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An Econometric Analysis of Time-Varying Risk Premia in Large Cross-Sectional Equity Datasets

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Abstract

Investments in financial assets exhibit different expected excess returns because of their different risks. We can split the risk of an asset in two components: idiosyncratic and systematic risks. The investors reduce the idiosyncratic risk component by diversification, i.e., by investing in a broad range of assets. On the contrary, the systematic component represents the risk that is common to all assets and cannot be eliminated by diversification. Therefore, in equilibrium the investors ask for a financial compensation to bear this kind of risk. In linear factor models, systematic risk is represented by a set of pervasive factors. In such a setting, the absence of arbitrage opportunities implies that the asset expected excess return is equal to risk premia multiplied by factor loadings. Thus, risk premia are the rewards per unit of systematic risk borne by investors. Since systematic risk is influenced by financial and macroeconomic variables, risk premia are expected to be time-varying. A natural question that arises is how we can estimate these time-varying risk premia.

In this thesis, we develop a new econometric methodology to estimate the time-varying risk premia implied by conditional linear asset pricing models. In contrast to the classical approach, we estimate risk premia from a large dataset of returns of individual stocks instead of portfolios. The aim is to avoid the potential bias and loss of information implied by sorting and grouping stocks into portfolios. When working with individual stock returns we face several econometric challenges. First, our datasets are characterized by large cross-sectional and time series dimensions, and this fact complicates the numerical implementation and the study of the statistical properties of the estimators. Second, in available datasets, we do not observe asset returns for all firms at all dates, i.e., the panel of stock returns is unbalanced. Third, data feature cross-sectional dependence because of the correlation structure in error terms. To address these challenges, we propose a new estimator that uses simple weighted two-pass regressions. Our estimation methodology accounts for the unbalanced characteristic of large panel data. We study the large sample properties of our estimators in a double asymptotics scheme that reflects the large dimensions of the dataset. In this setting, we test the asset pricing restrictions induced by the no-arbitrage assumption in large economies and we address consistent estimation of the large-dimensional variance-covariance matrix of the errors by sparsity methods. We apply our methodology to a dataset of about ten thousands US stocks with monthly returns from July 1964 to December 2009. The conditional risk premia estimates are large and volatile in crisis periods, and do not match risk premia estimates on standard sets of portfolios.

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Chapter 1

Introduction

A major aim of asset pricing theory is explaining the cross-sectional variation in expected excess returns between financial assets. Idiosyncratic risk and systematic risk both affect returns of assets. However, idiosyncratic risk is asset-specific, and is eliminated by diversification. On the opposite, systematic risk is common to all assets, and cannot be eliminated by diversification. Thus, assets feature different expected excess returns because of their different exposures to systematic risk. Consequently, modeling systematic risk factors is necessary and crucial in the asset pricing literature. Linear factor models represent systematic risk through a set of pervasive factors, and excess returns of assets follow a factor structure. According to this setting, the expected excess return of an asset is a linear function of its sensitivities to changes in each factor weighted by risk premia. The risk premium on a factor is the financial compensation asked by investors to bear a unit of systematic risk on that factor. A broad financial econometrics literature deals with linear factor models, and proposes approaches to estimate risk premia in several frameworks. In this chapter, we first review the literature on linear factor models, and present the standard estimation methodologies and the approaches to testing the asset pricing restrictions. Then, we describe the main theoretical and empirical contributions of the thesis.

1.1 Linear factor models

The standard classical asset pricing models assume that the excess return of an asset follows an unconditional linear factor model. Let $R_{i,t}$ denote the excess return of asset i at date t, where $i = 1, ..., n$ and $t = 1, ..., T$.

