigner's two results and they will have to repeat their measurement a considerable amount of times. b.: In the "all versus nothing" scenario of large d the friend's prediction is almost deterministic and fewer repetitions, compared to the qubit case, will suffice to falsify one of the two predictions.

her outcome but still using collapse dynamics to describe her own measurement mans that she associates a mixed state

$$\rho_{SF} = a^2|\uparrow,u\rangle\langle\uparrow,u|_{SF} + b^2|\downarrow,d\rangle\langle\downarrow,d|_{SF} \tag{2.22}$$

to her laboratory before Wigner's measurement, instead of one of the collapsed states $|z\rangle_S|f_z\rangle_F$. Wigner, who describes her measurement unitarily, obtains the following state after the friend's measurement

$$|\Phi\rangle_{SF} = a|\uparrow, u\rangle_{SF} + be^{i\phi_S}|\downarrow, d\rangle_{SF}. \tag{2.23}$$

The probability of the friend seeing result f_z at time t_1, after her but before Wigner's measurement, is the same for both descriptions:

	$P_W(f \text{ at } t_1)$	$P_F(f \text{ at } t_1)$
$f_1 = u:$	a^2	a^2
$f_1 = d:$	b^2	$b^2,$

After W's measurement, however, which we call t_2 the probability of F's perceived result will differ depending on whether one used collapse, i.e. Equation (2.22), or unitary dynamics, i.e. Equation (2.23), to describe F's measurement.

	$P_W(f \text{ at } t_2)$	$P_F(f \text{ at } t_2)$
$f_2 = u:$	$a^2(\alpha^4+\beta^4)+2b^2\alpha^2\beta^2+\chi(\alpha^2-\beta^2)$	$a^2(\alpha^4+\beta^4)+2b^2\alpha^2\beta^2$
$f_2 = d:$	$b^2(\alpha^4+\beta^4)+2a^2\alpha^2\beta^2-\chi(\alpha^2-\beta^2)$	$b^2(\alpha^4+\beta^4)+2a^2\alpha^2\beta^2,$

where $\chi = 2ab\alpha\beta\cos(\phi_S - \phi_{SF})$. Since, in this case, the friend's prediction again does not depend on which result she observes at time t_1 a classical record "s_F : $P_F(f \text{ at } t_2)$" would again not change Wigner's prediction and at time t_2 only one of them would be supported by collected statistics, $P^{rels}_{class}W(f \text{ at } t_2) = P_W(f \text{ at } t_2)$. Assuming that Wigner's description is correct the friend could keep the classical record "s_F : $P_F(f \text{ at } t_2)$" inside the laboratory and then register her perceived outcome at time t_2 in a classical record different from the one containing the prediction. Repeating this multiple times will allow her to collect statistics on her memory records after Wigner's measurement and establish $P_W(f \text{ at } t_2)$ as well as notice the fact that they differ from her own prediction all by herself.

In the extended Wigner's-friend setup by Frauchiger and Renner described in Section 1.2.2 the predictions leading to the contradiction depicted in Figure 1.4 cannot be exchanged between observers and superobservers. The crucial probability assignment is that made by friend F_1 upon observing outcome "tails" when she uses the state-update rule. Similar to the simple Wigner's-friend setup this friend's probability assignment is different from the probability assignment of W, who describes the friends' measurements unitarily:

$$\begin{array}{c|c|c} P_{F_1}(w|f_1): & O & F \\ \hline f_1 = H & \frac{1}{2} & \frac{1}{2} \\ f_1 = T & 0 & 1 \end{array} \qquad \begin{array}{c|c|c} P_W(w|f_1): & O & F \\ \hline f_1 = H & \frac{1}{6} & \frac{5}{6} \\ f_1 = T & \frac{1}{6} & \frac{5}{6}. \end{array} \tag{2.24}$$

Note that Wigner's prediction is not directly used for the contradiction in Figure 1.4, but it ensures that the halting round of the protocol, where both superobservers measure "ok" and in which the contradiction occurs, is also compatible with F_1 seeing "tail".

As one can see in the probability tables (2.24) F_1's deterministic prediction "s_{F_1}: $w = F$" arises only for the state $|t\rangle_C \otimes |T\rangle_{F_1}$ of her laboratory, while the state $|h\rangle_C \otimes |H\rangle_{F_1}$ would be correlated with a different prediction "s'_{F_1}: $P(w = F) = P(w = O) = \frac{1}{2}$" with $\langle s_{F_1} | s'_{F_1} \rangle = 0$. Hence, encoding this prediction into a classical register is analogous to the example of a classical record of the friend's result discussed in Section 2.2 and one finds that

Extended Wigner's-Friend Experiment

$$\begin{array}{c|c|c} & P^{rels}_{class}(O \mid f_1) = P_{F_1}(O|f_1) & P^{rels}_{class}(F \mid f_1) = P_{F_1}(F|f_1) \\ \hline f_1 = H & \frac{1}{2} & \frac{1}{2} \\ f_1 = T & 0 & 1. \end{array} \tag{2.25}$$

This means that in the presence of classical records of F_1's prediction the unitary description of the superobservers predicts the same conditional probabilities as those predicted by F_1 who employs collapse dynamics. Therefore, other than the simple Wigner's-friend experiment in Figure 1.2, the extended Wigner's-friend experiment in Figure 1.3 does not give rise to a scientific contradiction.

2.2.2 Observable quantities for Wigner-Bell setups

The so called Wigner-Bell inequalities discussed in Section 1.2.2 do not involve predictions made by the observers – Charlie and Debbie – about the results measured by the superobservers – Alice and Bob. It is the statistics of the superobservers' results themselves that preclude assigning a classical (i.e. locally causal) joint probability distribution to all the results observed by the four agents. While classical records stating that the friends observed definite results, i.e. "s_J: Result observed " where J can be either C for Charlie or D for Debbie, can be regarded as a motivation for assuming that such a joint probability distribution $P(A_x, C = A_z, B_x, D = B_z)$ exists, the respective results are by construction not jointly observable. Hence, Wigner-Bell tests do not necessarily require the observers to communicate with the superobservers to arrive at a contradiction. This allows for performing experiments within the current technological possibilities that confirm the violation of such Wigner-Bell and similar so-called *local friendliness* inequalities Bong et al. [2020]; Proietti et al. [2019]. The "friend" in these experiments is, however, the same type of quantum system as the one emitted by source. The observed statistics violate the respective inequality, which means the experimental results contradict the union of assumptions made to derive these inequalites.

The violation of a Wigner-Bell inequality, see Bong et al. [2020], only negates the existence of a joint probability distribution $P(A_x, C = A_z, B_x, D = B_z)$ whose marginals are the probabilities observed by the superobserver $P(A_x, B_x)$, $P(A_x, B_z)$, $P(A_z, B_x)$ and $P(A_z, B_z)$. As Richard Healey pointed out in Healey [2018], this does, however, not prevent someone from assigning definite values to the results of all four agents in single rounds of the experiment. Healey then proposed a similar setup, which is shown in Figure 2.4 and allegedly allows to predict, in every run, the correlation functions between the outcomes of different observers, that violate a Wigner-Bell inequality analogous to that in Equation (1.9). Like the setup in Section 1.2.2 Healey's setup features two Wigner's friend experiments (Charlie + Alice and Debbie + Bob) each receiving a part of an entangled state and the two observers Charlie and Debbie measure their part in bases $\{|\uparrow_c\rangle, |\downarrow_c\rangle\}$ and $\{|\uparrow_d\rangle, |\downarrow_d\rangle\}$ respectively. Other than in Section 1.2.2 the two superobservers Alice and Bob both undo their friends' measurement disentangling them from the respective subsystems. Afterwards the superobservers themselves measure the systems in different bases $\{|\uparrow_a\rangle, |\downarrow_a\rangle\}$ and $\{|\uparrow_b\rangle, |\downarrow_b\rangle\}$ respectively. Healey's setup makes use of effects of special relativity in order to combine the relevant quantities from different reference frames. As we argue in Baumann, Del Santo and Brukner [2019] these expressions do not correspond to any observable quantities and, as we will show below, can give rise to different values for the expression in Equation (1.9), when the relative-state formalism is used. First consider the protocol from the reference frame of Alice's lab:

0 State preparation: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle_{S_1}|\downarrow\rangle_{S_2} - |\downarrow\rangle_{S_1}|\uparrow\rangle_{S_2}\big)|r\rangle_C|r\rangle_D$. Where $|r\rangle_C$ and $|r\rangle_D$ are the initial ("ready") states of Charlie and Debbie, respectively.

1 Debbie measures: $|\Psi(0)\rangle \to |\Psi(1)\rangle = U_D|\Psi(0)\rangle$; where her measurement is described by unitary U_D acting as follows $U_D|\downarrow_d\rangle_{S_2}|r\rangle_D = |\downarrow_d\rangle_{S_2}|d\rangle_D$ (and equivalently for $|\uparrow_d\rangle_{S_2}$).

2 Charlie measures: $|\Psi(1)\rangle \to |\Psi(2)\rangle = U_C|\Psi(1)\rangle$; where his measurement is $U_C|\downarrow_c\rangle_{S_1}|r\rangle_C = |\downarrow_c\rangle_{S_1}|d\rangle_C$ (and equivalently for $|\uparrow_c\rangle_{S_1}$).

3 Alice undoes Charlie's measurement by applying $U_C^\dagger$:
$|\Psi(2)\rangle \to |\Psi(3)\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow_d\rangle_{S_1}|\downarrow_d\rangle_{S_2}|d\rangle_D - |\downarrow_d\rangle_{S_1}|\uparrow_d\rangle_{S_2}|u\rangle_D\big)|r\rangle_C = |\Psi(1)\rangle$.

4 Alice measures, corresponding to unitary $U_A|\downarrow_a\rangle_{S_1}|r\rangle_A = |\downarrow_a\rangle_{S_1}|d\rangle_A$ (and equivalently for $|\uparrow_a\rangle_{S_1}$)

5 Bob undoes Debbie's measurement by applying $U_D^\dagger$.

6 Bob measures corresponding to a unitary $U_B|\downarrow_d\rangle_{S_2}|r\rangle_B = |\downarrow_b\rangle_{S_2}|d\rangle_B$ (and equivalently for $|\uparrow_b\rangle_{S_2}$) resulting in the state $|\Psi(6)\rangle = U_A U_B|\Psi(0)\rangle$.

According to Healey, after step **2**, Alice predicts the correlation function

$$\begin{aligned} E(C,D) &= |\langle\psi_{S_1S_2}|(|\uparrow_c\rangle\langle\uparrow_c| - |\downarrow_c\rangle\langle\downarrow_c|) \otimes (|\uparrow_d\rangle\langle\uparrow_d| - |\downarrow_d\rangle\langle\downarrow_d|)|\psi_{S_1S_2}\rangle|^2 \\ &= -\cos(c-d) \end{aligned} \tag{2.26}$$

for the measurements outcomes observed by Charlie and Debbie, where $|\psi_{S_1S_2}\rangle = |\psi^-\rangle$ is the maximally entangled Bell-state of the two subsystems and $c-d$ is the relative angle between measurement directions $\vec{c}$ and $\vec{d}$. Similarly, after step **4**, she predicts the correlation function for her own outcome and the one observed by Debbie as

$$E(A,D) = -\cos(a-d)\,, \tag{2.27}$$

with $a-d$ again being the relative angle between the measurement directions $\vec{a}$ and $\vec{d}$. With respect to Bob's laboratory, in motion relative to Alice's, the same protocol looks as follows.

0* State preparation: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle_{S_1}|\downarrow\rangle_{S_2} - |\downarrow\rangle_{S_1}|\uparrow\rangle_{S_2}\big)|r\rangle_C|r\rangle_D$.

1* Charlie measures: $|\Psi(0)\rangle \to |\Psi(1^*)\rangle = U_C|\Psi(0)\rangle$.

2* Debbie measures: $|\Psi(1^*)\rangle \to |\Psi(2^*)\rangle = U_D|\Psi(1^*)\rangle$.

3* Bob undoes Debbie's measurement by applying $U_D^\dagger$: $|\Psi(2^*)\rangle \to |\Psi(3^*)\rangle = |\Psi(1^*)\rangle$.

4* Bob measures, which is described by unitary U_B.

5* Alice undoes Charlie's measurement by applying $U_C^\dagger$.

6* Alice measures corresponding to unitary U_A, resulting in the final state $|\Psi(6^*)\rangle = U_A U_B |\Psi(0)\rangle$.

Similar to Alice, after step **4***, Bob predicts

$$E(C,B) = -\cos(b-c)\,, \tag{2.28}$$

where $b-c$ is now the relative angle between Charlie's and Bob's measurement directions. Finally, after step **6**, Alice (or alternatively Bob, after step **6***) can compute the correlation function

$$E(A,B) = -\cos(a-b) \tag{2.29}$$

for their observed outcomes a and b, with $a-b$ being the relative angle between the superobservers' measurements. Expressions (2.26)-(2.29) are known to give a value of

$$S = |E(A,B) + E(A,D) + E(C,B) - E(C,D)| = 2\sqrt{2}, \tag{2.30}$$

if the angles are chosen appropriately.

The correlation functions $E(A,D)$ and $E(C,B)$ entering Equation (2.32) are computed from two different reference frames as well as in principle experimentally inaccessible. More concretely, no pair of observers that is involved in the argument can test (not even in principle) the violation of the proposed Bell's inequality. Without specifying how the computed expressions are related to observable quantities, one could equally assign different values to the correlation functions. For example, one can use the relative-state formalism and exploit the fact that the observers "ready" states are orthogonal to both the states of them having seen one of the two outcomes, i.e.; $\langle u|r\rangle_F = \langle d|r\rangle_F = 0$ for both $F = C$ and $F = D$. Hence, the reference frame dependent correlation functions are

$$\begin{aligned}
E(A,D) &= |\langle\Psi(t)|(|u\rangle\langle u|_A - |d\rangle\langle d|_A) \otimes (|u\rangle\langle u|_D - |d\rangle\langle d|_D)|\Psi(t)\rangle|^2 \\
&= -\cos(a-d) \\
E^*(A,D) &= |\langle\Psi(t^*)|(|u\rangle\langle u|_A - |d\rangle\langle d|_A) \otimes (|u\rangle\langle u|_D - |d\rangle\langle d|_D)|\Psi(t^*)\rangle|^2 = 0 \\
E(C,B) &= |\langle\Psi(t)|(|u\rangle\langle u|_C - |d\rangle\langle d|_C) \otimes (|u\rangle\langle u|_B - |d\rangle\langle d|_B)|\Psi(t)\rangle|^2 = 0 \\
E^*(C,B) &= |\langle\Psi(t^*)|(|u\rangle\langle u|_C - |d\rangle\langle d|_C) \otimes (|u\rangle\langle u|_B - |d\rangle\langle d|_B)|\Psi(t^*)\rangle|^2 \\
&= -\cos(b-c)\,,
\end{aligned} \tag{2.31}$$

where the $E(\cdot,\cdot)$ are computed in Alice's and the $E^*(\cdot,\cdot)$ in Bob's reference frame. Note that t, t^* can take any value between 0 and 6 here, since due to the inclusion of the observers' states the correlation functions are well-defined throughout

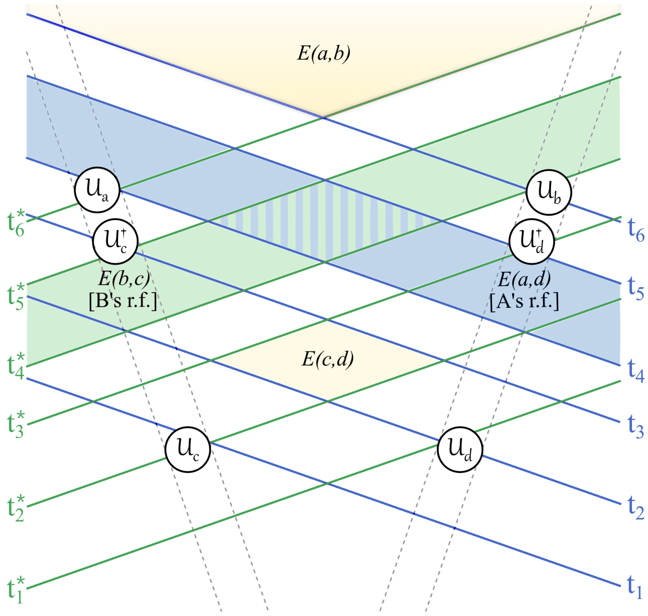


Figure 2.4. The protocol in Healey [2018]: Two laboratories (containing Alice-Charlie-system 1 and Bob-Debbie-system 2, respectively) are moving apart with constant velocity. The four agents perform a series of measurements. In every run of the protocol there supposedly exist four space-time regions wherein correlations between the results could in principle be established. The yellow areas are those where correlations can be agreed upon in the reference frames of both laboratories, whereas the blue (green) areas are those where correlations are relative to Alice's (Bob's) reference frame only. (picture taken from Baumann, Del Santo and Brukner [2019])

the whole protocol. Healey's argument requires to insert $E(A,D)$ and $E^*(C,B)$ in Equation (2.32). However, since these expressions are unobservable in principle, one could equally use $E^*(A,D)$ and $E(C,B)$ for evaluating the Wigner-Bell inequality, obtaining no violation

$$S = |E(A,B) + 0 + 0 - E(C,D)| \leq 2 \,. \tag{2.32}$$

If Alice and Bob were to verify their predictions and actually violate a Bell's inequality with collected data, one would have to adapt the protocol in a way that renders the setup the same as that proposed in Brukner [2018].

Something that can be excluded by Wigner-Bell setups is the observers being aware of changes of their perception induced by the superobservers' measurements. Consider the two entangled perceptions of the two friends in a Wigner-

Bell setup, see Figure 2.5. If one superobserver induces a change of the perceived result of the respective observer in one of the two entangled laboratories, the perception of the observer in the other laboratory must change as well. Therefore, if the observers were directly aware of these changes, an entangled observer would be instantaneously aware of the measurement choice made for a spacelike separated laboratory. Moreover, awareness of a change of perception enables the observer to communicate to the superobserver whether a change of the perceived result occurred. This could then also be accessed by the superobservers and enable superluminal signaling between them.

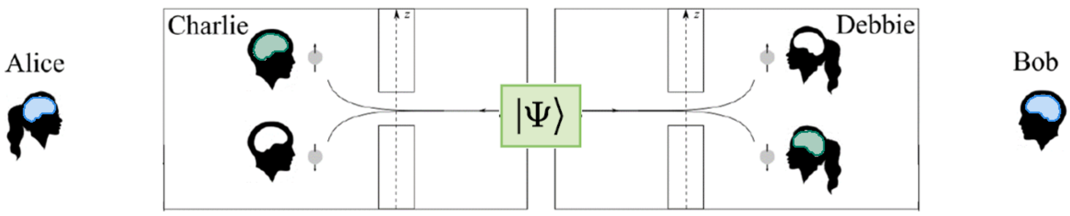


Figure 2.5. In a Wigner-Bell setup the perceptions of the two observers – Debbie and Charlie– are entangled due to their measurements on the two halves of an entangled pair. For example, given a suitable entangled state the perceptions of Debbie and Charlie will be perfectly anti-correlated – whenever one sees "up" the other sees "down". The two superobservers – Alice and Bob – can choose between two different measurements on the observers' laboratories, only one of which will alter the perception of Debbie or Charlie respectively. If the perception of one of the observers is changed by Alice's or Bob's measurement, however, so is the perception of the other observer due to the entanglement. If the observers were aware of such changes, this setup could be used for faster than light signaling, since an observer would be instantaneously aware of a spacelike separated superobserver's actions.

2.3 Probability rules for Wigner's-friend experiments

This sections summarizes the results published in Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz and Brukner [2019]. Formulating the Wigner's-friend experiment in term of the Page-Wootters formalism, see Section 1.4.1, has the advantage of removing the ambiguity in the dynamical descriptions of Wigner and his friend. Due to the constraint equation satisfied by the physical states there is only one specific history state for a given setup. All observers knowing the relevant specifics of this setup will compute probabilities of one event conditioned on another using this particular state and hence circumvent the problem of different state assignments by Wigner and his friend after the friend's measurement, i.e. subjective collapse. As we will see in Section 2.3.1, however, there remains some ambiguity due to different possible rules for calculating two-time conditional probabilities. The two former proposals for such a probability rule, compare Equations (1.24) and (1.25) in Section 1.4, turn out to give different probability assignments when adapted to encapsulated observers, which means that it can (in principle) be decided experimentally which is the correct rule. While the first approach, similar to Equation (1.24), leads to the probabilities one obtains when applying the state-update rule after the friend's measurement, i.e. objective collapse, the second proposal, similar to Equation (1.25), gives probabilities in agreement with full unitary dynamics, i.e. no-collapse. Once the agents have singled out one of the rules, they will all always agree on the probabilities they assign.

Further, note that, since in the Page-Wootters formalism (PWF) time evolution emerges from the correlations between the clock and the systems under consideration, two-time conditional probabilities can be evaluated regardless of the temporal ordering. Hence, the PWF allows for conditioning not only on past but also on future events, something otherwise only possible in the time-symmetric formulation of the theory of quantum measurements introduced by Yakir Aharonov, see Aharonov et al. [1964]; Aharonov and Vaidman [2002].

In the context of Winger's friend experiments we can, therefore, answer the following two questions:

1. What is the probability that W measures outcome w at clock time t_2, given that F has measured outcome f at a previous clock time t_1?

2. What is the probability that F measures outcome f at clock time t_1, conditioned on the fact that W will measure outcome w at a later clock time t_2?

We are interested in the case of $t_F < t_1 < t_W < t_2$ but the Page Wootters formulation of the Wigner's-friend experiment together with the fact that we condition on both the past and the future means that one can, in principle consider any ordering.

2.3.1 Different definitions of conditional probabilities for the Page-Wootters description of Wigner's friend

We propose three different definitions for conditional probabilities of the results observed by Wigner and his friend and apply them to the simple Wigner's-friend experiment, introduced in Sections 1.2.1 and 1.4.1. The operational meaning of these probabilities is by no means straight forward, since the state of the friend is in general modified by Wigner's measurement. Moreover, the friend's memory is, by construction, the only record of her observed result and, hence, the two results f and w are in general not jointly observable. The two exceptions are the cases where W either reads out F's observed result in his own measurement (i.e. projectors on the states $|z_f\rangle_S|f\rangle_F$) or performs the measurement which confirms the entangled state he assigns to F's laboratory, compare Sections 2.1.2 and 2.2. We will refer to the latter as a *non-disturbance measurement* and in this case Wigner and his friend can exchange their data at the end of the protocol and construct a joint, as well as conditional, probability distribution, which is equivalent to the one of the outcomes f that F observed at t_1 and and w that W observed at t_2. Note that, also in case of a non-disturbance measurement F cannot send her observed results to W before his measurement at t_W without changing the probabilities for Wigner's measurements, compare Section 2.2.

First, we adapt Carl Dolby's proposal Dolby [2004] for calculating two-time conditional probabilities within the Page-Wootters formalism, see Equation (1.24). In order to deal with the Page Wootters formulation of the Wigner's-friend setup in Section 1.4.1 we consider a van Neumann measurement model in general.

Definition 1 The conditional probability of result n at time t_2 given result m at time t_1 is

$$P_1(n \text{ when } t_2 \,|\, m \text{ when } t_1) = \frac{\langle\langle\Psi|\,|t_1\rangle\langle t_1|\otimes\Pi^m\hat{P}|t_2\rangle\langle t_2|\otimes\Pi^n\hat{P}|t_1\rangle\langle t_1|\otimes\Pi^m|\Psi\rangle\rangle}{\langle\langle\Psi||t_1\rangle\langle t_1|\otimes\Pi^m|\Psi\rangle\rangle}, \tag{2.33}$$

where Π^m and Π^n are the projectors on the respective observer's states (or in general the states of the measurement apparatus) in $\mathscr{H}_M$ and $\mathscr{H}_N$, and the physical projector $\hat{P}$ is given by Equation (1.20).

When applied to the setup in Section 1.4.1, in particular to the history state in Equation (1.29), Definition 1 gives the conditional probabilities listed in Table 2.2. These probabilities are always well-defined, genuinely two-time and

$P_1(w \text{ when } t_2 \,|\, f \text{ when } t_1)$

f \ w	yes	no
u	α^2	β^2
d	β^2	α^2

$P_1(f \text{ when } t_1 \,|\, w \text{ when } t_2)$

f \ w	yes	no
u	α^2	β^2
d	β^2	α^2

Table 2.2. The conditional probabilities of Wigner seeing result w at time t_2 given that the friend saw result f at time t_1 and of the friend seeing f at t_1 given that Wigner will see w at t_2 according to Definition 1. Since the two conditional probabilities are equal, applying Bayes' rule will in general not give rise to one joint probability expression.

correspond to collapse dynamics for measurements, compare the probability tables (1.6). Note that, Definition 1 does, in general, not give rise to a well-defined joint probability, since $P_1(n \text{ when } t_2 \,|\, m \text{ when } t_1) \cdot P(m \text{ when } t_1) \neq P(n \text{ when } t_2) \cdot P_1(m \text{ when } t_1 \,|\, n \text{ when } t_2)$. This is due to the fact that the numerator of Equation (2.33) implies an ordering of $|t_1\rangle\langle t_1| \otimes \Pi^m$ and $|t_2\rangle\langle t_2| \otimes \Pi^n$. Only if said ordering is irrelevant, corresponding to $[\hat{P}|t_1\rangle\langle t_1| \otimes \Pi^m \hat{P}, \hat{P}|t_2\rangle\langle t_2| \otimes \Pi^n \hat{P}] \,|\Psi\rangle\rangle = 0$, the joint probability is unique and given by $\langle\langle\Psi|\,|t_1\rangle\langle t_1| \otimes \Pi^m \hat{P}|t_2\rangle\langle t_2| \otimes \Pi^n \,|\Psi\rangle\rangle$.

The probability rule proposed by Giovanetti *et al.* Giovannetti et al. [2015] can directly be used for Wigner's-friend experiments, since it already considers purified (i.e. van Neumann) measurements. Directly using Equation (1.25) gives the following definition.

Definition 2a The conditional probability (in the cases where it is well-defined) of result n at time t_2 given result m at time t_1 is

$$P_{2a}(n \text{ when } t_2 \,|\, m \text{ when } t_1) = \frac{\langle\langle\Psi|t_2\rangle\langle t_2| \otimes \Pi^n \otimes \Pi^m \,|\Psi\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1| \otimes \Pi^m \,|\Psi\rangle\rangle}. \tag{2.34}$$

where Π^m and Π^n are again the projectors on the respective observer's states in $\mathscr{H}_M$ and $\mathscr{H}_N$.

The numerator of Equation (2.34) corresponds to a single joint measurement of F's and W's states after both their measurements have been performed. The denominator is given by the one-time probability of F finding outcome m at time t_1. When evaluated for arbitrary a, b, α, β and $\Delta\phi = \phi_S - \phi_{SF}$, Definition 2a gives expressions that are in general not normalized, see Table 2.3.

$$\frac{\langle\langle\Psi|t_2\rangle\langle t_2|\otimes\Pi^w\otimes\Pi^f|\Psi\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1|\otimes\Pi^f|\Psi\rangle\rangle}$$

f \ w	yes	no
u	$\frac{a^2\alpha^4+2ab\alpha^3\beta\cos(\Delta\phi)+b^2\alpha^2\beta^2}{a^2}$	$\frac{a^2\beta^4-2ab\alpha\beta^3\cos(\Delta\phi)+b^2\alpha^2\beta^2}{a^2}$
d	$\frac{b^2\beta^4+2ab\alpha\beta^3\cos(\Delta\phi)+a^2\alpha^2\beta^2}{b^2}$	$\frac{b^2\alpha^4-2ab\alpha^3\beta\cos(\Delta\phi)+a^2\alpha^2\beta^2}{b^2}$

Table 2.3. Definition 2a evaluated for arbitrary a, b, α, β and $\Delta\phi = \phi_S - \phi_{SF}$. These expressions constitute possible conditional probabilities of W seeing result $w \in \{\text{yes}, \text{no}\}$ at time t_2 given that F saw result $f \in \{u, d\}$ at time t_1 only if conditions (2.35) and (2.36) are both satisfied.

Requiring normalization means that $\sum_w \frac{\langle\langle\Psi|t_2\rangle\langle t_2|\otimes\Pi^w\otimes\Pi^f|\Psi\rangle\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1|\otimes\Pi^f|\Psi\rangle\rangle} = 1$ for all f. In that case the following two equations have to be fulfilled

$$\alpha^4 + \beta^4 + 2cos(\Delta\phi)(\alpha^3\beta - \alpha\beta^3)\frac{b}{a} + 2\alpha^2\beta^2\left(\frac{b}{a}\right)^2 = 1 \tag{2.35}$$

$$\alpha^4 + \beta^4 - 2cos(\Delta\phi)(\alpha^3\beta - \alpha\beta^3)\frac{a}{b} + 2\alpha^2\beta^2\left(\frac{a}{b}\right)^2 = 1, \tag{2.36}$$

which is trivially the case for $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$, corresponding to the case where Wigner's measurement reveals which outcome his friend observed. Otherwise, Equations (2.35) and (2.36) can be rewritten as quadratic equations in $\frac{b}{a}$ and $\frac{a}{b}$. They have the solutions

$$\left(\frac{b}{a}\right)_\pm = \frac{-\cos\Delta\phi(\alpha^2-\beta^2) \pm \sqrt{1-\sin^2\Delta\phi(\alpha^2-\beta^2)^2}}{2\alpha\beta} \tag{2.37}$$

and

$$\left(\frac{a}{b}\right)_\pm = \frac{\cos\Delta\phi(\alpha^2-\beta^2) \pm \sqrt{1-\sin^2\Delta\phi(\alpha^2-\beta^2)^2}}{2\alpha\beta}, \tag{2.38}$$

which exist for any $\alpha, \beta \neq 0$ and $\Delta\phi$. However, requiring that $\left(\frac{b}{a}\right)_\pm = 1/\left(\frac{a}{b}\right)_\pm$ singles out those combinations of solutions that have equal signs, which are then the common solutions to both Equations (2.35) and (2.36). For these setting does Definition 2a give normalized probabilities which are listed in Table 2.4.

$P_{2a}(w$ when $t_2\,|\,f$ when $t_1)$

$f \backslash w$	yes	no
u	$\frac{1+2\alpha^2+(\beta^4-\alpha^4)\cos(2\Delta\phi)\pm\chi}{4}$	$\frac{1+2\beta^2+(\alpha^4-\beta^4)\cos(2\Delta\phi)\mp\chi}{4}$
d	$\frac{1+2\beta^2+(\alpha^4-\beta^4)\cos(2\Delta\phi)\pm\chi}{4}$	$\frac{1+2\alpha^2+(\beta^4-\alpha^4)\cos(2\Delta\phi)\mp\chi}{4}$

with $\chi := 2\cos(\Delta\phi)\sqrt{1-(\alpha^2-\beta^2)^2\sin^2(\Delta\phi)}$

Table 2.4. The conditional probabilities of Wigner seeing result w at time t_2 given that the Friend saw result f at time t_1 according to Definition 2a. The different signs correspond to the different solutions of the quadratic equations in Equations (2.37) and (2.38).

Although Equation (2.34) depends on both times t_1 and t_2 and thus can be considered a two-time probability, it does not allow for conditioning on the future like Definition 1 did. Since the numerator in Equation (2.34) does not depend on t_1 and $\langle\langle\Psi|t_1\rangle\langle t_1|\Pi^w \otimes \Pi^f\,|\Psi\rangle\rangle = 0$, there is no sensible definition of $P_{2a}(f$ when $t_1\,|\,w$ when $t_2)$.

Following the original idea in Hellmann et al. [2007] of equating multiple measurements at different times by one measurement at a final time, however, suggests a different definition for conditional probabilities in Wigner's-friend setups.

Definition 2b The conditional probability of result n given m is

$$P_{2b}(n \text{ when } t_2\,|\,m \text{ when } t_2) = \frac{\langle\langle\Psi|t_2\rangle\langle t_2| \otimes \Pi^n \otimes \Pi^m\,|\Psi\rangle\rangle}{\langle\langle\Psi|t_2\rangle\langle t_2| \otimes \Pi^m\,|\Psi\rangle\rangle}, \tag{2.39}$$

where t_2 is some time after the second measurement has been performed.

This definition gives always well-defined one-time probabilities, which correspond to jointly observable statistics after full unitary evolution, i.e., no-collapse model, compare Equation (1.6). They are shown in Table 2.5 for arbitrary parameters a, b, α, β and $\Delta\phi$. Note that for non-Wigner's-friend scenarios Definitions 2a and 2b are equivalent, since in this case $\langle\langle\Psi|t_2\rangle\langle t_2| \otimes \Pi^m\,|\Psi\rangle\rangle =$

$P_{2b}(w \text{ when } t_2 \,|\, f \text{ when } t_2)$

f \ w	yes	no
u	$\frac{\frac{\alpha^2}{\beta^2}+2\frac{b\alpha}{a\beta}\cos(\Delta\phi)+\frac{b^2}{a^2}}{N_u}$	$\frac{\frac{\beta^2}{\alpha^2}-2\frac{b\beta}{a\alpha}\cos(\Delta\phi)+\frac{b^2}{a^2}}{N_u}$
d	$\frac{\frac{\beta^2}{\alpha^2}+2\frac{a\beta}{b\alpha}\cos(\Delta\phi)+\frac{a^2}{b^2}}{N_d}$	$\frac{\frac{\alpha^2}{\beta^2}+2\frac{a\alpha}{b\beta}\cos(\Delta\phi)+\frac{a^2}{b^2}}{N_d}$

with $N_u := \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} + 2\frac{b}{a}\cos(\Delta\phi)\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right) + 2\frac{b^2}{a^2}$
and $N_d := \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} + 2\frac{a}{b}\cos(\Delta\phi)\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) + 2\frac{a^2}{b^2}$

$P_{2b}(f \text{ when } t_2 \,|\, w \text{ when } t_2)$

f \ w	yes	no
u	α^2	β^2
d	β^2	α^2

Table 2.5. The conditional probabilities for results w of Wigner and f of the friend, both evaluated at time t_2, according to Definition 2b. The joint probability is well-defined and corresponds to the numerator in Equation (2.39).

$\langle\langle\Psi|t_1\rangle\langle t_1| \otimes \Pi^m |\Psi\rangle\rangle$ (see Appendix B.2). This is due to the fact that the friend's state, which encodes the result m at some time t_1, undergoes no further evolution until time t_2. In contrast to this, in a Wigner's-friend experiment F's state will be altered by W's measurement and reading it out at the end does in general not correspond to what was encoded there before W's measurement.

The fact that either Winger measuring which result his friend observed or the non-disturbance measurement are special cases where it is operationally meaningful to assign a joint probability to the results f and w lead us propose a third definition for two-time conditional probabilities within the Page-Wootters formalism, which gives well-defined probabilities for Wigner's-friend setups only in these special cases.

Definition 3 The conditional probability (in the cases where it is well-defined) of result n at time t_2 given result m at time t_1 is

$$P_3(n \text{ when } t_2 \,|\, m \text{ when } t_1) = \frac{\langle\langle\Psi|(|t_2\rangle\langle t_2| \otimes \Pi^n)P^{\mathrm{ph}}(|t_1\rangle\langle t_1| \otimes \Pi^m)\,|\Psi\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1| \otimes \Pi^m\,|\Psi\rangle\rangle}. \tag{2.40}$$

This probability rule gives well-defined probabilities only under the very restrictive condition that the measurement operators commute on the physical state when compared at the same instant of time, i.e.

$$\left[\mathscr{U}(t_1,t_2)\Pi^m\mathscr{U}^\dagger(t_1,t_2),\Pi^n\right]|\Psi\rangle\rangle = 0, \tag{2.41}$$

where $\mathscr{U}(t_2,t_1) = \langle t_2|P^{\mathrm{ph}}|t_1\rangle$. More concretely, evaluating Definition 3 for arbitrary parameters in the simple Winger's friend experiment gives the expressions listed in Table 2.6, which are in general neither real nor positive although they always sum up to one. As Richard Feynman already argued, however, one can consider those cases where some theory predicts non-positive probabilities as not physically realizable, rather than dismissing said theory merely because it predicts these probabilities.

> "If a physical theory for calculating probabilities yields a negative probability for a given situation under certain assumed conditions, we need not conclude the theory is incorrect. Two other possibilities of interpretation exist. One is that the conditions (for example, initial conditions) may not be capable of being realized in the physical world. The other possibility is that the situation for which the probability appears to be negative is not one that can be verified directly. A combination of these two, limitation of verifiability and freedom in initial conditions, may also be a solution to the apparent difficulty." Feynman [1987]

For Winger's friend scenarios this means understanding Definition 3 as identifying those cases where the two results under consideration are indeed jointly observable.

For the expressions in Table 2.6 to be real numbers we have to require

$$\Delta\phi = n\pi, \tag{2.42}$$

with n being an integer. Furthermore, the terms calculated using Definition 3 are all positive for $\alpha = 0, \beta = 1$ or $\alpha = 1, \beta = 0$, which is the measurement by W, where he reads out which result F observed and where all definitions agree on

$$\frac{\langle\langle\Psi(|t_2\rangle\langle t_2|\otimes\Pi^w)P^{\mathrm{ph}}(|t_1\rangle\langle t_1|\otimes\Pi^f)|\Psi\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1|\otimes\Pi^f|\Psi\rangle\rangle\rangle}$$

$f \backslash w$	yes	no
u	$\alpha^2+\frac{b}{a}\alpha\beta e^{-i(\Delta\phi)}$	$\beta^2-\frac{b}{a}\alpha\beta e^{-i(\Delta\phi)}$
d	$\beta^2+\frac{a}{b}\alpha\beta e^{-i(\Delta\phi)}$	$\alpha^2-\frac{a}{b}\alpha\beta e^{-i(\Delta\phi)}$

$$\frac{\langle\langle\Psi(|t_2\rangle\langle t_2|\otimes\Pi^w)P^{\mathrm{ph}}(|t_1\rangle\langle t_1|\otimes\Pi^f)|\Psi\rangle\rangle}{\langle\langle\Psi|t_2\rangle\langle t_2|\otimes\Pi^w|\Psi\rangle\rangle}$$

$f \backslash w$	yes	no
u	$\frac{a\alpha}{a\alpha+b\beta e^{-i(\Delta\phi)}}$	$\frac{b\beta}{b\beta+a\alpha e^{i(\Delta\phi)}}$
d	$\frac{a^2\beta^2-ab\alpha\beta e^{i(\Delta\phi)}}{a^2\beta^2+b^2\alpha^2-2ab\alpha\beta\cos(\Delta\phi)}$	$\frac{b^2\alpha^2-ab\alpha\beta e^{-i(\Delta\phi)}}{a^2\beta^2+b^2\alpha^2-2ab\alpha\beta\cos(\Delta\phi)}$

Table 2.6. Definition 3 evaluated for arbitrary a, b, α, β and $\Delta\phi$ possibly constituting the conditional probabilities of W seeing result $w \in \{\text{yes}, \text{no}\}$ at time t_2 given that F saw result $f \in \{u, d\}$ at time t_1 as well as that of F seeing f at t_1 given that W will see w at t_2. These expressions are real and positive and, hence, probabilities only if conditions (2.44) and either (2.47) or (2.48) are satisfied.

the probabilities. Moreover, for $\alpha, \beta \neq 0$, one obtains well-defined conditional probabilities for the result w at t_2 given the result f at t_1, if either

$$\alpha^2 - \frac{a}{b}\alpha\beta \geq 0 \quad \text{and} \quad \beta^2 - \frac{b}{a}\alpha\beta \geq 0 \tag{2.43}$$

or

$$\alpha^2 - \frac{b}{a}\alpha\beta \geq 0 \quad \text{and} \quad \beta^2 - \frac{a}{b}\alpha\beta \geq 0, \tag{2.44}$$

and conditional probabilities for f at t_1 given w at t_2, if either

$$1 - \frac{b\alpha}{a\beta} \geq 0 \quad \text{and} \quad 1 - \frac{a\beta}{b\alpha} \geq 0 \tag{2.45}$$

or

$$1 - \frac{b\beta}{a\alpha} \geq 0 \quad \text{and} \quad 1 - \frac{a\alpha}{b\beta} \geq 0, \tag{2.46}$$

The solution to both cases are the settings where Wigner's measurement is aligned

$P_3\,(w \text{ when } t_2\,|\,f \text{ when } t_1)$

$f \backslash w$	yes	no
u	1 (or 0)	0 (or 1)
d	1 (or 0)	0 (or 1)

$P_3\,(f \text{ when } t_1\,|\,w \text{ when } t_2)$

$f \backslash w$	yes	no
u	α^2 (or 0)	0 (or β^2)
d	β^2 (or 0)	0 (or α^2)

Table 2.7. The conditional probabilities of Wigner seeing result w at t_2 and the Friend seeing result f at time t_1 (above) and of the Friend seeing f at t_1 given that Wigner will see w at t_2 (below), according to Definition 3 for the non-disturbance case, i.e. $a = \alpha$ and $b = \beta$ (or $a = \beta$ and $b = -\alpha$). The probabilities are also well-defined in the case where Wigner's measurement reveals which result the friend observed.

with the initial state

$$a = \alpha, \qquad b = \beta \tag{2.47}$$

or

$$a = \beta, \qquad b = -\alpha. \tag{2.48}$$

In this case the state of the friend's laboratory is in an eigenstate of W's non-disturbance measurement and the conditional probabilities are either 0 or 1, when conditioning on the past, see Table 2.7.

Note that, where it is well-defined $P_3(w \text{ when } t_2\,|\,f \text{ when } t_1)$ coincides with $P_{2b}(w \text{ when } t_2\,|\,f \text{ when } t_2)$ which is in accordance with unitary dynamics for the friend's measurement. Furthermore, the respective one-time probability according to Definition 3 satisfies

$$\begin{aligned} P_3\,(f \text{ when } t_2\,\&\,w \text{ when } t_2) &= \langle\langle\Psi'|t_2\rangle\Pi^f\,\langle t_2|P^{\mathrm{ph}}|t_2\rangle\Pi^w\langle t_2|\Psi'\rangle\rangle \\ &= \langle\langle\Psi'|t_2\rangle\Pi^f \otimes \Pi^w\langle t_2|\Psi'\rangle\rangle = P_{2b}(w \text{ when } t_2\,\&\,f \text{ when } t_2). \end{aligned} \tag{2.49}$$

Hence, Definition 3 singles out those settings where the joint probability distribution of F's and W's outcomes is the same before and after Wigner's measurement

$$P_3(f \text{ when } t_2 \,\&\, w \text{ when } t_2) = P_3(f \text{ when } t_1 \,\&\, w \text{ when } t_2), \tag{2.50}$$

which means that the collected statistics at the end of the experiment can be identified with a two-time probability distribution.

As shown in Appendix B.2 all the above definitions of probability rules for the Page-Wootters formalism give the standard quantum probabilities when applied to non-Winger's friend scenarios. Moreover, a comprehensive comparison of all the definitions presented in this section can be found in Appendix B.3.