Linear factor models represent systematic risk by a vector of factors f_t, and assume that the excess return $R_{i,t}$ satisfies:

$$\begin{aligned} R_{i,t} &= a_i + b_i' f_t + \varepsilon_{i,t} \\ &= \beta_i' x_t + \varepsilon_{i,t}, \end{aligned} \tag{1.1}$$

where vector f_t gathers the values at time t of K factors, $x_t = (1, f_t')'$, and vector $\beta_i = (a_i, b_i')'$ contains the intercept and the factor sensitivities of asset i. The error term $\varepsilon_{i,t}$ represents the idiosyncratic risk. Let us define the error vector $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$. Then, in the simplest version of the model, the error terms ε_t are independent and identically distributed ($i.i.d.$) over time and such that $E[\varepsilon_t] = 0$ and the $n \times n$ variance-covariance matrix $\Sigma_\varepsilon = E[\varepsilon_t \varepsilon_t']$ is diagonal with $E\left[\varepsilon_{i,t}^2\right] = \sigma_{ii}$. Under these assumptions, the model has a static strict, or exact, factor structure: the error terms are cross-sectionally uncorrelated. Moreover, the expected excess return of an asset is a linear function of its factor loadings, b_i:

$$E[R_{i,t}] = b_i' \lambda, \tag{1.2}$$

where λ is the time-invariant vector of K risk premia. The asset pricing restriction (1.2) implies that the intercept a_i is a linear function of b_i:

$$a_i = b_i' \nu, \tag{1.3}$$

where

$$\nu = \lambda - E[f_t]. \tag{1.4}$$

The simplest and most popular linear factor model is the Capital Asset Pricing Model (CAPM) by Sharpe (1964), Lintner (1965) and Black (1972). The CAPM is an equilibrium model, i.e., it assumes an optimal behaviour of investors and asset prices are the result of an equilibrium on asset demand and supply. The investors, who are assumed to be risk averse, maximize their economic utilities considering only the mean and variance of the portfolio. The underlying factor model for asset returns uses the market portfolio return as the single systematic risk factor. In this context, the expected excess return of an asset is linear in the market loading (beta). Furthermore, the CAPM accounts for an economy with a fix and small number of assets. In order to derive the CAPM, we need to assume a particular form of the utility function of the investors. Lintner

(1965) derives the model assuming either quadratic utility function, or an exponential utility function with normality of returns. Sharpe (1964) and Lintner (1965) build the CAPM assuming that a risk-free asset exists in the economy and investors can lend or borrow unlimited amounts at the risk-free rate. On the contrary, Black (1972) proposes a version of the CAPM where a risk-free asset does not exist, and uses a proxy for the riskless asset, namely the zero-beta portfolio. The CAPM has received critiques about its assumptions that are not compatible with the empirical properties of asset returns. Roll (1977) highlights that the CAPM is not empirically testable because the market portfolio is not observable. Indeed, we cannot build a portfolio of all assets, and we have to use a proxy of the market portfolio in empirical analysis (see Kandel and Stambaugh (1987)). Ross (1976) develops the Arbitrage Pricing Theory (APT) based on a multifactor model and on the absence of arbitrage opportunity. In term of portfolio payoff, the no-arbitrage restriction ensures that positive portfolio payoffs have positive prices (see e.g., Hansen and Richard (1987), Duffie (2001) and Cochrane (2005)). The APT has been developed by assuming a large number of assets available in the economy, an exact static factor-structure for the error terms in (1.1), and the absence of asymptotic arbitrage opportunities. Ross (1976) proves that expected excess returns are approximately a linear function of factor loadings as in Equation (1.2), such that

$$\sum_{i=1}^{\infty} \left(E\left[R_{i,t}\right] - b_i'\lambda\right)^2 < \infty. \tag{1.5}$$

Building on the Ross (1976) critique, Fama and French (1993) find that expected returns are related also to firm characteristics, and state that a single factor is not enough to explain expected returns. They develop an asset pricing model that includes two other factors in addition to market excess returns: “small minus big” (SMB) market capitalization and “high minus low” (HML) book-to-market ratio.

The assumption of an exact factor structure in asset returns is often rejected empirically. To fill this gap, Chamberlain and Rothschild (1983, CR) introduce an approximate factor model that accounts for some cross-sectional dependence in the error terms, i.e. $E\left[\varepsilon_{i,t}\varepsilon_{j,t}\right] = \sigma_{ij} \neq 0$ for some $i, j = 1, ..., n$. An approximate factor model assumes that there is a large number n of assets and Σ_ε is a non-diagonal matrix with bounded eigenvalues as n increases, i.e., the proportion of non-zero correlations is small. This ensures that each of the K factors represents a pervasive source of systematic risk in the cross-section of returns. In this setting, CR generate no-arbitrage restrictions in large economies where the number of assets grows to infinity. In particular, if the asset excess returns follow an approximate factor model, then the expected excess return is approximately a linear function of the factor loadings as in Equation (1.5), i.e., most assets have small pricing errors. Shanken (1982) critiques this asset pricing restriction because it is not empirically testable. Indeed, the condition is always verified on a finite number of assets. Al-Najjar (1998) derives the empirically testable condition in (1.2) for a strict factor structure in a static, unconditional economy.