2.3.2 Different solutions of the Wigner's friend paradox within the Page-Wootters formalism

The three probability rules proposed in the previous Section 2.3.1 resolve the Wigner's friend paradox in different ways provided that Winger and his friend agree on a rule to calculate conditional probabilities for their consecutive measurement. Based on that, they can be naturally linked to different interpretations of quantum theory.

Definition 1 resolves the problem of differing probability assignments for Wigner's-friend scenarios insofar as it corresponds to describing F's measurement with collapse dynamics, i.e. objective collapse. Examples of interpretations and approaches most compatible with this probability rule are, for example Bub and Pitowsky [2010]; Diósi [2014]; Ghirardi et al. [1986]; Penrose [2000]. Note that models that purport a "real", physical collapse, such as those in Diósi [2014]; Ghirardi et al. [1986] and Penrose [2000], are not merely interpretations, but non-linear modifications of quantum theory that bring about a physical collapse of quantum system at a certain scale. Definition 1, on the contrary, is linear in the system state by construction. Just like the Schrödinger's equation with the standard Born rule, which is considered an effective approximation of the real dynamics in physical collapse theories, the Page-Wootters formalism together with Equation (2.33) can be regarded as an effective description of collapse dynamics, which gives the same probabilities, but does not provide a physical model for the collapse.

Natural candidates for interpretations that would endorse one of the Definitions 2, are those which endorse unitary dynamics also for measurements and therefore propose the no-collapse model for describing Wigner's-friend scenarios.

As already mentioned, the best known example for such interpretations are the many worlds interpretation of Everettian quantum theory DeWitt and Graham [2015] and Bohmian mechanics Dürr et al. [1996]. While Definition 2b is always well-defined and has a clear interpretation, namely the collectible statists of F's and W's perception at the end of the experiment, it fails in representing genuine two-time probabilities, since only t_2 appears in Equation (2.39). Definition 2a, on the other hand, while depending on both t_1 and t_2, gives well-defined, i.e., normalized, probabilities only under certain conditions, see Equations (2.35) and (2.36). Although both the measurement where W reads out F's observed result as well as the non-disturbance measurement satisfy these conditions and the probabilities agree with both Definition 2b and Definition 3, see Figure 2.6, for most cases there is no clear interpretation for the probabilities provided by Definition 2a. Formally for any given initial state of the quantum system and fixed measurement of the friend, there exists a measurement choice of Wigner (besides the non-disturbance one), for which Definition 2a gives well-defined probabilities. However, these probabilities in general neither correspond to collapse nor full unitary evolution up to the final time. David Wallace claims in Wallace [2012], that within the many worlds interpretation one can consistently adapt either a Lewisian view, where one can speak about multiple times within one decohered branch of the multiverse, or what he calls a Stage view, where one can only talk about single times. Neither of the two views is compatible with Definition 2a, since the friend is by construction not part of decohered history in a Wigner's-friend experiment, which rules out the Lewisian view, and yet it gives two-time probabilities, which are incompatible with the Stage view. Hence, only Definition 2b is compatible with the many worlds interpretation and aligns with the Stage view therein.

Definition 3 gives well-defined probabilities, i.e. real and non-negative, only under the condition in Equation (2.41) which singles out the cases when Wigner either measures what the friend observed or confirms the state of the friend's laboratory via a non-disturbance measurement. Remarkably, this purely formal condition singles out those settings for the simple Wigner's-friend setup, where one can attribute an operational meaning to the joint probability of the outcomes of F and W. This definition is, therefore in line with the considerations on Winger-Bell setups, see Section 1.2.2, stating that it is in general not possible to construct a joint probability distribution associated with measurement outcomes of encapsulated observers. As shown in Appendix B.1, condition (2.41) ensures that the outcomes of Wigner and his friend are part of a family of consistent quantum histories Griffiths [2003]. In the consistent histories framework probabilities of properties at different times, can only be assigned within such families

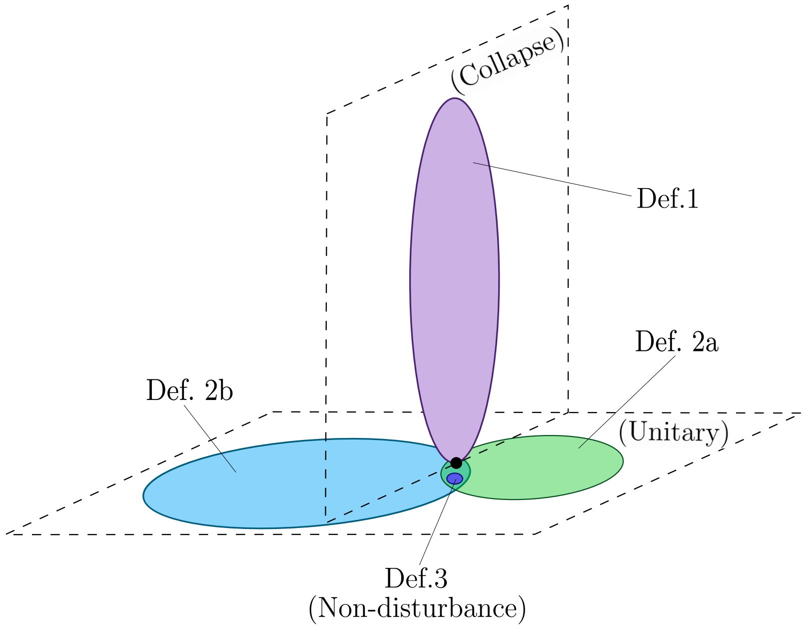


Figure 2.6. Suggestive representation of the probabilities according to Definitions 1-3 in a Wigner's-friend scenario (when they are well-defined). The two planes distinguish between joint probabilities that are in accordance with applying collapse or unitary dynamics for the friend's measurement. In the non-disturbance case (i.e., $\Delta\phi = n\pi$ and $a/\alpha = \pm b/\beta$) Definitions 2a, 2b and 3 all give the same conditional probabilities, that are in line with a unitary description of the friends measurement and different from the collapse ones given by Definition 1. There is, however, at least one instance where Definitions 1, 2a, and 2b coincide (i.e., $\Delta\phi = (n+1/2)\pi$, $a = b = 1/\sqrt{2}$ and arbitrary α) but Definition 3 is not defined. Note that, for non Wigner's-friend scenarios all the definitions give the standard quantum probabilities. (picture taken from Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz and Brukner [2019]).

of consistent histories. The reasoning about the extended Wigner's-friend setup in Frauchiger and Renner [2018], which is discussed in Detail in Section 1.2.2, is not allowed within this framework, because the results of the four agents are not all part of a family of consistent quantum histories, see Losada et al. [2019]. The consistent history interpretation is, therefore, a natural interpretation of quantum theory for the probability rule of Definition 3, although other interpretations, such as modern variants of the Copenhagen interpretation, might also be compatible with it. Since Definition 3 gives genuine two-time probabilities, in the few cases where they are defined, which agree with a unitary evolution during measurements it is incompatible with the many worlds interpretation for the same reasons that Definition 2a is.

Note that interpretations of quantum theory, which are likely to endorse the problematic subjective collapse model for Wigner's-friend setups, do not fit with the Page-Wootters formalism and, thus, with neither of the probability rules proposed in the previous section. Examples for such interpretations are relational quantum mechanics Rovelli [1996] and QBism Fuchs [2010], which then need to provide independent possible resolutions of the Wigner's friend paradox and especially consider those cases where the subjective collapse model may lead to an observable contradiction, as discussed in Section 2.2.1.

2.4 A no-go theorem for Wigner's friend's perception

In Allard Guérin et al. [2020] we extend our considerations about joint probabilities for the observed results of Wigner and his friend, to only the friend's observation at different times. More concretely, we formulated a no-go theorem for the joint probability distribution of the friend's perceived measurement outcomes before and after Wigner's measurement, at times t_1 and t_2 respectively, in the simple Winger's friend scenario, compare Sections 1.2.1 and 1.4.1. Such a joint probability distribution would, via division by the respective one-time probability, provide an answer to the friend's question: "Given that I saw outcome f_1 at time t_1, what is the probability (attributed by using quantum theory) that I will see outcome f_2 at time t_2?". Note that, recent work about the emergence of physical laws is based on the idea that the primary purpose of these laws is to give the conditional probability distributions relating events perceived by an observer at two subsequent times Müller [2020]. Such considerations place the friend's question above at the core of scientific theories. As shown bellow, however, a joint probability distribution $P(f_1, f_2)$ cannot simultaneously fulfill three assumptions, which seem natural within quantum theory. One conclusion of our no-go theorem is that treating a piece of information from the past as if it was still presently existing (even when one takes into account a possible subjective uncertainty) cannot in general be upheld within unitary quantum theory in the context of Wigner's-friend experiments.

Note that, since $P(f_1, f_2)$ does not correspond to anything directly observable – one cannot have direct perceptions about two different times– which of the assumptions of the no-go theorem is dropped is a matter of metaphysical preferences and will, as we discuss in Section 2.4.2, differ in different interpretations of quantum theory.

2.4.1 The no-go theorem and its assumptions

We will consider fully unitary quantum theory, which means that the overall state of the simple Wigner's-friend experiment evolves according to the relative state formalism. The initial state (at time t_0) of the whole setup is given by

$$|\Psi(t_0)\rangle = |\psi\rangle_S|r\rangle_F|r\rangle_W = \left(a|\uparrow\rangle_S + be^{i\phi_S}|\downarrow\rangle_S\right)|r\rangle_F|r\rangle_W. \tag{2.51}$$

At time t_F, the friend measures in the σ_z-basis, and for some time $t_1 > t_F$ the state is

$$|\Psi(t_1)\rangle = \left(a|\uparrow\rangle_S|u\rangle_F + be^{i\phi_S}|\downarrow\rangle_S|d\rangle_F\right)|r\rangle_W. \tag{2.52}$$

At time $t_W > t_1$, Wigner measures the friend and system in some entangled basis containing the states given by Equations (1.2) and (1.3), with the other two states corresponding to superpositions of $|\uparrow, d\rangle_{SF}$ and $|\downarrow, u\rangle_{SF}$ assumed to never be actualized in the experiment. Hence at time $t_2 > t_W$, the final state is

$$\begin{aligned}
|\Psi(t_2)\rangle =&(a\alpha + b\beta e^{-i\Delta\phi})|1\rangle_{SF}|1\rangle_W + (a\beta e^{i\phi_{SF}} - b\alpha e^{i\phi_S})|2\rangle_{SF}|2\rangle_W \\
=&\alpha(a\alpha + b\beta e^{-i\Delta\phi})|\uparrow\rangle_S|u\rangle_F|1\rangle_W \\
&+ \beta(a\alpha e^{i\phi_{SF}} + b\beta e^{i\phi_S})|\downarrow\rangle_S|d\rangle_F|1\rangle_W \\
&+ \beta(a\beta - b\alpha e^{-i\Delta\phi})|\uparrow\rangle_S|u\rangle_F|2\rangle_W \\
&- \alpha(a\beta e^{i\phi_{SF}} - b\alpha e^{i\phi_S})|\downarrow\rangle_S|d\rangle_F|2\rangle_W.
\end{aligned} \tag{2.53}$$

The state evolution given by Equations (2.51)-(2.53) also corresponds to the conditional Page Wootters states, i.e. $\langle t|\Psi\rangle\rangle$ with $t \in \{t_0, t_1, t_2\}$, for the Page Wootters formulation of the Winger's friend experiment presented in Sections 1.4.1 and 2.3. Note that the state $|\Psi(t_2)\rangle$ depends on the specific unitary realization of Wigner's measurement and that different purifications will lead to different final states.

One-time probabilities are calculated by using the projector Π^x onto the state of the respective observer seeing outcome x and the full state at the time t of interest, i.e.

$$P^{rels}(x) = \mathrm{Tr}\left(\Pi^x|\Psi(t)\rangle\langle\Psi(t)|\right), \tag{2.54}$$

compare Equation (2.5).

Our no-go theorem states that Wigner's friend cannot treat her perceived measurement outcome as having reality across multiple times without contradicting at least one of the following assumptions:

A1 The events f_1 and f_2, corresponding to the perceived measurement outcomes of F at times t_1 and t_2, respectively, can be combined into a joint event to which one assigns a probability distribution $P(f_1, f_2)$. The rules of the probability calculus further imply that $P(f_1) = \sum_{f_2} P(f_1, f_2)$ and $P(f_2) = \sum_{f_1} P(f_1, f_2)$.

A2 One-time probabilities are assigned according to the relative-state formalism

$$P(f) = \mathrm{Tr}(|f\rangle\langle f|_F |\Psi(t_i)\rangle\langle\Psi(t_i)|), \tag{2.55}$$

with $|\Psi(t_i)\rangle$ being the unitarily evolved global state according to Equations (2.51) - (2.53).

A3 The joint probability of F's perceived outcomes $P(f_1, f_2)$ has convex linear dependence on the initial state ρ_S of the system qubit.

Note that assumptions A1–A3 are not logically independent, for example, one cannot hold A3 without at the same time assuming A1.

In general we want to be able to consider cases where the system is initially in a mixed state ρ_S. Since such a state can always be decomposed as $\rho = \lambda|\psi\rangle\langle\psi| + (1-\lambda)|\phi\rangle\langle\phi|$, where $|\psi\rangle, |\phi\rangle$ are orthonormal states and $0 \leq \lambda \leq 1$, we obtain the following expression for the whole setup at different times

$$\Sigma(t) = \lambda|\Psi(t)\rangle\langle\Psi(t)| + (1-\lambda)|\Phi(t)\rangle\langle\Phi(t)|. \tag{2.56}$$

The states $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ are analogous to Equations (2.51)-(2.53) with initial system states $|\psi\rangle_S$ and $|\phi\rangle_S$ respectively. For mixed initial states ρ_S probabilities are then given by $P(x) = \mathrm{Tr}\,(\Pi^x \Sigma(t))$.

Assumption A1 is a consequence of requiring that quantum theory –like any other predictive theory– should provide a conditional probability distributions for the friend's perceptions before and after Wigner's measurement, i.e. $P(f_2|f_1)$, and the fact that it allows for calculating single event probabilities. Together they allow the construction of the joint probability distribution for the friend's perceived outcomes at two different times, $P(f_1, f_2)$. A1 can also be understood as a special case of what is termed *absoluteness of observed events* (AOE) in Bong et al. [2020]; Cavalcanti [2021], i.e. the assumption that "an observed event is a real single event, and not relative to anything or anyone". Since this idea is here applied to the same observer, it is conceptually different from original formulation of AOE, where it is about joint probability assignments for the measurement outcomes of multiple observers.

Assumption A2 provides the single-time probabilities for the friend's perception according to the relative state formalism $P(f_1) = \mathrm{Tr}\,(|f\rangle\langle f|_F|\Psi(t_1)\rangle\langle\Psi(t_1)|)$ and $P(f_2) = \mathrm{Tr}\,(|f\rangle\langle f|_F|\Psi(t_2)\rangle\langle\Psi(t_2)|)$. It can, in principle, be tested against objective collapse quantum theory in a Winger's friend experiment, where the two give different probabilistic predictions even for single event probabilities, see Bassi et al. [2013] and compare Sections 1.2.1 and 2.1.2.

Assumption A3 asks that the joint probabilities for events at multiple times depend linearly on the initial quantum state like single time probabilities do. It can be motivated by the fact that it holds in all typical laboratory situations. Alternatively A3 can be motivated operationally, in a way that is common in generalized probabilistic theories Hardy [2001]; Barrett [2007], by imagining a third agent is preparing the initial state of the system qubit, independently from the friend and Wigner. Said third party prepares one of two system states σ or τ with probabilities λ and $1-\lambda$ respectively, resulting in system state $\rho_S = \lambda\sigma + (1-\lambda)\tau$. Assuming A3 then implies that $P_\rho(f_1, f_2) = \lambda P_\sigma(f_1, f_2) + (1-\lambda)P_\tau(f_1, f_2)$, while denying it, and upholding assumption A1, means that a full specification of the initial state ρ_S alone is not sufficient for computing $P(f_1, f_2)$. Any convincing case against A3 should involve the prescription and justification of a specific non-linear two-time probability rule.

Theorem 1. *The conjunction of the assumptions A1-A3 cannot be satisfied for the simple Wigner's-friend experiment for a general choice of Wigner's measurement basis.*

The proof of Theorem 1 employs arguments about joint measurability due to the formal equivalence of the two problems. The physical interpretation, however, is different since we are concerned here with different perceptions of an agents at different times.

Proof. First, we define the isometries $V_i : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_F \otimes \mathscr{H}_W$, $i = 1, 2$ mapping the initial state of the system $|\psi\rangle_S$ to the corresponding state of the whole setup at time t_i as $V_i|\psi\rangle_S = |\Psi(t_i)\rangle_{SFW}$. Using Equations. (2.52) and (2.53), these isometries are

$$V_1 = |\uparrow, U, 0\rangle_{SFW}\langle\uparrow|_S + |\downarrow, D, 0\rangle_{SFW}\langle\downarrow|_S \tag{2.57}$$

$$V_2 = |1\rangle_{SF}|1\rangle_W\langle\phi_1|_S + |2\rangle_{SF}|2\rangle_W\langle\phi_2|_S \tag{2.58}$$

where $|\phi_1\rangle := \alpha|\uparrow\rangle + \beta e^{i\phi_{SF}}|\downarrow\rangle$ and $|\phi_2\rangle := \beta e^{-i\phi_{SF}}|\uparrow\rangle - \alpha|\downarrow\rangle$.

Extending assumption A3 to mixed states and using assumption A2, we have

$$P(f_1) = \mathrm{Tr}\left((|f_1\rangle\langle f_1|_F \otimes \mathbb{1}_{SW})V_1\rho V_1^\dagger\right) \tag{2.59}$$

$$= \mathrm{Tr}\left(V_1^\dagger(|f_1\rangle\langle f_1|_F \otimes \mathbb{1}_{SW})V_1\rho\right) = \mathrm{Tr}(E^1_{f_1}\rho), \tag{2.60}$$

where $E^1_{f_1} := V_1^\dagger(|f_1\rangle\langle f_1|_F \otimes \mathbb{1}_{SW})V_1$ can be thought of as the "Heisenberg picture" operator corresponding to measuring $|f_1\rangle\langle f_1|$ at time t_1. Note that, it is not exactly the Heisenberg operator since V_1 is not a unitary, but an isometry. It is easy to see that $\{E^1_{f_1}\}$ is a POVM. For time t_2 we analogously obtain

$$P(f_2) = \mathrm{Tr}(V_2^\dagger(|f_2\rangle\langle f_2|_F \otimes \mathbb{1}_{SW})V_2\rho) := \mathrm{Tr}(E^2_{f_2}\rho), \tag{2.61}$$

where $E^2_{f_2} := V_2^\dagger(|f_2\rangle\langle f_2|_F \otimes \mathbb{1}_{SW})V_2$ is also a POVM. Explicit calculations yield

$$E^1_u = |\uparrow\rangle\langle\uparrow|\,, \quad E^1_d = |\downarrow\rangle\langle\downarrow| \tag{2.62}$$

and

$$E^2_u = \alpha^2|\phi_1\rangle\langle\phi_1| + \beta^2|\phi_2\rangle\langle\phi_2|\,, \quad E^2_d = \beta^2|\phi_1\rangle\langle\phi_1| + \alpha^2|\phi_2\rangle\langle\phi_2|. \tag{2.63}$$

Assumptions A1 and A3 together imply that there exists a joint POVM $\{G_{f_1f_2}\}$ such that

$$P(f_1, f_2) = \mathrm{Tr}(G_{f_1f_2}\rho). \tag{2.64}$$

Requiring that the marginals of $P(f_1, f_2)$ obey assumption A2 for all states means that $\sum_{f_1} G_{f_1f_2} = E^1_{f_1}$ and $\sum_{f_2} G_{f_1f_2} = E^2_{f_2}$, which means that $\{E^1_{f_1}\}$ and $\{E^2_{f_2}\}$ are jointly measurable.

If (at least) one of two jointly measurable POVM's is sharp, then joint measurability is equivalent to commutativity. Moreover, the joint POVM $G_{f_1f_2} = E^1_{f_1}E^2_{f_2}$ is unique and gives the correct marginals, see Heinosaari et al. [2008]. Since we are considering two-outcome POVMs, and since E^1 is sharp, joint measurability means that in particular $[E^1_u, E^2_u] = 0$. Direct calculation, however, yields

$$[E^1_u, E^2_u] = (\alpha^2 - \beta^2)\alpha\beta e^{-i\phi_{SF}}|\uparrow\rangle\langle\downarrow| + (\beta^2 - \alpha^2)\alpha\beta e^{i\phi_{SF}}|\downarrow\rangle\langle\uparrow|. \tag{2.65}$$

which means that $E^1_{f_1}$ and $E^2_{f_2}$ are not jointly measurable for general choices of α, β, ϕ_{SF}. This concludes the proof.

□

While Theorem 1 holds in general, i.e. for arbitrary settings of W's measurement, there are special cases, which do allow for assumptions A1-A3 to be satisfied simultaneously. For two types of settings the commutator in Equation (2.65) vanishes and one can explicitly calculate probabilities $P(f_1, f_2)$ as well as $P(f_1|f_2)$. The conditional probability indicates whether the the friend's perception is altered during Winger's measurement. The two special cases are the one where Wigner reads out his friend's observed result, i.e. $\phi_{SF} = n\pi$ and $\alpha = 1, \beta = 0$ or vice versa, as well as the case where $\phi_{SF} = n\pi$ and $\alpha = \beta = \frac{1}{\sqrt{2}}$. In the first case, one obtains $E^1_f = E^2_f$ and the probability distribution that satisfies all assumptions A1-A3 is

$$P(f_1,f_2) = \mathrm{Tr}(E^1_{f_1}E^2_{f_2}\rho)$$

$$\begin{array}{c|c|c} f_1 \backslash f_2 & u & d \\ \hline u & a^2 & 0 \\ d & 0 & b^2, \end{array} \tag{2.66}$$

which gives conditional probabilities $P(f_2|f_1) = \delta_{f_1f_2}$. This means that the perception of friend is perfectly preserved.