The specification in equations (1.1) and (1.2) disregards conditional information, and assumes constant beta coefficients and risk premia. Since financial and macroeconomic variables influence systematic risk, assuming time-varying factor loadings and risk premia is preferable. Conditional factor models aim at capturing the time-varying influence of these variables in a simple setting. The excess return $R_{i,t}$ satisfies the conditional linear factor model:

$$R_{i,t} = a_{i,t} + b'_{i,t} f_t + \varepsilon_{i,t}, \tag{1.6}$$

where the intercept and the factor loadings are functions of lagged common observable instruments Z_{t-1}, i.e., $a_{i,t} = a_i\left(Z_{t-1}\right)$ and $b_{i,t} = b_i\left(Z_{t-1}\right)$ (Ferson and Harvey (1999)). The set of instruments can also contain variables that are specific to stock i. Avramov and Chordia (2006) allow $b_{i,t}$ to vary with common instruments Z_{t-1} (macroeconomic variables), and asset specific instruments $Z_{i,t-1}$ (firm-level size and book-to-market). Let $\mathcal{F}_{n,t}$ be the relevant information available in the economy with n assets at date t. Cochrane (1996) and Jagannathan and Wang (1996) relax the assumption of constant risk premia, and equation (1.2) becomes

$$E\left[R_{i,t} | \mathcal{F}_{n,t-1}\right] = b'_{i,t} \lambda_t, \tag{1.7}$$

where λ_t is the vector of risk premia at time t and is function of lagged instrumental variables Z_{t-1}, i.e., $\lambda_t = \lambda\left(Z_{t-1}\right)$. A broad finance literature deals with conditional factor models, see e.g., Shanken (1990), Cochrane (1996), Ferson and Schadt (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Petkova and Zhang (2005). Ghysels (1998) discusses pros and cons of modeling time-varying factor loadings. In particular, he shows that the potential misspecification of the time-varying specification makes the conditional model less preferable than the unconditional one. Indeed, in case of misspecification, the pricing errors with constant loadings b_i could be smaller than using time-varying loadings $b_{i,t}$.

1.2 Estimation approaches

Let us consider the unconditional linear factor model in equation (1.1) where f_t is a vector of observable factors. In this framework, we review the econometric approaches to estimate the vector of risk premia in equation (1.2). Three methodologies are proposed in the literature: the two-pass cross-sectional regression method, the Maximum Likelihood, and the Generalized Method of Moments.

i. *Two-pass cross-sectional regression*

The two-pass cross-sectional regression approach is the most popular methodology to estimate equity risk premia in an unconditional linear multi-factor setting. Lintner (1965) proposes to estimate risk premia in two steps. In the first step, the intercept a_i and factor loadings b_i are estimated by time-series Ordinary Least Squares (OLS) regression on model (1.1) for each asset i:

$$\hat{\beta}_i = \left(\hat{a}_i, \hat{b}_i'\right)' = \left(\sum_t x_t x_t'\right)^{-1} \sum_t x_t R_{i,t}.$$

In the second step, the vector of risk premia is estimated by a cross-sectional OLS regression of average excess returns on the $\hat{b}_i$, that is

$$\hat{\lambda} = \left(\sum_i \hat{b}_i \hat{b}_i'\right)^{-1} \sum_i \hat{b}_i \bar{R}_i,$$

where $\bar{R}_i = \dfrac{1}{T} \sum_t R_{i,t}$ is the average excess return of asset i.