In the second case, Wigner measures in a Bell-like basis, for which one obtains $E^2_f = \frac{1}{2}\mathbb{1}$ leading to probabilities

$$P(f_1,f_2) = \mathrm{Tr}(E^1_{f_1}E^2_{f_2}\rho)$$

$$\begin{array}{c|c|c} f_1 \backslash f_2 & u & d \\ \hline u & \frac{a^2}{2} & \frac{a^2}{2} \\ d & \frac{b^2}{2} & \frac{b^2}{2} \end{array} \tag{2.67}$$

The corresponding conditional probability distribution is $P(f_2|f_1) = \frac{1}{2}\ \forall f_1, f_2$, which means that F's perception is flipped during W's measurement with probability $\frac{1}{2}$, regardless of which result the friend initially observed and independently of initial state ρ. This case is surprising since it includes the scenario where the system is initially in state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. In this case W performs a non-disturbance measurement when choosing the Bell-like basis, which does not change the state of the friend's laboratory.

2.4.2 Implications of the no-go theorem for different interpretations

While previous no-go theorems Brukner [2018]; Bong et al. [2020] were concerned with Wigner's view of the friend's state, the no-go theorem above shows that even from the friend's perspective, treating her own perception of an outcome as persistent throughout a Wigner's-friend experiment is problematic. More concretely, she cannot assign a joint probability to her observed outcomes before and after Wigner's measurement fulfilling all three of our assumptions in previous section. Different interpretations of quantum theory advocate for dropping different assumptions out of A1-A3 in reaction to Theorem 1.

For example, the many-worlds interpretation would probably reject A1 for the reasons mentioned in Section 2.3.2. Since there is by construction not enough decoherence present in a Wigner's-friend experiment for associating the friend with a distinct decohered branch of the multiverse, there is no way to meaningfully talk about the friend's perception at two different times, see Wallace [2012]. Another class of interpretations that are likely to reject A1 are those based on operational approaches like Brukner [2017]; Oeckl [2019]. These approaches would allow for the assignment of probabilities only where one can in principle perform a corresponding measurement, which is not the case for the joint event (f_1, f_2). In a Wigner's-friend experiment there cannot be a reliable record of f_1 that remains available after time t_2 without changing the probabilities of measurements performed at times $t > t_1$, see Section 2.2.

Assumption A2, on the other hand will be denied by any interpretation that purports collapse dynamics, either objectively or subjectively. In the first case the overall state after F's measurement would no longer be given by Equation (2.52) and probabilities calculated using Equation 2.54 for $t > t_F$ would be verifiably wrong. Alternatively, subjective collapse interpretations like QBism would claim that the friend, but not Wigner, is justified to use the state-update rule after her measurement and therefore calculate $P(f_2)$ different from Equation (2.55). This will, however, conflict with Wigner's probability assignment and in special cases lead to observable contradictions, compare Section 2.2.

In Bohmian mechanics assumption A1 holds and, as already shown in Bohm [1952a,b], single-time probabilities are the same as those given by unitary quantum theory, which means that A2 is also satisfied. Hence, our no-go theorem implies that A3 cannot hold in Bohmian mechanics, which agrees with the fact that the Bohmian guidance equation is in general non-linear in the density operator Bell [2004]; Luis and Sanz [2015].

Chapter 3

Superpositions of the order of operations

This Chapter contains our results on the topic of indefinite causal order, which are published in Baumann and Brukner [2016] and Baumann et al. [2021]. The process-matrix framework Oreshkov et al. [2012] was invented to describe exotic causal structures, like superpositions of space-times and superpositions of the order of events. However, many processes arising in this framework have no clear physical meaning. We investigated which classes of operations, for two agents, will always result in correlations which are compatible with definite causal order, see Section 3.1 as well as what kind of process, causal and non-causal, are compatible with a Page-Wootters history state, see Sections 3.2 and 3.3.

3.1 Effective bipartite causality

In Baumann and Brukner [2016] we showed that, in the case of two agents, quantum operations that contain a measurement of the input in some fixed basis give rise to effective process matrices which correspond to a definite causal order. Consider agents A and B performing operations corresponding to CJ-matrices $\{M_i^{A_1A_2}\}$ and $\{ M_j^{B_1B_2} \}$ respectively. As discussed in Section 1.3, the joint probabilities for these operations are given by Equation (1.10). However, for any given sets $\{M_{i_k}^{X_1X_2}\}$ of operations, we can define the effective process matrix W_{eff}, which returns the same joint probabilities as the original process matrix W, by

$$\forall i_k : \mathrm{Tr}\left(W \left(\bigotimes_{k=1}^{n} M_{i_k}^{X_1^k X_2^k} \right) \right) \stackrel{!}{=} \mathrm{Tr}\left(W_{eff} \left(\bigotimes_{k=1}^{n} M_{i_k}^{X_1^k X_2^k} \right) \right). \tag{3.1}$$

This definition of effective process matrices is useful if the agents' operations are restricted to certain classes.

In Oreshkov et al. [2012] the authors showed that if two agents' operations are classical, meaning they perform so-called *measure-and-prepare operations* where the inputs are measured in a fixed basis and the outputs are re-prepared in a fixed basis, the causal relations between these operations are always compatible with a global causal order. This means that the effective process matrix defined by Equation (3.1) is a convex mixture of causally ordered processes

$$W_{eff} = pW^{B\preceq A} + (1-p)W^{A\preceq B}, \tag{3.2}$$

with $p \in [0,1]$ and $W^{A\preceq B}$ containing only terms where B's operation cannot influence A's and vice versa for $W^{B\preceq A}$. Processes of the form (3.2) are called *causally separable* Oreshkov and Giarmatzi [2016] and do not lead to violations of causal inequalities, compare Section 1.3.

We show, if two agents are restricted to measuring their input in fixed bases, there always exists a causally separable effective process matrix. This means that two such agents cannot violate any casual inequality. The CJ matrices of such operations are of the form

$$\begin{aligned} M_i^{A_1A_2} &= \sum_n p(i|n)|n\rangle\langle n|^{A_1} \otimes \rho_i^{A_2} \\ &\text{and} \\ M_j^{B_1B_2} &= \sum_m p(j|n)|m\rangle\langle m|^{B_1} \otimes \rho_j^{B_2}, \end{aligned} \tag{3.3}$$

where $p(k|l)$ is the probability to prepare state $\rho_k^{X_2}$ given the input is measured to be in state $|l\rangle$. All process matrices for two agents can be written as

$$W = \sum w_{nn'rr'mm'ss'}|n\rangle\langle n'|^{A_1} \otimes |r\rangle\langle r'|^{A_2} \otimes |m\rangle\langle m'|^{B_1} \otimes |s'\rangle\langle s'|^{B_2}, \tag{3.4}$$

where the sum is taken over all the indices. The effective process matrix for operations of the form (3.3) is partially diagonal in the two agents' input Hilbert spaces:

$$W_{eff} = \sum w^{eff}_{nii'mjj'}|n\rangle\langle n|^{A_1} \otimes |i\rangle\langle i'|^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes |j\rangle\langle j'|^{B_2}, \tag{3.5}$$

where $\{|i\rangle\}$ and $\{|j\rangle\}$ are arbitrary orthonormal bases of $\mathscr{H}^{A_2}$ and $\mathscr{H}^{B_2}$ and $w^{eff}_{nii'mjj'}$is shorthand notation for $w_{nnii'mmjj'}$. It is straight forward to check that Equations (3.5) and (3.4) give the same probabilities when applied to operations in Equation (3.3) only. Note that this effective process matrix can be related to the original one by the update rule

$$W \rightarrow W_{eff} = \sum_{n,m} P_{(n,m)} W P_{(n,m)}, \tag{3.6}$$

where $P_{(n,m)} = |n\rangle\langle n|_{A_1} \otimes |m\rangle\langle m|_{B_1}$, which is analogous to the von Neumann-Lüder rule Lüders [1950] for the state update under a non-selective measurement. While W might exhibit indefinite casual order, once the agents perform measurements on their inputs in fixed bases, the process matrix is causally separable and given by Equation (3.6). This is similar to the projection of a general bipartite state measured in a product basis, i.e. $\sum_{n,m} P_n \otimes P_m \rho P_n \otimes P_m$ being a separable state even if the original state ρ was entangled. However, there is no such analogy for single selective measurements, since although $P_n \otimes P_m \rho P_n \otimes P_m$ is a valid quantum state, $P_{(n,m)} W P_{(n,m)}$ is in general *not* a process matrix. The latter may contain forbidden causal loops.

Further, note that any bipartite process matrix can also be written as

$$W = \frac{1}{d}\left(\mathbb{1}^{A_1A_2B_1B_2} + \sum_i c_i W_i^{A_1A_2B_1B_2}\right) = \frac{1}{d}((1+\lambda_0)\mathbb{1} + \kappa_1 + \kappa_2), \tag{3.7}$$

where $d = \dim(\mathscr{H}^{\mathrm{A}_1}) \cdot \dim(\mathscr{H}^{\mathrm{B}_1})$ and $\lambda_0 \in [-1, 0]$ is the minimal eigenvalue of $\sum_i c_i W_i^{A_1A_2B_1B_2}$. We have that $\kappa_1 + \kappa_2 \geq 0$ and κ_1 acting trivially on $\mathscr{H}^{B_2}$ and κ_2 acting trivially on $\mathscr{H}^{A_2}$. Note, however, that κ_1 and κ_2 themselves are in general not positive semi-definite themselves.

For the effective process matrix in Equation (3.5) we can use terms $P_{(n,m)} = |n\rangle\langle n|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes \mathbb{1}^{B_2}$ to add and subtracted from κ_1 and κ_2 for every pair (n,m) such that the sum $\kappa_1 + \kappa_2$, and hence W_{eff}, remains unchanged. This allows us to arrive at matrices $\overline{\kappa}^{A_1A_2B_1} \geq 0$ and $\overline{\kappa}^{A_1B_1B_2} \geq 0$ where each is positive semi-definite on its own. Since we only acted on the subspace of $\mathscr{H}^{A_1} \otimes \mathscr{H}^{B_1}$ we still find that $\overline{\kappa}^{A_1A_2B_1}$ acts trivially on $\mathscr{H}^{B_2}$ while $\overline{\kappa}^{A_1B_1B_2}$ acts trivially on $\mathscr{H}^{A_2}$. Hence we can rewrite

$$W_{eff} = \frac{1}{d}(\overline{\kappa}^{A_1A_2B_1} + \overline{\kappa}^{A_1B_1B_2}) = pW^{A \preceq B} + (1-p)W^{B \preceq A}, \tag{3.8}$$

where $p = \mathrm{Tr}(\overline{\kappa}^{A_1A_2B_1})/d'$ and $1-p = \mathrm{Tr}(\overline{\kappa}^{A_1B_1B_2})/d'$ with $d' = d \cdot \dim(\mathscr{H}^{\mathrm{A}_2}) \cdot \dim(\mathscr{H}^{\mathrm{B}_2})$.

The systematic modification of the eigenvalues of κ_1 and κ_2 requires that

$$[\kappa_1, \kappa_2] = [\kappa_1, P_{(n,m)}] = [P_{(n,m)}, \kappa_2] = 0 \tag{3.9}$$

and the joint eigenvectors of κ_1, κ_2 and $P_{(n,m)}$ to be product vectors

$$|\psi^{(n,m)}\rangle = |n\rangle^{A_1} \otimes |a^{(n,m)}\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |b^{(n,m)}\rangle^{B_2} \tag{3.10}$$

with $\{|n\rangle\}$ and $\{|m\rangle\}$ being orthonormal bases of the input spaces $\mathscr{H}^{A_1}$ and $\mathscr{H}^{B_1}$ and $\{|a^{(n,m)}\rangle\}$ and $\{|b^{(n,m)}\rangle\}$ bases of $\mathscr{H}^{A_2}$ and $\mathscr{H}^{B_2}$ respectively. The latter two might, however, be different for every pair (n, m). If Equations (3.9) and (3.10) are fulfilled the eigenvalues of $\kappa_1 + \kappa_2$ are given by $m(n, a, m, b) = m_1(n, a, m) + m_2(n, m, b)$ and adding and subtracting $P_{(n,m)}$ for every (n, m) allows for creating positive eigenvalues $m_1'(n, a, m)$ and $m_2'(n, m, b)$ while keeping the sum invariant. This procedure eventually leads to the expression in Equation (3.8). This argument is a generalization of the proof provided in the supplementary information of Oreshkov et al. [2012], where the authors considered classical operations, the CJ matrices of which are fully diagonal in the so called pointer basis. Hence also the effective process matrix for classical operations is diagonal in that basis.

Note that while conditions (3.9) and (3.10) are sufficient for causal separability in the bipartite case, they are by no means necessary. Consider, for example, the causally separable process matrix

$$W_0 = \frac{p}{d}(\mathbb{1} - \sigma_z^{A_1} \otimes \sigma_z^{A_2} \otimes \sigma_x^{B_1}) + \frac{1-p}{d}(\mathbb{1} + \frac{1}{2}\sigma_z^{A_1} \otimes \sigma_x^{B_2} + \frac{1}{2}\sigma_x^{A_1} \otimes \sigma_z^{B_1} \otimes \sigma_z^{B_2}), \tag{3.11}$$

with non-commuting terms for κ_1 and κ_2. Moreover, there is no $P_{(n,m)}$ that commutes with both terms, since it would have to commute with both σ_x and σ_z on the subspaces $\mathscr{H}^{A^1}$ and $\mathscr{H}^{B^1}$. Hence neither condition (3.9) nor (3.10) is fulfilled, although W_0 is causally separable per definition.

The effective arise of a definite causal order is, however, not a consequence of classicality in general. For three or more parties there exist (effective) process matrices, that violate causal inequalities, even if all operations involved are classical, see Baumeler et al. [2014]; Baumeler and Wolf [2016].

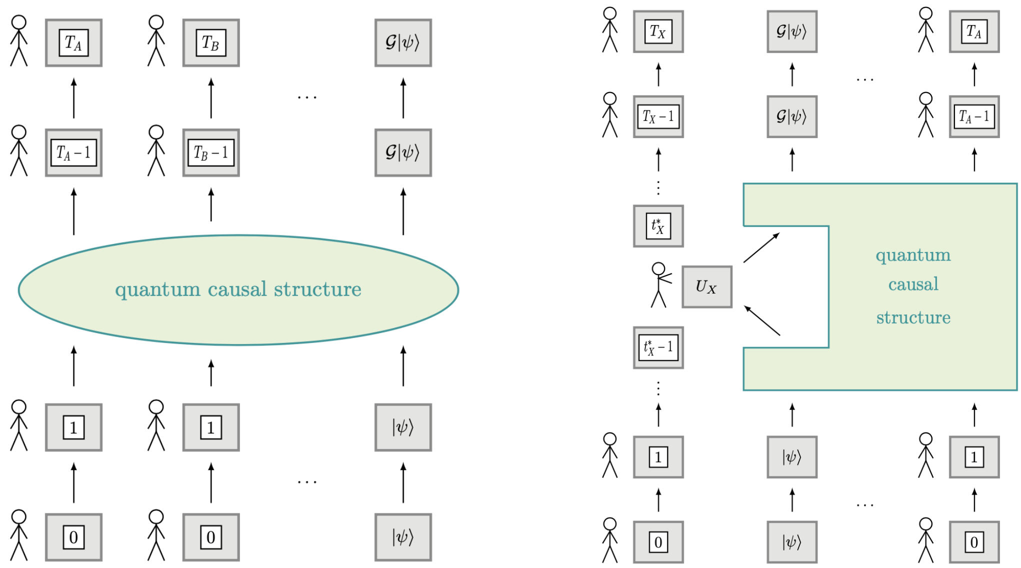
3.2 Page-Wooters formulation of indefinite causal order

In Castro-Ruiz et al. [2020], the authors considered a generalized Page-Wootters formalism using several clocks, compare Section 1.4, and found that history states arising from solving a Hamiltonian constraint can give rise to indefinite causal order. The authors studied the time evolution according to the perspectives of different clocks and showed how to recover the so-called gravitational quantum switch Zych et al. [2019]. However, it is in general not clear which non-causal processes can be implemented within such a framework or which concrete process (if any) is implemented by a given history state.

In Baumann et al. [2021] we, therefore, systematically combined the process-matrix framework with a generalization of the Page-Wootters formalism by introducing multiple discrete quantum clocks, each of which is associated with an agent. This allowed us to describe scenarios where different definite causal orders are coherently controlled, it also hints at the possibility that certain non-causal processes might not be implementable within this setting, which can be regarded as an argument against these processes being compatible with the known physical laws. We presented a general definition of what it means for a history state to implement a process matrix. The various discrete clocks and the corresponding agents can be thought of as initially being part of definite causal structure before they experience some quantum casual structure, where the global order of events is no longer well-defined. Finally they all return to a definite causal structure. This setting is reminiscent of pure process matrices and we, indeed, arrive at a refined version of the perspectival circuits from the causal reference frame picture Allard Guérin and Brukner [2018], compare Section 1.3.2.

The main idea of our paper Baumann et al. [2021] is to associate each agent $X \in \{A, B, \dots N\}$ a discrete quantum clock with Hilbert spaces $\mathscr{H}_{c_A}$, $\mathscr{H}_{c_B}, \dots \mathscr{H}_{c_N}$. We denote all clock variables collectively as $\mathscr{H}_c$. Analogous to the picture associated with pure processes, we are interested in scenarios with a well-defined global causal past and future with potentially indefinite casual order in between. We formalize this by requiring that all the clocks experience at least one well-synchronized time step at the beginning and in the end, see Figure 3.1. Again, analogous to the pure process formalism, each agent has access to an ancillary degree of freedom, denoted by Hilbert spaces $\mathscr{H}_{A'}$, $\mathscr{H}_{B'}, \dots \mathscr{H}_{N'}$, to implement their quantum operation as unitaries $U_X \in \mathscr{L}(\mathscr{H}_X \otimes \mathscr{H}_{X'})$. These ancilla systems are assumed to undergo trivial time evolution, except at the moment when they are part of the corresponding quantum operation, and are collectively denoted as $\mathscr{H}_{S'} := \mathscr{H}_{A'} \otimes \mathscr{H}_{B'} \otimes \dots \mathscr{H}_{N'}$. The main quantum system, part of which the agents act upon, is in the initial state $|\psi\rangle_S$ of Hilbert space $\mathscr{H}_S$. In addition to the agents' ancillas it constitutes the input from the global past.

The Page-Wootters history states, objects in the Hilbert space $\mathscr{H}_c \otimes \mathscr{H}_S \otimes \mathscr{H}_{S'}$,



(a) global perspective (b) agent's perspective

Figure 3.1. Scenarios with potentially indefinite causal order described by a Page-Wooters history state: In the beginning and at the end, all agents are part of a definite causal structure, which is represented by their clocks ticking in synchronization. In between, however, they experience a quantum causal structure, in which the clocks and the system might get entangled with each other, see (a). Each agent, inside his or her laboratory, experiences the progression of time according to his or her clock. At time $t_X^* - 1$, (part of) the system enters the laboratory and agent X applies unitary operation U_X to this (part of the) system and the respective ancilla X' (not shown here). At time t_X^* the agent sends away his or her output, compare (b).

can be written as

$$
\begin{aligned}
|\Psi\rangle\rangle &= \sum_{t_{A_1}=0,\dots t_{A_N}=0}^{T_{A_1}\dots T_{A_N}} |t_{A_1},\dots t_{A_N}\rangle \otimes |\psi(t_{A_1}\dots t_{A_N})\rangle_S \\
&= \sum_{t_{A_1}=0,\dots t_{A_N}=0}^{T_{A_1}\dots T_{A_N}} |t_{A_1}\rangle \dots |t_{A_N}\rangle \otimes M_{t_{A_1}\dots t_{A_N}} |\phi\rangle, \qquad (3.12)
\end{aligned}
$$

where we initially set all clocks to 0 for notational convenience and $T_A, T_B, \dots T_N$ are the times the agents' clocks show at the end of the scenario depicted in Figure 3.1. Note that these final times can be different for different agents due to effects of "time dilation". Matrices $M_{t_{A_1}\dots t_{A_N}}$ describe what happens to the system $|\phi\rangle \in \mathscr{H}_S$ for every collection of clock states $|t_{A_1}\rangle \dots |t_{A_N}\rangle$. The perspective of agent X is associated with projecting the history state onto that agents clock. More precisely $\langle t_X \,|\Psi\rangle\rangle$ gives the state agent X assigns to everything other than his or her own clock at time t_X.

Putting the above in more formal language leads to the following axioms, which set up the general framework for so-called non-causal Page-Wootters circuits discussed in Section 3.3. First, the reference state the agents' ancilla systems are initialized to is denoted as $|0\rangle$ for all agents and independent of the system state $|\psi\rangle_S$.

S.1 $|\psi(0,0,\dots)\rangle = |\psi\rangle_S|0\rangle_{S'}$, where $|0\rangle_{S'} = |0\rangle_{A'} \otimes |0\rangle_{B'} \otimes \dots |0\rangle_{N'}$ is the collection of all the agents' ancillas and $|\psi\rangle_S$ is a free parameter.

The synchronized time step that all agents experience in the beginning and at the end is captured by the second axiom, where we further assume that during these well-synchronized time-steps the system undergoes trivial evolution.