Fama and MacBeth (1973) suggest an alternative procedure to Lintner (1965) approach. They modify the second step. Instead of estimating a single cross-sectional regression on average excess returns, they run a cross-sectional OLS regression at each time t, and take the average of the cross sectional regression coefficients, i.e., $\hat{\lambda} = \dfrac{1}{T} \sum_t \hat{\lambda}_t$, where $\hat{\lambda}_t = \left(\sum_i \hat{b}_i \hat{b}_i'\right)^{-1} \sum_i \hat{b}_i R_{i,t}$. This estimator of the risk premia is exactly the same obtained by Lintner (1965). However, Fama and MacBeth (1973) treat the cross-sectional estimates $\left\{\hat{\lambda}_t,\ t = 1, ..., T\right\}$ as a sample of realizations of the risk premia estimator, and propose to estimate the asymptotic variance-covariance matrix of $\sqrt{T}\left(\hat{\lambda} - \lambda\right)$ with $\hat{\Sigma}_{\lambda,FMB} = \dfrac{1}{T} \sum_t \left(\hat{\lambda}_t - \hat{\lambda}\right)\left(\hat{\lambda}_t - \hat{\lambda}\right)'$.

An alternative estimation procedure can be obtained if we consider the asset pricing restriction written as in (1.3). From (1.4), an equivalent expression of the estimator of risk premia vector is

$$\hat{\lambda} = \hat{\nu} + \bar{f}, \tag{1.8}$$

where $\hat{\nu} = \left(\sum_i \hat{b}_i \hat{b}_i'\right)^{-1} \sum_i \hat{b}_i \hat{a}_i$ and $\bar{f} = \frac{1}{T}\sum_t f_t$ is the sample moments of $E\left[f_t\right]$.

In the classical literature, risk premia are estimated from a dataset of portfolio returns and the cross-sectional dimension n is usually (much) smaller than the time series dimension T. For instance, we might have $n \simeq 20 - 30$ portfolios where monthly returns are observed over $T \simeq 400 - 500$ months. Thus, the large sample properties of the estimators have been studied in a series of papers by keeping fixed the cross-sectional dimension n and letting the time series dimension T increase to infinity (see e.g., Shanken (1992), Jagannathan and Wang (1998), Shanken and Zhou (2007); see also the reviews in Jagannathan, Skoulakis and Wang (2009), Goyal (2012)). In this setting, the asymptotic distribution of risk premia merits consideration. We will denote by $\Rightarrow$ the convergence in distribution. Let us consider equations (1.1) and (1.3), and the corresponding estimators. In the homoskedasticity case (i.e., the covariance between assets i and j is constant over time), Shanken (1992) shows that

$$\sqrt{T}\left(\hat{\lambda} - \lambda\right) \Rightarrow N\left(0, \Sigma_\lambda\right), \text{ where } \Sigma_\lambda = \left(1 + \lambda' \Sigma_f^{-1} \lambda\right) \Sigma^* + \Sigma_f,$$

with $\Sigma^* = \left(\frac{1}{n}B'B\right)^{-1} \frac{1}{n}B'\Sigma_\varepsilon B \left(\frac{1}{n}B'B\right)^{-1}$, $B = (b_1, ..., b_n)'$ and $\Sigma_f = V\left[f_t\right]$, for $T \to \infty$ and n fixed. From equation (1.8), we understand that the asymptotic variance of risk premia consists of two components: the asymptotic variability of the second step estimator $\hat{\nu}$, i.e., $\left(1 + \lambda'\Sigma_f^{-1}\lambda\right)\Sigma^*$, and the asymptotic variability of observable factors, i.e., Σ_f. Jagannathan and Wang (1998) extend the results to the heteroskedasticity case (see Section 2.3).