S.2 For agent A, and analogous for all other agents: $|\psi(0,\dots,t_X,\dots)\rangle \neq 0$ only for $t_X = 0\ \forall X \neq A$ and $|\psi(T_A,\dots,t_X,\dots)\rangle \neq 0$ only for $t_X = T_X\ \forall X \neq A$ and $|\psi(1,1\dots 1)\rangle = |\psi(0,0,\dots,0)\rangle$ and
$|\psi(T_A-1,T_B-1,T_C-1,\dots)\rangle = |\psi(T_A,T_B,T_C,\dots)\rangle$.

As mentioned in Section 1.4.2, in the usual Page-Wooters formalism with infinite dimensional systems, the physical Hilbert space is not a proper subspace of the kinematical Hilbert space which necessitates the definition of a new inner product when considering perspectival states, see Hoehn and Vanrietvelde [2018]; Hoehn et al. [2019]. This allows to absorb renormalization factors for the perspectival states in an inner product definition. Consider the example of the history state of two clocks, where one ticks at twice the rate of the other clock

$$|\Psi\rangle\rangle = \int \mathrm{d}t_A |t_A\rangle_{c_A} \otimes |2t_A\rangle_{c_B}\,. \tag{3.13}$$

To obtain agent A's perspective, we consider

$${}_{c_B}\langle t_B|\Psi\rangle\rangle = \int \mathrm{d}t_A |t_A\rangle\langle t_B|2t_A\rangle = \frac{1}{2}\int \mathrm{d}t'_B |1/2\ t'_B\rangle\langle t_B|t'_B\rangle = \frac{1}{2}|1/2\ t_B\rangle,$$

where the prefactor $\frac{1}{2}$, which is a consequence of the change of the integration variable, leads to a not normalized state after projecting onto A's clock. The according definition of the inner product for A's perspectival states will ensure that these states are normalized with respect to this new inner product. For the case of finite dimensional discrete clocks, where the physical Hilbert space is a proper subspace of the kinematical Hilbert space, the renormalization of perspectival states needs to be taken care of in a different manner. A different kind of renormalization issue for discrete clocks arises from the process of discretization itself, when continuous times $|t+\delta t\rangle$ and $|t\rangle$ for small δt will in general get mapped to the same discrete time state. The example in Equation (3.13) cannot be naively discretized to

$$|\psi\rangle\rangle = \sum_t |t\rangle_{c_A} \otimes |2t\rangle_{c_B}, \tag{3.14}$$

if we require that agent B can assign a state for each time value of their clock, since ${}_{C_B}\langle t|\psi\rangle\rangle = 0$ for odd integers t. Instead an acceptable discretization of Equation (3.13) is

$$|\psi\rangle\rangle = \sum_t |t\rangle_{c_A} \otimes |\lfloor t/2 \rfloor\rangle_{c_B} = |0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle + |2\rangle \otimes |1\rangle + |3\rangle \otimes |1\rangle + \dots \tag{3.15}$$

where $\lfloor\cdot\rfloor$ means the rounded down integer of the argument. This discretization allows for interpreting the clock states $|t\rangle$ as the number of ticks the agents see. Since A's clock runs twice as fast, B's first tick appear only after A's clock ticked twice already. Projecting onto the two different clocks gives the states

$${}_{c_A}\langle 0|\Psi\rangle\rangle = |0\rangle_{c_B}, \quad {}_{c_A}\langle 1|\Psi\rangle\rangle = |0\rangle_{c_B}, \quad {}_{c_A}\langle 2|\Psi\rangle\rangle = |1\rangle_{c_B}, \quad {}_{c_A}\langle 3|\Psi\rangle\rangle = |1\rangle_{c_B}, \dots$$

and

$${}_{c_B}\langle 0|\Psi\rangle\rangle = |0\rangle_{c_A} + |1\rangle_{c_A}, \qquad {}_{c_B}\langle 1|\Psi\rangle\rangle = |2\rangle_{c_A} + |3\rangle_{c_A}, \dots$$

The perspectival states of B are not normalized, but can easily be renormalized by the introduction of a *normalization operator* $N^{(B)}_{t_B} = \frac{1}{\sqrt{2}}\mathbb{1}$. We, therefore, define the perspectival states $|\psi_X(t_X)\rangle$ an agent X sees when their clock reads time t_X as

$$|\psi_X(t_X)\rangle := N^{(X)}_{t_X}\langle t_X|\Psi\rangle\rangle = \langle t_X|_{c_X} \otimes N^{(X)}_{t_X}|\Psi\rangle\rangle, \tag{3.16}$$

where $N^{(X)}_{t_X} \in \mathscr{L}(\mathscr{H}_{c_{\setminus X}} \otimes \mathscr{H}_S \otimes \mathscr{H}_{S'})$ is the normalization operator that relates the perspective-neutral history state to the perspectival state of agent X at time t_X. We denote by $\mathscr{H}_{c_{\setminus X}}$ is the Hilbert space of all clocks except that of agent X.

Our next three axioms concern said normalization operators.

N.1 $N_{t_X}^{(X)}$ is an invertible, linear, positive operator. It is independent of the input state $|\psi\rangle_S$ and the local operations $U_A, U_B, \ldots U_N$.

This captures the idea that the normalization operators should be a generalization of normalization constants and that the physics of the scenario should already be encoded in the history state and agents' operations and not in the normalization operator. Without this restriction, we could use $N_{t_X}^{(X)}$ to, for example, introduce copies of the initial state $|\psi\rangle_S$ and thereby violate the no-cloning principle Wootters and Zurek [1982].

N.2 The normalization operator has the form

$$N_{t_X}^{(X)} = \sum_{t_A,\ldots,\widehat{t_X},\ldots t_N} |t_A, \ldots \widehat{t}_X, \ldots t_N\rangle\langle t_A, \ldots \widehat{t}_X, \ldots t_N| \otimes n^{(X)}_{t_A,\ldots \widehat{t}_X,\ldots t_N} \otimes \mathbb{1}_{S'} \quad (3.17)$$

where $\widehat{t_X}$ indicates that the clock of agent X is not summed over while all other clocks are. The operator $n^{(X)}_{t_A,\ldots \widehat{t}_X,\ldots t_N}$ is a linear, invertible and positive operator acting on $\mathcal{H}_S$ but not on the ancillas $\mathcal{H}_{S'}$.

This means that the normalization operator of agent X does not perturb the clocks of the other agents. Further, the previous requirement of well-synchronized time-steps in the beginning and at the end implies that the respective normalization operators should just be the identity.

N.3 $N_1^{(X)} = N_0^{(X)} = \mathbb{1}$ as well as $N_{T_X-1}^{(X)} = N_T^{(X)} = \mathbb{1}\ \forall X$.

The perspectival states of an agent at different times should be unitarily related to each other. This means that we assume that for all relevant t_X, t'_X there exists a unitary operator $\mathcal{U}_X(t_X, t'_X)$ such that

$$|\psi_X(t_X)\rangle = \mathcal{U}_X(t_X, t'_X)|\psi_X(t'_X)\rangle. \quad (3.18)$$

Just like in standard quantum theory we require the following:

U.1 $\mathcal{U}_X(t, t')$ is a unitary operator, independent of the initial state $|\psi\rangle_S$.

U.2 $\mathcal{U}_X(t, t')\mathcal{U}_X(t', t'') = \mathcal{U}_X(t, t'')$, $\forall t, t', t''$.

Finally, we call the time agent X receives his or her part of the system $t_X^* - 1$ and assume that he or she applies quantum operation $U_X \in \mathcal{L}(\mathcal{H}_X \otimes \mathcal{H}_{X'})$ at t_X^*. While the agents enforce evolution of their input and respective ancilla via their unitary operation, all other degrees of freedom should evolve in an uncorrelated way.

U.3 X's quantum instrument is used at the so called *time of action* t_X^*, i.e.

$$\mathcal{U}_X(t_X^*, t_X^* - 1) = U_X \otimes \mathrm{Rest}^{(X)}. \tag{3.19}$$

Furthermore at other times $t \neq t_X^*$ the evolution operator $\mathcal{U}_X(t, t-1)$ is independent of U_X and only acts as the identity on $\mathcal{H}_{X'}$.

With the above axioms we can write down the perspective of an agent, for example A, and define what it means for a process to be implemented in our framework:

$$\begin{aligned} &|T_B, \dots T_N\rangle_{c\setminus A} \otimes |\psi(T_A, T_B, \dots T_N)\rangle \\ &\quad = \big(\mathcal{U}_A(T, t_A^*)(U_A \otimes \mathrm{Rest}^{(A)})\, \mathcal{U}_A(t_A^* - 1, 0)\big)\big(|0, \dots 0\rangle_{c\setminus A} \otimes |\psi(0, 0, \dots 0)\rangle\big) \\ &\quad =: |T_B, \dots T_N\rangle \otimes \mathcal{G}(U_A, U_B \dots)|\psi(0, 0, \dots)\rangle. \end{aligned} \tag{3.20}$$

The map $\mathcal{G}(U_A, U_B, \dots U_N)$ is a unitary that is multilinear in the local operations and the above decomposition shows that the only change in the state of the ancilla A' is caused by U_A.

Note that Equation (3.20) represents a refined picture of causal reference frames, which explicitly includes the quantum clocks of the agents, compare to Equation (1.12). The causal past and future Π_X and Φ_X of the original framework, see Section 1.3.1, correspond to $\mathcal{U}_X(t_X^* - 1, 0)$ and $\mathcal{U}_X(T_X, t_X^*)$ respectively. However, while the causal past and future unitaries in Allard Guérin and Brukner [2018] are allowed to be arbitrary as long as they combine to give the process $\mathcal{G}$ via Equation (1.12), in our setting the history state induces further compatibility constraints on the perspectives of the agents. For example, consider the form of a history state in Equation (3.12) and in particular agent A's perspective

$$|\psi(t_A, t_B, \dots t_N)\rangle = M_{t_A, t_B, \dots t_N} |\psi(0, 0, \dots 0)\rangle, \tag{3.21}$$

where

$$M_{t_A, t_B, \dots t_N} = {}_{c\setminus A}\langle t_B, \dots t_N | (N_{t_A}^{(A)})^{-1}\, \mathcal{U}_A(t_A, 0) | 0, \dots 0\rangle_{c\setminus A}. \tag{3.22}$$

It follows from our axioms that $M_{t_A, \dots t_N}$ is constant in U_A for $t_A < t_A^*$ and linear in U_A for $t_A \geq t_A^*$, because the same is true for the respective $\mathcal{U}_A$. We can alternatively relate the time evolutions the agents see via

$$\mathcal{U}_B(t_B, 0)|0, 0, \dots 0\rangle_{c\setminus B} = \sum_{t_A, t_C, \dots t_N} N_{t_B}^{(B)} |t_A, t_C, \dots t_N\rangle_{c\setminus B} M_{t_A, \dots t_N}, \tag{3.23}$$

where due to the dependence of $M_{t_A,\dots t_N}$ on U_A, $\mathscr{U}_B(t_B,0)|0,0,\dots\rangle$ is a sum of functions linear or constant in U_A. Analogous equations and arguments can be made for all agents and, hence, all $\mathscr{U}_X$ must be affine linear functions of the other agents' operations.

Further considerations of our framework, in particular in relation to the original Page-Wootters formalism as presented in Section 1.4, can be found in Appendix C.1.

3.3 Causal and non-causal Page-Wootters circuits

We now present concrete examples of causal and non-causal processes in our framework. First, we give the details of the scenario with two clocks with different ticking speeds, briefly discussed in the previous section. Then, we show that non-causal processes, where the causal order is coherently controlled, can always be implemented in our framework and, finally, we argue that the time-reversed Lugano process, introduced in Section 1.4, and in particular its causal reference frame decomposition in Allard Guérin and Brukner [2018] cannot be implemented in our framework.

3.3.1 Two agents with clocks ticking at different rates

As a first example we consider a more sophisticated version of the example discussed in Section 3.2, where two agents A and B with clocks which tick at different rates perform operations U_A and U_B on subsystems S_A and S_B of an initially shared quantum system. Between the application of the agents' operations there is some free evolution V of the system, see Figure 3.2. In the beginning and at the end, the two clocks tick at the same rate, as required by our axioms introduced in the previous section. This scenario is reminiscent of the famous twin paradox and captured by the following history state $|\Psi\rangle\rangle \in \mathscr{H}_{c_A} \otimes \mathscr{H}_{c_B} \otimes \mathscr{H}_{S_A} \otimes \mathscr{H}_{S_B}$.

$$
\begin{aligned}
|\Psi\rangle\rangle =& |0_A,0_B\rangle_c \otimes |\phi\rangle + |1_A,1_B\rangle_c \otimes |\phi\rangle + |2_A,2_B\rangle_c \otimes (U_A \otimes \mathbb{1})|\phi\rangle \\
&+ |2_A,3_B\rangle_c \otimes (U_A \otimes \mathbb{1})|\phi\rangle + |3_A,4_B\rangle_c \otimes V(U_A \otimes \mathbb{1})|\phi\rangle \\
&+ |3_A,5_B\rangle_c \otimes V(U_A \otimes \mathbb{1})|\phi\rangle + |4_A,6_B\rangle_c \otimes (\mathbb{1} \otimes U_B)V(U_A \otimes \mathbb{1})|\phi\rangle \\
&+ |4_A,7_B\rangle_c \otimes \mathscr{G}(U_A,U_B)|\phi\rangle + |5_A,8_B\rangle_c \otimes \mathscr{G}(U_A,U_B)|\phi\rangle \\
&+ |6_A,9_B\rangle_c \otimes \mathscr{G}(U_A,U_B)|\phi\rangle
\end{aligned}
\tag{3.24}
$$

where $\mathcal{G}(U_A, U_B) = (\mathbb{1} \otimes U_B)V(U_A \otimes \mathbb{1})$ is a causally ordered process with A acting in the casual past of B. The perspectival states for the two agents including the non-trivial normalization operators are

$$
\begin{aligned}
|\psi_A(0)\rangle &= |0_B\rangle_{c_B} \otimes |\phi\rangle, & |\psi_B(0)\rangle &= |0_A\rangle_{c_A} \otimes |\phi\rangle,\\
|\psi_A(1)\rangle &= |1_B\rangle_{c_B} \otimes |\phi\rangle, & |\psi_B(1)\rangle &= |1_A\rangle_{c_A} \otimes |\phi\rangle,\\
|\psi_A(2)\rangle &= \frac{1}{\sqrt{2}}(|2_B\rangle + |3_B\rangle)_{c_B} \otimes (U_A \otimes \mathbb{1})|\phi\rangle, & |\psi_B(2)\rangle &= |2_A\rangle_{c_A} \otimes (U_A \otimes \mathbb{1})|\phi\rangle\\
&\quad \text{with } N_2^{(A)} = \frac{1}{\sqrt{2}}\mathbb{1}_S, & &= |\psi_B(3)\rangle,\\
|\psi_A(3)\rangle &= \frac{1}{\sqrt{2}}(|4_B\rangle + |5_B\rangle)_{c_B} \otimes V(U_A \otimes \mathbb{1})|\phi\rangle & |\psi_B(4)\rangle &= |3_A\rangle_{c_A} \otimes V(U_A \otimes \mathbb{1})|\phi\rangle\\
&\quad \text{with } N_3^{(A)} = \frac{1}{\sqrt{2}}\mathbb{1}_S, & &= |\psi_B(5)\rangle,\\
|\psi_A(4)\rangle &= \frac{1}{\sqrt{2}}(|6_B\rangle + |7_B\rangle)_{c_B} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle & |\psi_B(6)\rangle &= |4_A\rangle_{c_A} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle\\
&\quad \text{with } N_4^{(A)} = \frac{1}{\sqrt{2}}\mathbb{1}_S, & &= |\psi_B(7)\rangle,\\
|\psi_A(5)\rangle &= |8_B\rangle_{c_B} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle, & |\psi_B(8)\rangle &= |5_A\rangle_{c_A} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle,\\
|\psi_A(6)\rangle &= |9_B\rangle_{c_B} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle, & |\psi_B(9)\rangle &= |6_A\rangle_{c_A} \otimes \mathcal{G}(U_A, U_B)|\phi\rangle.
\end{aligned}
\tag{3.25}
$$

The non-trivial normalization operators appear for those times where the clock of agent A ticks slower than the clock of agent B. Note that the final time read by A's clock is less than that of B's clock, i.e. $T_A < T_B$, which corresponds to A being the twin that leaves earth, travels at relativistic speed and returns to find his or her twin older than they are themselves. The perspectival states in Equations (3.25) can be related to one another by the following unitaries

$$
\begin{aligned}
& & \mathcal{U}_B(1,0) &= T_{c_A} \otimes \mathbb{1}_S,\\
\mathcal{U}_A(1,0) &= T_{c_B} \otimes \mathbb{1}_{\mathbb{S}}, & \mathcal{U}_B(2,1) &= T_{c_A} \otimes (U_A \otimes \mathbb{1})_S,\\
\mathcal{U}_A(2,1) &= (T_2')_{c_B} \otimes (U_A \otimes \mathbb{1})_S, & \mathcal{U}_B(3,2) &= \mathbb{1},\\
\mathcal{U}_A(3,2) &= (T^2)_{c_B} \otimes V_S, & \mathcal{U}_B(4,3) &= T_{c_A} \otimes V_S,\\
\mathcal{U}_A(4,3) &= (T^2)_{c_B} \otimes (\mathbb{1} \otimes U_B)_S, & \mathcal{U}_B(5,4) &= \mathbb{1},\\
\mathcal{U}_A(5,4) &= (T_6')_{c_B} \otimes \mathbb{1}_S, & \mathcal{U}_B(6,5) &= T_{c_A} \otimes (\mathbb{1} \otimes U_B)_S,\\
\mathcal{U}_A(6,5) &= T_{c_B} \otimes \mathbb{1}_S, & \mathcal{U}_B(7,6) &= \mathbb{1}\\
& & \mathcal{U}_B(8,7) &= T_{c_A} \otimes \mathbb{1}_S = \mathcal{U}_B(9,8),
\end{aligned}
\tag{3.26}
$$

where T is the unitary that makes the clock of the other agent tick, i.e. $T : |t\rangle \mapsto |t+1\rangle$ and T_i' is any unitary that acts as $|i-1\rangle \mapsto 1/\sqrt{2}(|i\rangle + |i+1\rangle)$, $1/\sqrt{2}(|i\rangle + |i+1\rangle) \mapsto |i+2\rangle$. As one can see in Equations (3.26), from A's perspective B's clock seems to tick at double the rate in the middle of the process, while from the point of view of B, A's clock seems partially frozen in time. The times of action of the two agents are $t_A^\star = 2$ and $t_B^\star = 6$ respectively.

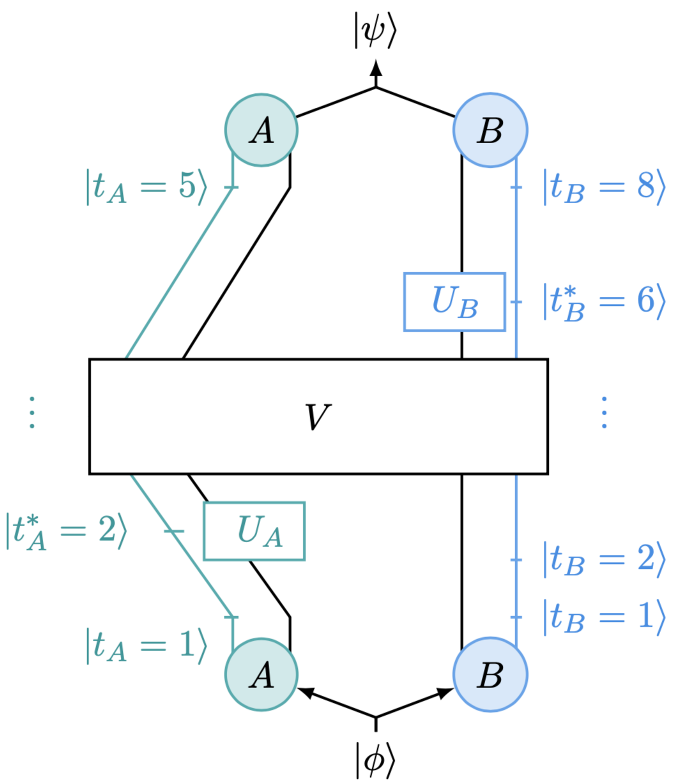


Figure 3.2. A "twin-paradox"-like process: Agents A and B each receive a part of the input system $|\phi\rangle$ and experience one synchronized time step. Then A's clock starts ticking slower than B's and A applies unitary U_A to her part of the system. This is followed by some free unitary evolution V of the system, which is independent of the two agents. Then B applies unitary U_B to his or her subsystem before, at the end of the protocol, the two clocks tick in synchronization once more.

3.3.2 Coherent control of causal order

An important class of non-causal processes are those where the casual order is coherently controlled. More precisely, for each value of the control system $|k\rangle \in \mathcal{H}_{S_C}$ one implements a process with definite causal order, i.e. a quantum comb, $\tilde{\mathcal{G}}_k$ and said definite causal order is different for at least two different values k. Consider M such pure combs $\tilde{\mathcal{G}}_k$, $1 \leq k \leq M$, and an M-dimensional control system $\in \mathcal{H}_{S_C}$.

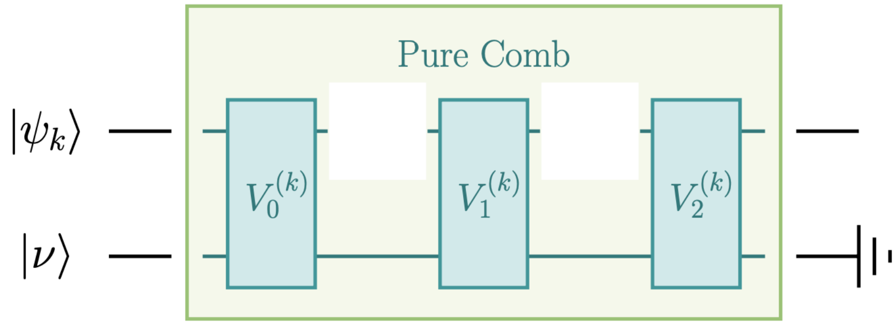


Figure 3.3. A bipartite quantum comb: A quantum comb is a higher-order quantum map that takes (purified) quantum operations as an input, signified by the two white slots. It can be modeled by a sequence of unitary operations with memory in case of a pure comb or with an additional environment, which is traced out at the end. We regard the dilated environment input $|\nu\rangle$ as an extension of the input system $|\psi_k\rangle$. The process we implement in our framework corresponds to everything that happens after the input is received and before the outcome is discarded, which constitutes a sequence of comb unitaries with memories and the agents operations and, hence, a pure quantum comb. (Picture taken from Baumann et al. [2021])

As shown in Yokojima et al. [2021] quantum combs can always be represented by a sequence of channels with memory, and a general (mixed) quantum comb $\tilde{\mathcal{G}}_k$ is given by unitaries $V_0^{(k)}, \ldots, V_N^{(k)}$ together with an environment input state $|\nu^{(k)}\rangle$ and an environment output, which is traced out at the end. In order to put quantum combs in controlled superposition the dimensions of input and output of both the agents' laboratories and the whole process in general should all be independent of the control value k. We assume that the input and output space of an agent have the same dimension and that the memories of the combs are chosen such that their dimensions are independent of the comb index

k. Note that, if this assumption is not satisfied, these dimensions can extended by the use of ancillas such that afterwards the combs dimensions do satisfy these requirements.

We further assume that the environment inputs $|\nu^{(k)}\rangle_{E_k}$ to the different combs are the same for all combs $\tilde{\mathscr{G}}_k$, i.e. $|\nu^{(k)}\rangle_{E_k} = |\nu\rangle_E \ \forall k$, and incorporate it into the overall input to the process, i.e.$|\tilde{\psi}\rangle_{\tilde{S}} = |\psi\rangle_S \otimes |\nu\rangle_E$. Moreover, since the trace over the environment is taken at the very end, it is enough to consider pure quantum combs, for which there is no environment, see Figure 3.3 for the bipartite example. We denote the agents by $A_1 \otimes \cdots \otimes A_N$ and their ancillas $A'_1 \otimes \cdots \otimes A'_N$. The definite causal order in comb $\tilde{\mathscr{G}}_k$ corresponds to a certain permutation π_k of the agents' operations. More precisely, the j-th agent in comb k is given by $A_{\pi_k(j)}$ and their unitary by $U_{\pi_k(j)}$. Note that while the agents' unitaries U_j act on the agents' ancillas denoted by A'_j they do not affect the memories of the comb, which we denote by E_j. Conversely the comb unitaries $V_j^{(k)}$ do not affect ancillas $A'_1 \otimes \cdots \otimes A'_N$ while acting on the comb memories, see Figure 3.4. Hence, we can write the pure comb $\tilde{\mathscr{G}}_k$ as the following sequence of unitaries

$$\begin{aligned} &\tilde{\mathscr{G}}_k(U_1, U_2, \dots U_N) = \qquad (3.27)\\ &(V_N^{(k)} \otimes \mathbb{1}_{S'}) U_{\pi_k(N)} \otimes \mathbb{1}_{E_{\pi_k(N)}} (V_{N-1}^{(k)} \otimes \mathbb{1}_{S'}) \dots (V_1^{(k)} \otimes \mathbb{1}_{S'}) U_{\pi_k(1)} \otimes \mathbb{1}_{E_{\pi_k(1)}} (V_0^{(k)} \otimes \mathbb{1}_{S'}), \end{aligned}$$

and the process $\mathscr{G}$ where the application of M such combs is coherently controlled as

$$\begin{aligned} \mathscr{G}(U_1, \dots, U_N) &= \sum_{k=1}^{M} |k\rangle\langle k|_C \otimes \tilde{\mathscr{G}}_k(U_1, \dots, U_N) \qquad (3.28)\\ &= \sum_{k=1}^{M} |k\rangle\langle k|_C \otimes V_N^{(k)} U_{\pi_k(N)} V_{N-1}^{(k)} U_{\pi_k(N-1)} \dots V_1^{(k)} U_{\pi_k(1)} V_0^{(k)}, \qquad (3.29) \end{aligned}$$

leaving identity operations on the ancillary systems implicit for notational convenience. It is easy to see that $\mathscr{G}$ is a unitary, map multilinear in the agents' operations, and hence, a pure process matrix.

A history state describing the process given in Equation (3.29) requires an input state $|\psi\rangle_S \in \mathscr{H}_{Sc} \otimes \mathscr{H}_{Sp}$ from the global past featuring a control system ($\in \mathscr{H}_{Sc}$) and a target system ($\in \mathscr{H}_{Sp}$). We separate this history state into three parts

$$|\Psi\rangle\rangle = |\Psi_{\text{desync}}\rangle\rangle + |\Psi_{\text{combs}}\rangle\rangle + |\Psi_{\text{resync}}\rangle\rangle, \qquad (3.30)$$

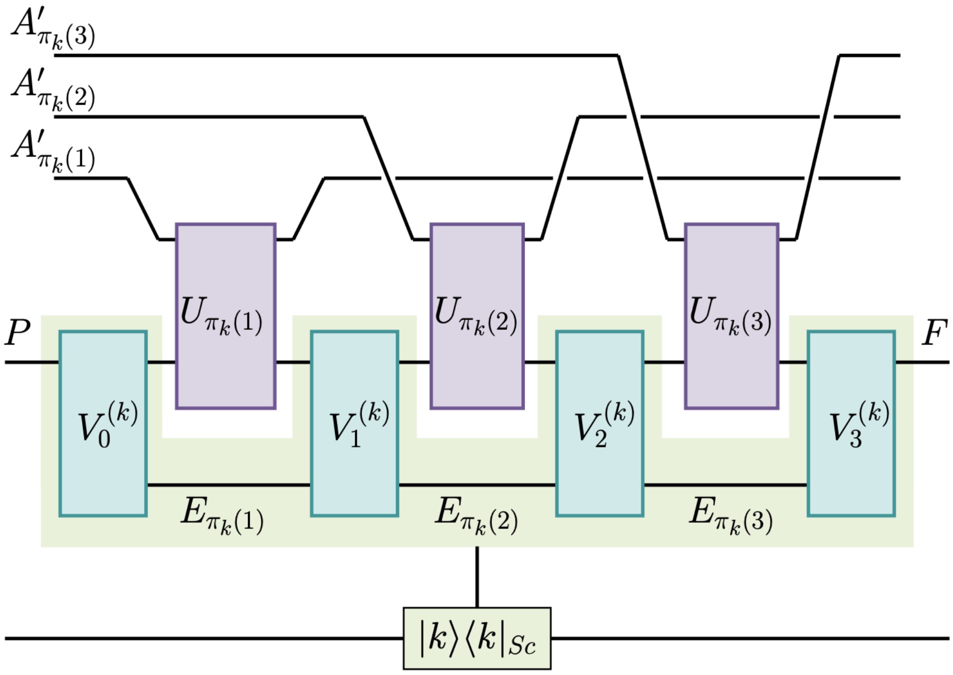


Figure 3.4. Coherently controlled causal order for three parties: The value k of the control system determines which pure quantum comb $\tilde{\mathscr{G}}_k$ gets implemented. The order of the agents A_j in comb $\tilde{\mathscr{G}}_k$ is described by permutation π_k. The comb is given by a sequence of unitaries $V_j^{(k)}$ with memories, where comb memory $E_{\pi_k(j)}$ is parallel to agent $A_{\pi_k(j)}$ and independent of k. The agents operations U_j act on the agents' ancilla A'_j but not on the ancillas of other agents or the comb memory.

namely one that describes the desynchronization of the initially synchronized clocks of the agents $|\Psi_{\text{desync}}\rangle\rangle$, one that implements the different quantum combs depending on the value of the control $|\Psi_{\text{combs}}\rangle\rangle$, and finally one where the agents' clocks get resynchronized $|\Psi_{\text{resync}}\rangle\rangle$.

During the desynchronization of the clocks we use the control degree of freedom to make sure that the agents are put into the right order, while the target system undergoes trivial evolution. Hence, we can write

$$\begin{aligned} |\Psi_{\text{desync}}\rangle\rangle = & |0,\dots,0\rangle_c \otimes |\psi\rangle_S + |1,\dots,1\rangle_c \otimes |\psi\rangle_S + |2,\dots,2\rangle_c \otimes |\psi\rangle_S \\ & + \sum_{k=1}^{M}\sum_{j=3}^{T_0} |t_1^{(k)}(j),\dots,t_N^{(k)}(j)\rangle_c \otimes (|k\rangle\langle k|_{Sc} \otimes \mathbb{1}_{Sp})|\psi\rangle_S, \end{aligned} \tag{3.31}$$

where we choose the $t_i^{(k)}(j)$ such that different k give rise to different orderings of the agents. We choose them such that consecutive agents are always two ticks apart. This turns out to be useful during the application of the combs where between the actions of two consecutive agents the comb unitaries $V_j^{(k)}$ get applied. We start from $|0,0,\ldots,0\rangle \otimes |\psi\rangle_S$ and first have all the clocks perform two synchronized steps to $|2,2,\ldots,2\rangle \otimes |\psi\rangle_S$ in compliance with axiom **S.2**. Then the clock of the fastest agent for each k, i.e. $\pi_k(1)$, continues to tick at the same rate as before

$$t(j)^{(k)}_{\pi_k(1)} = j, \tag{3.32}$$

while the other agents' clocks get frozen in time one after the other. We let all the other clocks tick on while agent m's clock gets frozen for times $2(m-2)\cdot N+2 \leq j \leq 2(m-2)\cdot N + 2(m-1)+2$, see Figure 3.5 for the example of four clocks.

$$\forall 2 \leq m \leq N: \tag{3.33}$$

$$t(j)^{(k)}_{\pi_k(m)} = \begin{cases} j & \text{if } j \leq 2(m-2)N+2 \\ 2(m-2)\cdot N+2 & \text{if } 2(m-2)N+2 \leq j \leq 2(m-2)N+2(m-1)+2 \\ j-2(m-1) & \text{if } j \geq 2(m-2)N+2(m-1)+3 \end{cases}$$

The largest value for j is $T_0 := 2(N-2)N + 2(N-1) + 4 + 2(N+1) = 2N^2+4$, which includes $2(N+1)$ additional well-synchronized ticks of all clocks. The latter ensures that the clock freezes are far away from the application of the combs for all k.

$\pi_k(1)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\pi_k(2)$	0	1	2	2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\pi_k(3)$	0	1	2	3	4	5	6	7	8	9	10	10	10	10	10	11	12	13	14	15	16	17	18	19	20	21	22
$\pi_k(4)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	18	18	18	18	18	18	19	20

Figure 3.5. The systematic desynchronization of four clocks: The fastest clock keeps ticking at the same rate while the other clocks are frozen one after the other for longer and longer time intervals starting with the second fastest clock. After the shown times, we include 10 additional well-synchronized ticks of all the clocks together.

In order to obtain the agents' perspectives we define

$$\alpha_k(t) = \| \langle t|_{c_1} \sum_{j=2}^{T_0} |t(j)^{(k)}_1, t(j)^{(k)}_2, \ldots, t(j)^{(k)}_N\rangle_c \|, \tag{3.34}$$

and can then write the normalization operator, for example, for agent A_1 as

$$N_t^{(A_1)} = \sum_k \frac{1}{\alpha_k(t)} |k\rangle\langle k|_{S_c}. \tag{3.35}$$

Note that, for any $1 < t < T_{A_1}$, where T_{A_1} is the largest time A_1 sees during the desynchronization phase, $\alpha_k(t) \neq 0$ because no time is skipped during desynchronization and, hence, $N_t^{(A_1)}$ is well-defined. This normalization operator gives the following perspectival states for A_1:

$$|\psi^{A_1}(t)\rangle = \sum_k |\xi_k(t)\rangle_c |k\rangle_{S_c} |\psi_k\rangle_{S_p}, \tag{3.36}$$

where $|\xi_k^1(t)\rangle_c$ is proportional to $\langle t|_{c_1} \sum_{j=2}^{T_0} |t(j)_1^{(k)}, t(j)_2^{(k)}, \ldots, t(j)_N^{(k)}\rangle_c$ and normalized. It is clear that there exists a unitary relating $|\psi^{A_1}(t)\rangle$ and $|\psi^{A_1}(t+1)\rangle$ which is of the form

$$\mathcal{U}_{A_1}(t, t+1) = \sum_k \mathcal{V}_c^{1,k} \otimes |k\rangle\langle k|_{S_c} \otimes \mathbb{1}_{S_p}, \tag{3.37}$$

where we can choose $\mathcal{V}_c^{1,k}$ to be any unitary mapping $|\xi_k^1(t)\rangle_c \mapsto |\xi_k^1(t+1)\rangle$. Analogous expressions exist for all other agents $A_2 \ldots A_N$.

Next, for each control value k, the unitaries of the comb $\tilde{\mathcal{G}}_k$ are applied one after the other while all the agents' clocks tick in synchronization. The corresponding parts of the history state are denoted $|\Psi_{combs}\rangle\rangle$ and the starting point is

$$\sum_{k=1}^{M} |T_0, T_0 - 2, \ldots, T_0 - 2(N-1)\rangle_{c_{\pi_k(1)}, \ldots, c_{\pi_k(N)}} \otimes (|k\rangle\langle k| \otimes \mathbb{1})|\psi\rangle_S, \tag{3.38}$$

with

$$|t_1, t_2, \ldots, t_N\rangle_{c_{\pi_k(1)}, \ldots, c_{\pi_k(N)}} := \mathcal{U}_{\pi_k} |t_1, t_2, \ldots, t_N\rangle_c, \tag{3.39}$$

where $\mathcal{U}_{\pi_k}$ is the unitary implementing the permutation on the Hilbert spaces of the local clocks. All agents see the following sequence of unitaries:

$$\begin{aligned}
&(T_0 + 1, T_0): && T^{\otimes(N-1)} \otimes V_0^{(k)} \\
&(T_0 + 2, T_0 + 1): && T^{\otimes(N-1)} \otimes U_{\pi_k(1)} \\
&(T_0 + 3, T_0 + 2): && T^{\otimes(N-1)} \otimes V_1^{(k)} \\
& && \vdots \\
&(T_0 + 2N + 1, T_0 + 2N): && T^{\otimes(N-1)} \otimes U_{\pi_k(N)} \\
&(T_0 + 2N + 2, T_0 + 2N + 1): && T^{\otimes(N-1)} \otimes V_N^{(k)},
\end{aligned} \tag{3.40}$$

where T is the unitary that makes a clock tick as introduced in Section 3.3.1. As we see, the time of action is $t^* = T_0 + 2$ for all agents. In order to explicitly write down the perspectival unitaries we define

$$\begin{aligned} &W_{A_j}^{(k)}(2m_j^{(k)} + 2x + 1, 2m_j^{(k)} + 2x) = V_x^{(k)}, \\ &W_{A_j}^{(k)}(2m_j^{(k)} + 2y, 2m_j^{(k)} + 2y - 1) = U_{\pi_k(y)}, \\ &W_{A_j}^{(k)}(p+1, p) = \mathbb{1} \text{ for other values of } p, \end{aligned} \tag{3.41}$$

where $p, m_j^{(k)} x, y$ are integers with $p \geq 0$, $\pi_k(N - m_j^{(k)}) = j$, $N \geq x \geq 0$ and $N \geq y \geq 1$. If we further denote the earliest time appearing in $|\Psi_{combs}\rangle\rangle$ as $\tau := T_0 - 2(N-1)$, the unitary time evolution that agent j sees is given by

$$\mathscr{U}_{A_j}(\tau + p + 1, \tau + p) = \sum_{k=1}^{M} |k\rangle\langle k|_{Sc} \otimes T^{\otimes(N-1)} \otimes W_{A_j}^{(k)}(p+1, p), \tag{3.42}$$

and the corresponding part of the history state can be written as follows

$$\begin{aligned} &|\Psi_{combs}\rangle\rangle = \qquad (3.43) \\ &\sum_{k=1}^{M} |T_0 + 1, T_0 - 1, \ldots, T_0 - 2(N-1) + 1\rangle_{c_{\pi_k(1)}, \ldots c_{\pi_k(N)}} \otimes \Big[|k\rangle\langle k| \otimes V_0^{(k)}\Big]|\psi\rangle_S \\ &+ \sum_{k=1}^{M}\sum_{y=1}^{N} |T_0 + 2y, T_0 - 2 + 2y, \ldots, T_0 - 2(N-1) + 2y\rangle_{c_{\pi_k(1)}, \ldots c_{\pi_k(N)}} \\ &\qquad \otimes \Big[|k\rangle\langle k| \otimes \Big(U_{\pi_k(y)} V_{y-1}^{(k)} \ldots U_{\pi_k(1)} V_0^{(k)}\Big)\Big]|\psi\rangle_S \\ &+ \sum_{k=1}^{M}\sum_{x=1}^{N} |T_0 + 1 + 2x, T_0 - 1 + 2x, \ldots, T_0 - 2(N-1) + 1 + 2x\rangle_{c_{\pi_k(1)}, \ldots c_{\pi_k(N)}} \\ &\qquad \otimes \Big[|k\rangle\langle k| \otimes \Big(V_x^{(k)} U_{\pi_k(x)} \ldots U_{\pi_k(1)} V_0^{(k)}\Big)\Big]|\psi\rangle_S. \end{aligned}$$

This shows that all the combs get applied in accordance with their value k, each agent has a well-defined time of action, namely $T_0 + 2$, and the other agents' unitaries appear at most linearly in each agent's perspective.

Finally, we have to resynchronize the clocks and have them perform one well-synchronized tick at the very end. These terms are subsumed under $|\Psi_{\text{resync}}\rangle\rangle$. If we denote $T_1 := T_0 + 2N + 1$, the starting point for this last part is

$$\begin{aligned} &\sum_{k=1}^{M} |T_1, T_1 - 2, \ldots, T_1 - 2(N-1)\rangle_{c_{\pi_k(1)}, \ldots c_{\pi_k(N)}} \\ &\qquad \otimes \Big[|k\rangle\langle k| \otimes \Big(V_N^{(k)} U_{\pi_k(N)} \ldots U_{A_{\pi_k(1)}} V_0^{(k)}\Big)\Big]|\psi\rangle_S. \end{aligned}$$

We repeat the desynchronization procedure in reversed order, meaning $t^{(k)}_{\pi_k(m)} \mapsto t^{(k)}_{\pi_k(N+1-m)}$, which gives

$$
\begin{aligned}
&\left|\Psi_{resync}\right\rangle\!\Big\rangle = \\
&\sum_{k=1}^{M}\sum_{j=0}^{2(N+1)} |T_1+1+j, T_1-1+j,\ \ldots,\ T_1+1-2(N-1)+j\rangle_{c_{\pi_k(1)},\ \ldots c_{\pi_k(N)}} \\
&\qquad\qquad \otimes\Big[|k\rangle\langle k| \otimes \Big(V_N^{(k)} U_{\pi_k(N)} \ldots U_{\pi_k(1)} V_0^{(k)}\Big)\Big]|\psi\rangle_S + \\
&\sum_{k=1}^{M}\sum_{j=0}^{T_0} |T_1+2(N+1)+2+t(j)^{(k)}_{\pi_k(N)},\ \ldots,\ T_1+t(j)^{(k)}_{\pi_k(1)}\rangle_{c_{\pi_k(1)},\ \ldots c_{\pi_k(N)}} \\
&\qquad\qquad \otimes\Big[|k\rangle\langle k| \otimes \Big(V_N^{(k)} U_{\pi_k(N)} \ldots U_{\pi_k(1)} V_0^{(k)}\Big)\Big]|\psi\rangle_S.
\end{aligned}
\tag{3.44}
$$

Just as during the desynchronization process, we first insert $2(N+1)$ well-synchronized ticks to make sure that for all k the clock-freezes do not overlap with the application of the combs, and then freeze each clock individually until they all show the same time again. Like in $|\Psi_{\text{desync}}\rangle\rangle$ nothing happens on the system and we can write the perspectival states and unitaries as

$$
|\psi^{A_j}(t)\rangle = \sum_k |\xi^j_k(t)\rangle_c \tilde{\mathscr{G}}_k(U_1, U_2, \ldots U_N)|k\rangle_{S_c}|\psi_k\rangle_{S_p} \tag{3.45}
$$

and

$$
\mathscr{U}_{A_j}(t, t+1) = \sum_k \mathscr{V}_c^{j,k} \otimes |k\rangle\langle k|_{S_c} \otimes \mathbb{1}_{S_p}, \tag{3.46}
$$

where $|\xi^j_k(t)\rangle_c$ and $\mathscr{V}_c^{j,k}$ are defined analogous to those for Equations (3.36) and (3.37).

With this protocol we can implement any process describing a coherent superposition of quantum combs, and hence, any coherent control of causal order. Note that the above protocol was chosen to be illustrative and as simple as possible for an arbitrary amount of agents and should rather be regarded as a proof-of-principle than an actual prescription for implementing processes describing coherent control of causal order. It will in general *not* be the simplest and most efficient way to implement any given such process. In fact, Appendix C.2 contains the explicit implementation of the quantum switch in our formalism and does not strictly adhere to the protocol described here.