Despite its simplicity, the two-pass approach suffers from the Error-In-Variable (EIV) problem: the risk premia estimator contains an estimation error through the estimated factor loadings $\hat{b}_i$. The EIV implies a biased second pass estimator for finite T. If we ignore the estimated errors in the $\hat{b}_i$, we understate the asymptotic variance of risk premia. Indeed, the estimator $\hat{\Sigma}_{\lambda,FMB}$ proposed in Fama and MacBeth (1973) is not a consistent estimator for Σ_λ. On the contrary, Shanken (1992) shows that the estimator $\hat{\Sigma}_\lambda = \left(1 + \hat{\lambda}'\hat{\Sigma}_f^{-1}\hat{\lambda}\right)\hat{\Sigma}^* + \hat{\Sigma}_f$, that contains the term of correction $\hat{\lambda}'\hat{\Sigma}_f^{-1}\hat{\lambda}$, is consistent for $T \to \infty$. Shanken (1992) also proposes a modified OLS estimator that accounts for an EIV correction and proves that it is consistent when n goes to infinity and T is fixed. Furthermore, λ is usually estimated by the OLS estimator, that is efficient when the error terms are cross-sectional *i.i.d.*. These assumptions are clearly not satisfied empirically. Therefore, in order to increase efficiency, Shanken and Zhou (2007) propose to estimate λ by applying a Generalized Least Squares (GLS) or a Weighted Least Squares (WLS) estimation approach in the second pass.

ii. *Maximum Likelihood (ML)*

The ML approach is asymptotically efficient under the classical assumption that errors and factors are mutually independent, independent across time and normally distributed. Plugging the asset pricing restriction (1.2) in equation (1.1), the constrained linear factor model is

$$R_{i,t} = b_i'\lambda + b_i'\left(f_t - E\left[f_t\right]\right) + \varepsilon_{i,t}. \tag{1.9}$$

The ML estimator maximizes the likelihood function over the parameters b_i for $i = 1, ..., n$, λ and the covariance matrix Σ_ε of the errors. Since the constraint is not linear (i.e., b_i and λ enter multiplied), Gibbons (1982) suggests to apply the Gauss-Newton procedure, i.e., an iterative algorithm that makes linear the constraint using a Taylor expansion and solves the maximization problem. Kandel (1984) and Shanken (1985) extend the results by Gibbons (1982). In particular, Shanken (1982) shows that the ML estimator and the GLS estimator are asymptotically equivalent as $T \to \infty$. This result stresses the fact that the EIV problem is not eliminated estimating the parameters in a single step as in the ML approach.

iii. *Generalized Method of Moments (GMM)*

The GMM approach (Hansen (1982), Hansen and Singleton (1982)) allows to estimate the parameters in a single step accounting for serial correlation and conditional heteroskedasticity in error terms. MacKinlay and Richardson (1991), and Zhou (1995) propose to estimate (1.1) and (1.2) by the GMM methodology. A crucial point of this approach is the definition of the moments conditions $E\left[g\left(x_t, \theta\right)\right] = 0$, say, where vector x_t involves the observable variables f_t and $R_{i,t}$ for each $i = 1, ..., n$, and the vector of parameters θ contains transformations of λ, $E\left[f_t\right]$ and b_i for each $i = 1, ..., n$. We derive the moments conditions from equation $E\left[R_{i,t}\right] = b_i'\lambda$ for any $i = 1, ..., n$

where $b_i = \Sigma_f^{-1} Cov\left[f_t, R_{i,t}\right]$. By appropriately rearranging terms, the moments conditions are

$$E\left[R_{i,t}\right] = b_i'\lambda \Leftrightarrow E\left[R_{i,t}\left(1 - \lambda'\Sigma_f^{-1} f_t + \lambda'\Sigma_f^{-1} E\left[f_t\right]\right)\right] = 0 \Leftrightarrow E\left[R_{i,t}\left(1 - \theta' f_t\right)\right] = 0,$$

where $\theta = \dfrac{\Sigma_f^{-1}\lambda}{1 + \lambda'\Sigma_f^{-1} E\left[f_t\right]}$. In particular, $(1 - \theta' f_t)$ is the stochastic discount factor. Thus, the number of moments conditions is equal to the cross-sectional dimension n. The GMM estimator is

$$\hat{\theta} = \arg\min_{\theta} g_T'\left(\theta\right)\Omega_T g_T\left(\theta\right),$$

where $g_T\left(\theta\right)$ is the sample average of $g\left(x_t, \theta\right)$, i.e, $g_T\left(\theta\right) = \dfrac{1}{T}\sum_t g\left(x_t, \theta\right)$, and Ω_T is the $n \times n$ optimal weighting matrix, i.e., the inverse of the asymptotic variance-covariance matrix of moment conditions (see Hansen (1982)). The GMM estimator is hard to compute when the number of assets is large. Indeed, one faces the numerical inversion of a large dimensional weighting matrix.