3.3.3 About the Lugano process

The Lugano process introduced in Section 1.3.2 is an example of a pure, non-causal process that is *not* an example of coherent control of causal order and is known to violate causal inequalities. In Allard Guérin and Brukner [2018] the authors considered the time-reversed version of the Lugano process, which is defined by

$$\mathcal{G}(U_A, U_B, U_C)|jjj\rangle = U_A \otimes U_B \otimes U_C|jjj\rangle \tag{3.47}$$
$$\mathcal{G}(U_A, U_B, U_C)|j01\rangle = XU_A \otimes U_B \otimes U_C|j01\rangle \tag{3.48}$$
$$\mathcal{G}(U_A, U_B, U_C)|1j0\rangle = U_A \otimes XU_B \otimes U_C|1j0\rangle \tag{3.49}$$
$$\mathcal{G}(U_A, U_B, U_C)|01j\rangle = U_A \otimes U_B \otimes XU_C|01j\rangle \tag{3.50}$$

where $j \in \{0,1\}$. Defining projectors $P_A = \sum_j |j01\rangle\langle j01|$, $P_B = \sum_j |1j0\rangle\langle 1j0|$, $P_C = \sum_j |01j\rangle\langle 01j|$ and $P_\perp = \sum_j |jjj\rangle\langle jjj|$ the time-reversed Lugano process is

$$\begin{aligned}\mathcal{G}(U_A, U_B, U_C)|\phi\rangle =& (U_A \otimes U_B \otimes U_C P_\perp + XU_A \otimes U_B \otimes U_C P_A \\ &+ U_A \otimes XU_B \otimes U_C P_B + U_A \otimes U_B \otimes XU_C P_C)|\phi\rangle.\end{aligned} \tag{3.51}$$

Due to the lack of a control system it is not possible to adapt the history state procedure described in the previous section to the reversed Lugano process. Instead, one can similarly try to use the projectors P_A,P_B,P_C and $P_\perp$ to define a controlled operation that desynchronizes the clocks. Further the clocks can act as control systems to define another controlled operation that applies the unitary operations in Equations (3.47)- (3.50). However, the re-synchronization cannot be done independently of the unitaries U_A, U_B and U_C once we obtained a term in the history state of the form

$$\begin{aligned}|\Psi\rangle\rangle = \cdots + |\gamma_\perp\rangle_c \otimes (U_A \otimes U_B \otimes U_C P_\perp)|\phi\rangle_S + |\gamma_A\rangle_c \otimes (XU_A \otimes U_B \otimes U_C P_A)|\phi\rangle_S \\ + |\gamma_B\rangle_c \otimes (U_A \otimes XU_B \otimes U_C P_B)|\phi\rangle_S + |\gamma_C\rangle_c \otimes (U_A \otimes U_B \otimes XU_C P_C)|\phi\rangle_S + \dots\end{aligned}$$

with some clock states $|\gamma_\perp\rangle_c$, $|\gamma_A\rangle_c$, $|\gamma_B\rangle_c$ and $|\gamma_C\rangle_c$, which represent the different temporal orderings of the agents. The states $U_A \otimes U_B \otimes U_C P_\perp|\phi\rangle_S$, $XU_A \otimes U_B \otimes U_C P_A|\phi\rangle_S$, $U_A \otimes XU_B \otimes U_C P_B|\phi\rangle_S$ and $U_A \otimes U_B \otimes XU_C P_C|\phi\rangle_S$ all depend on U_A, U_B, U_C in different and non-trivial ways. Hence, any overall map using them to resynchronize the clocks will non-trivially depend on U_A, U_B and U_C as well, which in turn leads to a non-trivial dependence of $\mathcal{U}_X(t_X, t_X - 1)$ on U_X for all $X \in \{A, B, C\}$. The latter, however, is a violation of our axiom **U.3**, which excludes such a history state from our framework.

The causal reference frame decomposition of the reverse Lugano process, which is shown in Figure 1.8 for agent A, uses perspectival circuits with gates that are not affine-linear in the unitaries of the other agents, namely $U_B^\dagger X U_B$ and $U_C^\dagger X U_C$. The same is true for the casual reference frames of agents B and C with non-linear gates involving U_A, U_C and U_A, U_B respectively. This however, means that hypothetical, corresponding perspectival states are forbidden in our framework due to the requirement of affine-linearity discussed in Section 3.2.

Note, however, that the two impossible implementations discussed above, namely the causal-reference-frame decomposition of Allard Guérin and Brukner [2018] and the desynchronization-resynchronization protocol similar to that for processes with coherent control of causal order, are not necessarily the only strategies to describe the time-reverse Lugano process within our non-causal Page-Wootters framework.

Chapter 4

Conclusion

We obtained various results on the two topics of encapsulated observers and indefinite causal order, which conceptually share the idea of agents in superposition.

The ability of agents to make predictions about each other's measurement outcomes allows for situations where they verifiably disagree when referring to the same experiment. Due to the non-persistence of the friend's perception in a Wigner's-friend-type setup and the fact that communication between the observer and superobserver has to be very restricted severely limit the situations where such verifiable disagreement can arise. An interesting line for future research would therefore be to identify and classify those Wigner's-friend-type setups that can give rise to actual contradictions and study them in detail when explicitly incorporating classical record systems and communication channels into the setups. In more elaborate Wigner's-friend-type scenarios comprising multiple memory registers of the friend there probably exist more complex operations on the collection of these registers that preserve the coherences relevant in Wigner's measurement. We would further like to model Wigner's-friend-type experiments as two different perspectives on the same process using the formalism of quantum reference frames de la Hamette and Galley [2020]; Giacomini et al. [2019]; Vanrietvelde et al. [2020]. Collapse dynamics should then be a consequence of switching to the friend's reference frame, while unitary dynamics (for the friend's measurements) are a consequence of switching to Wigner's frame of reference. Unambiguous probabilities would be determined by the perspective neutral description and induce the transformations for the respective observables in question.

Limits to the physicality of processes with indefinite causal order are still sought for, especially in the context of possible violations of causal inequalities.

We showed how to consider in general non-causal processes within the Page-Wootters formalism. While we considered discrete clocks, our framework can be adapted to continuous clocks, extending the approach of Castro-Ruiz et al. [2020] to a systematic operational protocol. As we focussed on the history states in our approach, the relation between the physical projector and the perspectival unitaries $\mathcal{U}_X(t', t)$ is still an open question. Resolving it might reveal further constraints on the history states, possibly restricting the set of process matrices that can be considered physical. If one could show that some processes do not fit into our framework, this would be important evidence that such processes should not be considered physical.

Appendix A

A.1 General formulation of the relative state formalism

The relative state formalism can easily describe general (mixed) states ρ and generalized measurements corresponding to positive operator valued measures (POVMs). The latter can be represented by Krauss operators $\{K_a^\dagger K_a\}$, where K_a is associated with outcome a. The probability of outcome a is given by

$$P_\rho^{clps}(a) = \mathrm{Tr}(K_a \rho K_a^\dagger), \tag{A.1}$$

and associated with the state-update rule for the general case is

$$\rho \xrightarrow[\text{result: } a]{} \frac{1}{P_\rho^{clps}(a)} K_a \rho K_a^\dagger. \tag{A.2}$$

The action of a Kraus operator K_a on ρ can always be written as

$$\rho \to \langle a| U_{s,x} \left(\rho \otimes |a_0\rangle\langle a_0|\right) U_{s,x}^\dagger |a\rangle, \tag{A.3}$$

where $\{|a\rangle\}$ form an orthonormal basis in an ancillary space $\mathscr{H}_X$, $|a_0\rangle\langle a_0|$ is the initial state of the auxiliary system and $U_{x,s}$ is a unitary operator on $\mathscr{H}_S \otimes \mathscr{H}_X$. Using representation (A.3), the measurement in the relative-state formalism is given by a unitary correlating the ancillary system with the observer's memory:

$$\begin{aligned} U_O : \quad & \mathscr{H}_X \to \mathscr{H}_X \otimes \mathscr{H}_O \\ & |a\rangle_X \otimes |r\rangle_O \mapsto |a\rangle_X \otimes |A_a\rangle_O \quad \forall a\,, \end{aligned} \tag{A.4}$$

where $|r\rangle_O$ is the observer's pre-measurement state. A general state ρ then evolves as follows:

$$\rho \otimes |r\rangle\langle r|_O \;\to\; \rho_{tot} = \sum_{cc'aa'} \rho_{cc'aa'} |c\rangle\langle c'|_S \otimes U_O |a\rangle\langle a'|_X \otimes |r\rangle\langle r|_O U_O^\dagger, \tag{A.5}$$
$$\text{with } \rho_{cc'aa'} = \langle c| \otimes \langle a| U_{s,x} \left(\rho \otimes |a_0\rangle\langle a_0|\right) U_{s,x}^\dagger |a'\rangle \otimes |c'\rangle.$$

Straight-forward calculation yields

$$P^{rels}_{\rho}(a) = \mathrm{Tr}(\mathbb{1}_{S,X} \otimes |A_a\rangle\langle A_a|_O \cdot \rho_{tot}) = \sum_c \rho_{ccaa} = \mathrm{Tr}(K_a \rho K_a^\dagger) = P^{clps}_{\rho}(a).$$

Next consider two observers performing consecutive measurements on one system, namely $M_{O_1} : \{K_a^\dagger K_a\}$ and $M_{O_2} : \{K_b^\dagger K_b\}$. We obtain conditional probabilites

$$\begin{aligned} P^{rels}_{\rho}(b|a) &= \frac{1}{P_\rho(a)} \mathrm{Tr}(\mathbb{1} \otimes |A_a\rangle\langle A_a|_{O_1} \otimes |B_b\rangle\langle B_b|_{O_2} \cdot \rho'_{tot}) = \frac{1}{P_\rho(a)} \sum_c \rho_{ccaabb} \\ &= \frac{1}{P_\rho(a)} \mathrm{Tr}(K_b K_a \rho K_a^\dagger K_b^\dagger) = \mathrm{Tr}\left(K_b \left(\frac{K_a \rho K_a^\dagger}{P_\rho(a)} \right) K_b^\dagger \right) = P^{clps}_{\rho}(b|a), \end{aligned}$$

with

$$\rho'_{tot} = \sum_{\substack{cc'aa' \\ bb'}} \rho_{cc'aa'bb'} |c\rangle\langle c'| \otimes U_{O_1} |a\rangle\langle a'| \otimes |r\rangle\langle r|_{O_1} U^\dagger_{O_1} \otimes U_{O_2} |b\rangle\langle b'| \otimes |r\rangle\langle r|_{O_2} U^\dagger_{O_2},$$

where $\rho_{cc'aa'bb'} = \langle cab| U_{s,y} (U_{s,x} \rho \otimes |a_0\rangle\langle a_0|_X U^\dagger_{s,x}) \otimes |b_0\rangle\langle b_0|_Y U^\dagger_{s,y} |c'a'b'\rangle$

and $|cab\rangle = |c\rangle_S \otimes |a\rangle_X \otimes |b\rangle_Y$ with $\mathscr{H}_X$ and $\mathscr{H}_Y$ being the purifying auxilary systems for M_{O_1} and M_{O_2} respectively. Hence, on the same level of observation, the relative-state formalism gives the same probabilistic predictions as the standard Born and state-update rules also for mixed states and generalized measurements.

A.2 The extended Wigner's-friend experiment in terms Bohmian mechanics

A description of the extended Wigner's-friend experiment, which is discussed in Section 1.2.2, in terms of generalized Bohmian mechanics was presented in Sudbery [2017]. There the author obtains the following joint probabilities for A's and W's outcomes (a, w) calculated for different agents:

$P(a,w)$	$P(o,O)$	$P(o,F)$	$P(f,O)$	$P(f,F)$
F_1	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$
F_2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{20}$	$\frac{9}{20}$
W	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{4}$

(A.6)

The last line representing W equals the joint probabilities $P^{rels}(a,w)$ calculated according to the relative-state formalism.

$$P^{rels}(a,w) = \mathrm{Tr}(\mathbb{1} \otimes |A_a\rangle\langle A_a| \otimes |W_w\rangle\langle W_w| \cdot |\Phi_{tot}\rangle\langle\Phi_{tot}|) \tag{A.7}$$

The probability distribution of A can be obtained by renormalizing the conditional probabilities $P^{rels}(w|a)$ calculated according to Equation (2.9). The friends' distributions arise when one takes the conditional probabilities $P^{rels}(a|z)$ for F_2, who measures result z, and $P^{rels}(w|c)$ for F_1, who measures result c, and renormalizes them to give distribution for both results.

In addition to that Bohmian mechanics is known to give the same probabilistic predictions as the standard quantum formalism in non-Wigner's-friend scenarios, see for example Dürr et al. [1996]. This together with the above comparison strongly suggests that Bohmian mechanics can be regarded as another interpretation of the relative-state formalism, which is usually associated with a many-worlds interpretation, according to Definition 2.

Appendix B

B.1 Consistent histories for Wigner's-friend setups

In the consistent histories framework sequences of physical properties are assigned to a closed quantum system. These sequences are represented by tensor products of orthogonal projectors (i.e. a *quantum history*)

$$Y^{\mathrm{i}} = \rho_0 \otimes P_1^{\mathrm{i}_1} \cdots \otimes P_f^{\mathrm{i}_f}, \tag{B.1}$$

where ρ_0 is the initial state and each $P_k^{\mathrm{i}_k}$ corresponds to some physical property i_k at a certain time k. A consistent family of histories is a complete set of histories $\{Y^i\}$ which satisfy the consistency condition

$$\mathrm{Tr}\left(K^\dagger(Y^{\mathrm{i}})\rho_0 K(Y^{\mathrm{i}'})\right) = 0 \quad \text{for } \mathrm{i} \neq \mathrm{i}' \tag{B.2}$$

where $\mathrm{i} = (\mathrm{i}_1 \dots \mathrm{i}_f)$ denotes the whole history and K is the so called chain operator defined by

$$K(Y^{\mathrm{i}}) := P_0^{\mathrm{i}_1} \cdot P_0^{\mathrm{i}_2} \cdots P_0^{\mathrm{i}_f}, \tag{B.3}$$

with $P_0^{\mathrm{i}_k} = U(t_0, t_k) P_k^{\mathrm{i}_k} U(t_k, t_0)$. Only within a consistent family the dynamics of quantum theory describe the respective properties over time.

For the simple Wigner's-friend setup in Figure 1.9, the properties of interest are the results observed by Wigner and his friend, i.e. $\mathrm{i} = (\mathrm{f}, \mathrm{w})$. The initial state is given by $\rho_0 = |\psi_S\rangle\langle\psi_S| \otimes |r\rangle\langle r|_F \otimes |r\rangle\langle r|_W$. Condition (2.41) from the main text, under which probability Definition 3 in Section 2.3.1 gives proper probabilities, implies that

$$\begin{aligned} &\mathrm{Tr}\left(K^\dagger(Y^{(f,w)})\rho_0 K(Y^{(f',w')})\right) \\ &= \delta_{ff'}\delta_{ww'}\,\mathrm{Tr}\left(\rho(t_2)\Pi^w U(t_2,t_1)\Pi^f U(t_1,t_2)\right), \end{aligned} \tag{B.4}$$

where $\rho(t_2) = \mathscr{U}(t_2, t_0)\rho_0\mathscr{U}(t_0, t_2)$. Hence, the consistency condition is satisfied if condition (2.41) is satisfied.

The solutions to the conditions on Definition 2a, however, in general do *not* satisfy Equation (B.2). Consider the concrete counterexample of initial state $|\psi_S\rangle = \sqrt{\frac{1}{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\text{yes}\rangle_{\text{SF}} = \alpha|\uparrow, \text{u}\rangle_{\text{SF}} + \text{i}\beta|\downarrow, \text{d}\rangle_{\text{SF}}$. In this case one obtains

$$\text{Tr}\left(K^\dagger(Y^i)\rho_0 K(Y^{i'})\right) = \pm\delta_{ww'}\frac{i}{2}\alpha\beta \quad \text{for } f \neq f'. \tag{B.5}$$

Note that the above example, still satisfies the so called weak consistency condition

$$\text{Re}\left[\text{Tr}\left(\text{K}^\dagger(\text{Y}^\text{i})\rho_0\text{K}(\text{Y}^{\text{i}'})\right)\right] = 0, \tag{B.6}$$

where Re[·] refers to the real part of the argument. In contrast to the consistency condition of (B.2), however, Equation (B.6) has been shown to be highly problematic concerning trivial combination of independent subsystems as well as dynamical stability, see Diosi [2004].

B.2 Standard quantum theory in the Page-Wootters formalism

Here we consider non-Wigner's-friend scenarios where two measurements are performed on the same quantum system in terms of our adapted Page-Wootters formalism presented in Section 1.4.1. The constraint Hamiltonian takes the form

$$\hat{H}' = \hat{p}_t + H_S + \delta(\hat{T} - t_M)\hat{K}_{SM} + \delta(\hat{T} - t_N)\hat{K}_{SN}, \tag{B.7}$$

where now M and N are apparatus or observer states, which encode the results of the respective measurements. This gives the physical state

$$\begin{aligned} |\Psi'\rangle\rangle = &\int_{-\infty}^{t_M} dt\, |t\rangle U_S(t, t_0)|\psi(t_0)\rangle_S |r\rangle_M |r\rangle_N \\ &+ \int_{t_M}^{t_N} dt\, |t\rangle \sum_m U_S(t, t_M)\Pi_m|\psi(t_M)\rangle_S |m\rangle_M |r\rangle_N \\ &+ \int_{t_N}^{\infty} dt\, |t\rangle \sum_{m,n} U_S(t, t_N)\Pi_n U_S(t_N, t_M)\cdot\Pi_m|\psi(t_M)\rangle_S |m\rangle_M |n\rangle_N, \end{aligned} \tag{B.8}$$

with both Π_m and Π_n acting on $\mathscr{H}_S$. Moreover, we have

$$\langle t|P^{\text{ph}}|t_0\rangle|\phi(t_0)\rangle = \mathscr{U}(t, t_0)|\phi(t_0)\rangle \tag{B.9}$$

for arbitrary $|\phi(t_0)\rangle \in \mathscr{H}_S \otimes \mathscr{H}_M \otimes \mathscr{H}_N$, where

$$\mathscr{U}(t,t_0) = \begin{cases} U_S(t,t_0) & t_0 < t \\ U_S(t,t_M)U_M U_S(t_M,t_0) & t_0 < t_M < t \\ U_S(t,t_N)U_N U_S(t_N,t_0) & t_0 < t_N < t \\ U_S(t,t_N)U_N U_S(t_N,t_M)U_M U_S(t_M,t_0) & t_0 < t_M < t_N < t \end{cases} \tag{B.10}$$

with $U_M = e^{-i\hat{K}_{SM}}$ and $U_N = e^{-i\hat{K}_{SN}}$ being the measurement unitaries that entangle the measured system with the respective apparatus or observers:

$$U_X|\psi\rangle_S|r\rangle_X = \sum_x \Pi_x|\psi\rangle_S|x\rangle_X,$$

and $(x,X) \in \{(m,M),(n,N)\}$.

According to probability Definition 1 in Section 2.3.1 the conditional probability of result n at time $t_2 \geq t_N$ given result m at time $t_1 \geq t_M$ is

$$\frac{\langle\langle\Psi' \mid t_1\rangle\Pi^m\langle t_1|P^{\mathrm{ph}}|t_2|\Pi\rangle^n\langle t_2|\hat{P}|t_1|\Pi\rangle^m\langle t_1|\Psi'\rangle\rangle}{\langle\langle\Psi' \mid t_1\rangle\Pi^m\langle t_1|\Psi'\rangle\rangle}, \tag{B.11}$$

with Π^m and Π^n acting on $\mathscr{H}_M$ and $\mathscr{H}_N$ respectively. From Equation (B.8) we see that $|\phi(t_1)\rangle := \langle t_1|\Psi'\rangle\rangle = \sum_{m'} U_S(t_1,t_M)\Pi_{m'}|\psi(t_M)\rangle_S|m'\rangle_M|r\rangle_N$, and the denominator in Equation (B.11) is

$$\begin{aligned}\langle\phi(t_1)|\Pi^m|\phi(t_1)\rangle &= \sum_{m',m''}\langle m''|\Pi^m|m'\rangle_M\langle\psi(t_M)|\Pi_{m''}\Pi_{m'}|\psi(t_M)\rangle_S \\ &= |\langle m|\psi_S(t_M)\rangle|^2. \end{aligned} \tag{B.12}$$

Moreover, since

$$\mathscr{U}(t_2,t_1)\Pi^m|\phi(t_1)\rangle = \sum_{n'} U_S(t_2,t_N)\Pi_{b'}U_S(t_N,t_M)\Pi_m|\psi(t_M)\rangle_S|m\rangle_M|n'\rangle_N,$$

the numerator in Equation (B.11) gives

$$\begin{aligned}&\langle\phi(t_1)|\Pi^m\mathscr{U}^\dagger(t_2,t_1)\Pi^n\mathscr{U}(t_2,t_1)\Pi^m|\phi(t_1)\rangle = \\ &\sum_{n',n''}\langle n''|\Pi^n|n'\rangle_N\langle\psi(t_M)|\Pi_m U_S(t_M,t_N)\Pi_{n''}U_S(t_N,t_N)\Pi_{n'}U_S(t_N,t_M)\Pi_m|\psi(t_M)\rangle_S \\ &\quad= \langle\psi(t_M)|\Pi_m U_S(t_M,t_N)\Pi_n U_S(t_N,t_M)\Pi_m|\psi(t_M)\rangle_S \\ &\quad= |\langle n|U_S(t_N,t_M)|m\rangle|^2|\langle m|\psi(t_M)\rangle|^2\end{aligned} \tag{B.13}$$

and, hence, we obtain

$$P_1\,(n \text{ when } t_2\,|\,m \text{ when } t_1) = |\langle n|U_S(t_N, t_M)|m\rangle|^2, \tag{B.14}$$

which are the standard quantum probabilities for two subsequent measurements on a quantum system S.

Definition 2a and Definition 2b in Section 2.3.1 are equal for non-Wigner's-friend scenarios since

$$\begin{aligned}\Big\langle\Big\langle\Psi'\,\Big|\,t_2\Big\rangle\Pi^m\Big\langle t_2\,\Big|\,\Psi'\Big\rangle\Big\rangle &= \sum_{\substack{m',n\\m'',n'}}\Big[\langle n'|n\rangle_N\langle m''|\Pi^m|m'\rangle_M\langle\psi(t_M)|\Pi_{m''}U_S(t_M,t_N)\Pi_{n'}\\&\qquad\cdot U_S(t_N,t_2)U_S(t_2,t_N)\Pi_n U_S(t_N,t_M)\Pi_{m'}|\psi(t_M)\rangle_S\Big]\\&= \langle\psi_S(t_M)|\Pi_m U_S(t_M,t_N)\sum_n \Pi_n U_S(t_N,t_M)\Pi_m|\psi_S(t_M)\rangle\\&= |\langle m|\psi_S(t_M)\rangle|^2 = \Big\langle\Big\langle\Psi'\,\Big|\,t_1\Big\rangle\Pi^m\Big\langle t_1\,\Big|\,\Psi'\Big\rangle\Big\rangle.\end{aligned}$$

From Equation (B.8) we obtain

$$|\phi(t_2)\rangle := \langle t_2|\Psi'\rangle\rangle = \sum_{m',n'} U_S(t,t_N)\Pi_{n'}U_S(t_N,t_M)\Pi_{m'}|\psi(t_M)\rangle_S|m'\rangle_M|n'\rangle_N,$$

and the numerator in both Definitions 2a and 2b gives

$$\begin{aligned}&\Big\langle\Big\langle\Psi'\,\Big|\,t_2\Big\rangle\langle t_2|\otimes\Pi^n\otimes\Pi^m\,\Big|\,\Psi'\Big\rangle\Big\rangle = \langle\phi(t_2)|\Pi^n\otimes\Pi^m|\phi(t_2)\rangle\\&\quad= \sum_{\substack{m',n'\\m'',n''}}\Big[\langle n''|\Pi^n|n'\rangle_N\langle m''|\Pi^m|m'\rangle_M\\&\qquad\cdot\langle\psi(t_M)|\Pi_{m''}U_S(t_M,t_N)\Pi_{n''}\Pi_{n'}U_S(t_N,t_M)\Pi_{m'}|\psi(t_M)\rangle_S\Big]\\&\quad= |\langle b|U_S(t_N,t_M)|m\rangle|^2|\langle m|\psi(t_M)\rangle|^2.\end{aligned}$$

Therefore, Defintions 2 give the standard quantum probabilities

$$P_{2a}\,(n \text{ when } t_2\,|\,m \text{ when } t_1) = |\langle n|U_S(t_N,t_M)|m\rangle|^2 = P_{2b}\,(n \text{ when } t_2\,|\,m \text{ when } t_2).$$

According to Definition 3 in Section 2.3.1 the conditional probability of result n at time $t_2 \geq t_N$ given result m at time $t_1 \geq t_M$ is

$$\frac{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\langle t_1|P^{\mathrm{ph}}|t_2\rangle\Pi^n\langle t_2||\Psi'\rangle\rangle}{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\langle t_1\,|\,\Psi'\rangle\rangle}. \tag{B.15}$$

The denominator is the same as in Definition 1 and Definition 2a and given by Equation (B.12). The numerator of Equation (B.15) gives

$$
\begin{aligned}
&\langle\phi(t_1)|\Pi^m \mathscr{U}(t_1,t_2)\Pi^n|\phi(t_2)\rangle \\
&= \Big[{}_N\langle r|_M\langle m|_S\langle\psi(t_M)|\Pi_m U_S(t_M,t_N)U_N^\dagger U_S(t_N,t_2) \\
&\qquad\qquad \cdot\sum_{m'} U_S(t_2,t_N)\Pi_n U_S(t_N,t_M)\Pi_{m'}|\psi(t_M)\rangle_S|m'\rangle_M|n\rangle_N\Big] \\
&= \sum_{m',n'}\Big[\langle n'|n\rangle_N\langle m|m'\rangle_M\langle\psi(t_M)|\Pi_m U_S(t_M,t_N)\Pi_{n'}\Pi_n U_S(t_N,t_M)\Pi_{m'}|\psi(t_M)\rangle_S\Big] \\
&= |\langle n|U_S(t_N,t_M)|m\rangle|^2|\langle m|\psi(t_M)\rangle|^2,
\end{aligned}
$$

and, hence, also Definition 3 recovers the standard quantum probabilities for non-Wigner's-friend scenarios.

B.3 The probability rules for the simple Wigner's-friend setup

	standard QT	positive	normalized	two-time	sym. joint	Conditions
Def. 1	✓	✓	✓	✓	✗	
Def. 2a	✓	✓	✗	✓	✓	normalized, if $\alpha^4+\beta^4+2\cos(\Delta\phi)(\alpha^3\beta-\alpha\beta^3)\frac{b}{a}+2\alpha^2\beta^2\left(\frac{b}{a}\right)^2=1$ and $\alpha^4+\beta^4-2\cos(\Delta\phi)(\alpha^3\beta-\alpha\beta^3)\frac{a}{b}+2\alpha^2\beta^2\left(\frac{a}{b}\right)^2=1$.
Def. 2b	✓	✓	✓	✗	✓	
Def. 3	✓	✗	✓	✓	✗	real for $\Delta\phi=n\pi$, positive for $\alpha=1,\beta=0$ or $\alpha=0,\beta=1$ as well as non-disturbance: $a=\alpha, b=\beta$ or $a=\beta, b=-\alpha$.

Table B.1. Comparison of the different proposed conditional probability rules for Wigner's-friend experiments: From the left to the right, the columns indicate whether each definition: (i) reduces to standard quantum theory for non-Wigner's-friend scenarios; (ii) is positive; (iii) is normalized; (iv) is a genuine two-time expression; (v) gives rise to a well-defined joint probability distribution (i.e. $P(A)P(B|A)=P(B)P(A|B)$). Definitions 2a and 3 become probabilities only under certain conditions, indicated by the blue ✗. The last column then specifies these conditions.

Appendix C

C.1 Notes on the Page-Wootters formulation of indefinite causal order

While we focussed on the history states when adapting the Page-Wootters formalism to describe, in general, non-causal processes, see Section 3.2, we here briefly discuss constraint operators and physical projectors in our framework since they are central objects in the original Page-Wootters formalism presented in Section 1.4. By construction, the physical states discussed in Section 3.2 form a proper subspace $\mathscr{H}_{\mathrm{ph}} \subset \mathscr{H}_c \otimes \mathscr{H}_S \otimes \mathscr{H}_{S'}$ by construction. Any linear combination of physical states, e.g. $\alpha|\Psi\rangle\rangle + \beta|\Psi'\rangle\rangle$ is again a physical state, namely that associated with input state $\alpha|\psi\rangle_S + \beta|\psi'\rangle_S$. Hence, we can define a constraint operator as

$$\hat{C} := \mathbb{1} - \hat{P}, \tag{C.1}$$

where $\hat{P}$ is the orthogonal projector onto $\mathscr{H}_{\mathrm{ph}}$. Note, however, that $\hat{P}$ in our framework can in general not be written analogous to Equation (1.33), which is used for standard quantum circuits with one clock. More specifically, for an orthonormal basis $|\psi_j\rangle_S$, the corresponding history states are

$$\begin{aligned}|\Psi_j\rangle\rangle &= \sum_{t_A, t_B, \ldots} |t_A, t_B, \ldots\rangle \otimes |\psi_j(t_A, t_B, \ldots)\rangle_S = \sum_{t_A=0}^{T_A} |t_A\rangle (N_{t_A}^{(A)})^{-1} |\psi_{A,j}(t_A)\rangle \\ &= \sum_{t_A=0}^{T_A} |t_A\rangle (N_{t_A}^{(A)})^{-1}\, \mathscr{U}_A(t_A, 0)|0, 0, \ldots\rangle \otimes |\psi_j\rangle_S.\end{aligned}$$

However, these may fail to be orthogonal due to the normalization operators

$$\langle\langle \Psi_k | \Psi_j \rangle\rangle = \tag{C.2}$$
$$\sum_{t_A=0}^{T_A} \langle \psi_k |_S \otimes \langle 0,0,\ldots 0| \mathscr{U}_A(t_A,0)^\dagger [(N_{t_A}^{(A)})^{-1}]^\dagger (N_{t_A}^{(A)})^{-1} \mathscr{U}_A(t_A,0) |0,0,\ldots 0\rangle \otimes |\psi_j\rangle_S.$$

We can, however, write $\hat{P}$ in a form more reminiscent of the original Page-Wootters framework, compare Equation (1.20), as

$$\hat{P} = \frac{1}{T} \sum_{k=0}^{T-1} \exp\left(-2\pi i \hat{C} \frac{k}{T}\right), \tag{C.3}$$

where T is an integer (we could take $T = T_A$). This can be seen by noting that $\hat{C}$ is a hermitian matrix with only eigenvalues 0 or 1. If $|\phi_0\rangle$ is an eigenvector of $\hat{C}$ with $\hat{C}|\phi_0\rangle = 0$, we have $\hat{P}|\phi_0\rangle = |\phi_0\rangle$, while if $\hat{C}|\phi_1\rangle = |\phi_1\rangle$ we have $\hat{P}|\phi_1\rangle = \frac{1}{T}\sum_{k=0}^{T-1} e^{-2\pi i \frac{k}{T}} |\phi_1\rangle = 0$, showing $\hat{P} = \mathbb{1} - \hat{C}$.

C.2 The Page-Wootters quantum switch

Here, we discuss the probably best known non-causal process, the bipartite quantum switch Chiribella et al. [2013], which is shown in Figure C.1. It is an example of coherent control of causal order and we can implement it similar to the general procedure presented in Section 3.3.2. A history state of the quantum switch is given by

$$\begin{aligned}
|\Psi\rangle\rangle = {} & |0_A, 0_B\rangle_c \otimes |\phi\rangle + |1_A, 1_B\rangle_c \otimes |\phi\rangle + |2_A, 2_B\rangle_c \otimes |\phi\rangle \\
& + |3_A, 2_B\rangle_c \otimes (|0\rangle\langle 0| \otimes \mathbb{1})|\phi\rangle + |2_A, 3_B\rangle_c \otimes (|1\rangle\langle 1| \otimes \mathbb{1})|\phi\rangle \\
& + |4_A, 3_B\rangle_c \otimes (|0\rangle\langle 0| \otimes U_A)|\phi\rangle + |3_A, 4_B\rangle_c (|1\rangle\langle 1| \otimes U_B)|\phi\rangle \\
& + |5_A, 4_B\rangle_c \otimes (|0\rangle\langle 0| \otimes U_B U_A)|\phi\rangle + |4_A, 5_B\rangle_c \otimes (|1\rangle\langle 1| \otimes U_A U_B)|\phi\rangle \\
& + |5_A, 5_B\rangle_c \otimes (|0\rangle\langle 0| \otimes U_B U_A + |1\rangle\langle 1| \otimes U_A U_B)|\phi\rangle \\
& + |6_A, 6_B\rangle_c \otimes \mathscr{G}(U_A, U_B)|\phi\rangle + |7_A, 7_B\rangle_c \otimes \mathscr{G}(U_A, U_B)|\phi\rangle,
\end{aligned} \tag{C.4}$$

where $\mathscr{G}(U_A, U_B) = |0\rangle\langle 0| \otimes U_B U_A + |1\rangle\langle 1| \otimes U_A U_B$ is the pure process matrix of the quantum switch. From Equation (C.4) we obtain the following perspectival

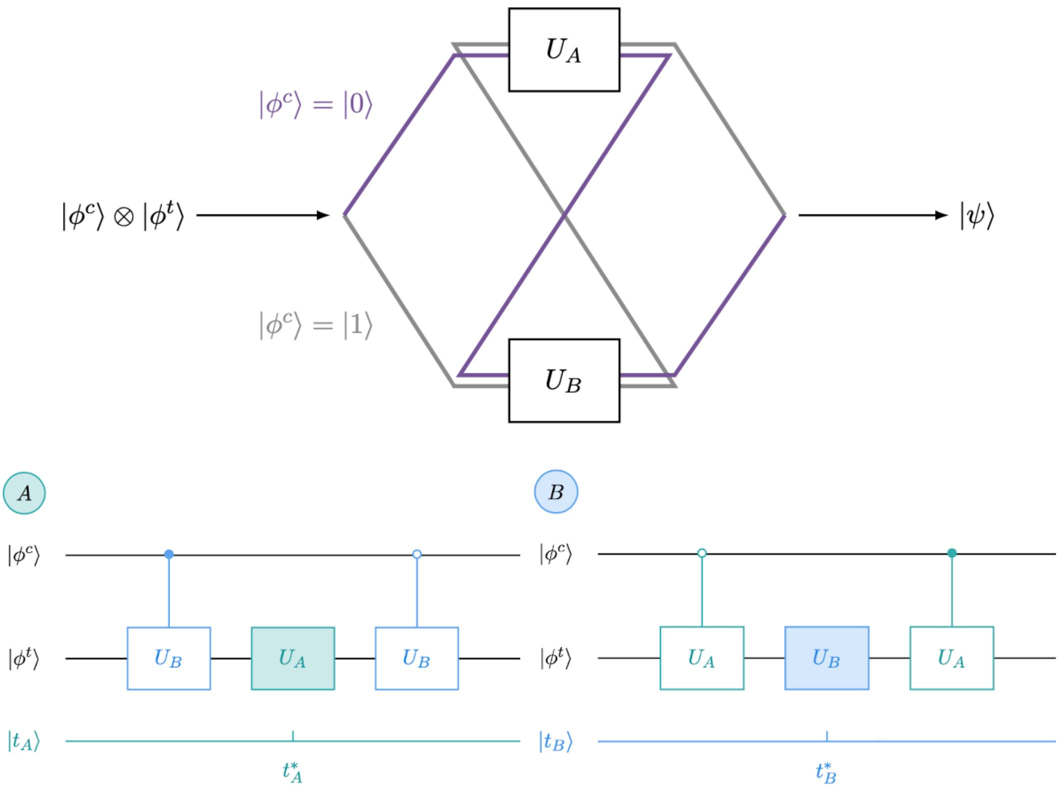


Figure C.1. The bipartite quantum switch: Depending on the value of a control qubit the two unitaries U_A, U_B are applied to the target system in different order (top). According to the perspectives of the two agents, A and B, each agent applies their own unitary to the target system at time t_A^* and t_B^* respectively, while the other agent's unitary is applied either before or after that depending on the value of the control system (bottom). The perspectival circuits are equivalent to the causal reference frames given in Allard Guérin and Brukner [2018].

states

$$
\begin{aligned}
|\psi_A(0)\rangle &= |0_B\rangle_{c_B} \otimes |\phi\rangle & |\psi_B(0)\rangle &= |0_A\rangle_{c_A} \otimes |\phi\rangle \\
|\psi_A(1)\rangle &= |1_B\rangle_{c_B} \otimes |\phi\rangle & |\psi_B(1)\rangle &= |1_A\rangle_{c_A} \otimes |\phi\rangle \\
|\psi_A(2)\rangle &= |2_B\rangle_{c_B} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})|\phi\rangle & |\psi_B(2)\rangle &= |2_A\rangle_{c_A} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})|\phi\rangle \\
&\quad + |+_{2,3}\rangle_{c_B} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})|\phi\rangle & &\quad + |+_{2,3}\rangle_{c_A} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})|\phi\rangle \\
\text{with } N_2^{(A)} &= |0\rangle\langle 0|_{Sc} + \frac{1}{\sqrt{2}}|1\rangle\langle 1|_{Sc} & \text{with } N_2^{(B)} &= \frac{1}{\sqrt{2}}|0\rangle\langle 0|_{Sc} + |1\rangle\langle 1|_{Sc} \\
|\psi_A(3)\rangle &= |2_B\rangle_{c_B} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})|\phi\rangle & |\psi_B(3)\rangle &= |2_A\rangle_{c_A} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})|\phi\rangle \\
&\quad + |4_B\rangle_{c_B} \otimes (|1\rangle\langle 1| \otimes U_B)|\phi\rangle & &\quad + |4_A\rangle_{c_A} \otimes (|0\rangle\langle 0| \otimes U_A)|\phi\rangle \\
|\psi_A(4)\rangle &= |3_B\rangle_{c_B} \otimes (|0\rangle\langle 0| \otimes U_A)|\phi\rangle & |\psi_B(4)\rangle &= |3_A\rangle_{c_A} \otimes (|1\rangle\langle 1| \otimes U_B)|\phi\rangle \\
&\quad + |5_B\rangle_{c_B} \otimes (|1\rangle\langle 1| \otimes U_A U_B)|\phi\rangle & &\quad + |5_A\rangle_{c_A} \otimes (|0\rangle\langle 0| \otimes U_B U_A)|\phi\rangle \\
|\psi_A(5)\rangle &= |+_{4,5}\rangle_{c_B} \otimes (|0\rangle\langle 0| \otimes U_B U_A)|\phi\rangle & |\psi_B(5)\rangle &= |+_{4,5}\rangle_{c_A} \otimes (|1\rangle\langle 1| \otimes U_A U_B)|\phi\rangle \\
&\quad + |5_B\rangle_{c_B} \otimes (|1\rangle\langle 1| \otimes U_A U_B)|\phi\rangle & &\quad + |5_A\rangle_{c_A} \otimes (|0\rangle\langle 0| \otimes U_B U_A)|\phi\rangle \\
\text{with } N_5^{(A)} &= \frac{1}{\sqrt{2}}|0\rangle\langle 0|_{Sc} + |1\rangle\langle 1|_{Sc} & \text{with } N_5^{(B)} &= |0\rangle\langle 0|_{Sc} + \frac{1}{\sqrt{2}}|1\rangle\langle 1|_{Sc} \\
|\psi_A(6)\rangle &= |6_B\rangle_{c_B} \otimes \mathscr{G}(U_A, U_B)|\phi\rangle & |\psi_B(6)\rangle &= |6_A\rangle_{c_A} \otimes \mathscr{G}(U_A, U_B)|\phi\rangle \\
|\psi_A(7)\rangle &= |7_B\rangle_{c_B} \otimes \mathscr{G}(U_A, U_B)|\phi\rangle & |\psi_B(7)\rangle &= |7_A\rangle_{c_A} \otimes \mathscr{G}(U_A, U_B)|\phi\rangle,
\end{aligned} \tag{C.5}
$$

where $|+_{i,j}\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |j\rangle) \in \mathscr{H}_{c_X}$ is an equal superposition of two states, corresponding to times i and j, of the same clock. These perspectival states can be related to each other by unitaries

$$
\begin{aligned}
\mathscr{U}_A(1,0) &= T_{c_B} \otimes \mathbb{1}_S & \mathscr{U}_B(1,0) &= T_{c_A} \otimes \mathbb{1}_S \\
\mathscr{U}_A(2,1) &= T_{c_B} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S & \mathscr{U}_B(2,1) &= T_{c_A} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})_S \\
&\quad + (T_2')_{c_B} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})_S & &\quad + (T_2')_{c_A} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S \\
\mathscr{U}_A(3,2) &= \mathbb{1}_{c_B} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S & \mathscr{U}_B(3,2) &= \mathbb{1}_{c_A} \otimes (|1\rangle\langle 1| \otimes \mathbb{1}) \\
&\quad + (T_2')_{c_B} \otimes (|1\rangle\langle 1| \otimes U_B)_S & &\quad + (T_2')_{c_A} \otimes (|0\rangle\langle 0| \otimes U_A)_S \\
\mathscr{U}_A(4,3) &= T_{c_B} \otimes (\mathbb{1} \otimes U_A)_S & \mathscr{U}_B(4,3) &= T_{c_A} \otimes (\mathbb{1} \otimes U_B)_S \\
\mathscr{U}_A(5,4) &= (T_4')_{c_B} \otimes (|0\rangle\langle 0| \otimes U_B)_S & \mathscr{U}_B(5,4) &= (T_4')_{c_A} \otimes (|1\rangle\langle 1| \otimes U_A)_S \\
&\quad + \mathbb{1}_{c_B} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})_S & &\quad + \mathbb{1}_{c_A} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S \\
\mathscr{U}_A(6,5) &= (T_4')_{c_B} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S & \mathscr{U}_B(6,5) &= (T_4')_{c_A} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})_S \\
&\quad + T_{c_B} \otimes (|1\rangle\langle 1| \otimes \mathbb{1})_S & &\quad + T_{c_A} \otimes (|0\rangle\langle 0| \otimes \mathbb{1})_S \\
\mathscr{U}_A(7,6) &= T_{c_B} \otimes \mathbb{1}_S & \mathscr{U}_B(7,6) &= T_{c_A} \otimes \mathbb{1}_S,
\end{aligned} \tag{C.6}
$$

where T and T_i' are the same as introduced in Section 3.3.1, i.e $T : |t\rangle \mapsto |t+1\rangle$ and T_i' is any unitary that acts as $|i-1\rangle \mapsto 1/\sqrt{2}(|i\rangle + |i+1\rangle)$, $1/\sqrt{2}(|i\rangle + |i+1\rangle) \mapsto |i+2\rangle$. It is straightforward to see that all our axioms are fulfilled. For both A and B the time of action is $t_A^\star = 4 = t_B^\star$ and depending on the value of the control the other agent applies their unitary either before or after $t_A^\star$ or $t_B^\star$ respectively. Note that the unitaries in Equations (C.6) are not unique but were chosen such that the perspectives of the agents resemble the causal reference frames of the two agents presented in Allard Guérin and Brukner [2018].

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