Let us now consider the conditional factor model in equations (1.6) and (1.7). Despite its empirical relevance, statistical inference in time-varying risk premia models has received less attention in the literature. The two-pass cross sectional approach allows to estimate time-varying factor loadings $\beta_{i,t} = \left(a_{i,t},\ b_{i,t}\right)'$ in (1.6) by rolling short-window regression methodology (see e.g., Ferson and Harvey (1991) and Lewellen and Nagel (2006)). Then, one computes the cross-sectional regression in (1.7) using the estimated $\beta_{i,t}$. However, conditional factor models imposes conditions on the information set $\mathcal{F}_{n,t}$. Thus, Jagannathan and Wang (1996) model expected returns and factor loadings as functions of instrumental variables, and define moment conditions in a GMM framework.

1.3 Test of asset pricing restrictions

Let us consider the unconditional linear factor model in (1.1) and the asset pricing restriction in (1.2), or (1.3). Gibbons, Ross and Shanken (1985, GRS) derive a test for the null hypothesis when the factors are traded portfolios. In this case, the risk premia vector λ is equal to the expected return of the factors $E\left(f_t\right)$, i.e., $\nu = 0$ by (1.4). The intercept a_i is the pricing error on asset i. Let us consider the following null hypothesis

$$\mathcal{H}_0: \; a_i = 0 \text{ for all assets } i.$$

The GRS's statistic to test this null hypothesis is $\hat{\xi}_T = T\hat{a}'\hat{\Omega}^{-1}\hat{a}$ where $\hat{a} = [\hat{a}_1, ..., \hat{a}_n]'$ and the variance-covariance matrix $\hat{\Omega}$ is a consistent estimator for $\Omega = \left(1 + \lambda'\Sigma_f^{-1}\lambda\right)\Sigma_\varepsilon$. Under the null hypothesis $\mathcal{H}_0$, we have $\hat{\xi}_T \Rightarrow \chi^2_n$ as $T \to \infty$ and n is fixed. GRS derive this result using the CAPM, where f_t is a scalar factor and corresponds to the market portfolio return. In this framework, assuming that the error terms ε_t are *i.i.d.* over time, GRS interpret the statistic ξ_T in term of a Sharpe ratio. In particular, ξ_T captures the difference between the Sharpe ratio of a portfolio of n assets and the Sharpe ratio of the market portfolio. Larger values of this deviation imply rejection of $\mathcal{H}_0$.

Let us consider the generalization of the GRS's statistic when the factor are not tradable portfolios. The null hypothesis becomes

$$\mathcal{H}_0: \; \text{there exists a vector } \nu \in \mathbb{R}^K \text{ such that } a_i = b_i'\nu \text{ for all assets } i.$$

The test statistic is $\hat{\xi}_T = T\hat{e}'\hat{\Omega}^{-1}\hat{e}$, where $\hat{e}$ is the vector $n \times 1$ of the error terms $\hat{e}_i = \hat{a}_i - \hat{b}_i'\nu$. Under the null hypothesis, $\hat{\xi}_T \Rightarrow \chi^2_{n-K}$ as $T \to \infty$. The GRS's statistic is usually computed in a classical setting where the excess returns of portfolios are used as base assets.

The GRS's statistic for the null hypothesis $\mathcal{H}_0$ is based on the sum of squared residuals of the second-pass cross-sectional regression. This sum of squared residuals is related to the coefficient of determination ρ^2 of the cross-sectional regression. Lewellen, Nagel and Shanken (2010) emphasize that high cross-sectional ρ^2 does not always imply that the model makes a good job and provide suggestions to improve empirical tests by simulation analysis. Kan, Robotti and Shanken (2012) study the asymptotic distribution of the cross-sectional ρ^2 when n is fixed and $T \to \infty$. In particular, they consider the misspecification problem of factor models and provide test statistics based on ρ^2 to compare and measure the misspecification between the models.

Alternative test statistics are proposed in the literature (see the review in Jagannathan, Schaumburg and Zhou (2010)). Shanken (1985) proposes a statistic based on the maximum-likelihood estimation approach. Shanken (1985, 1992) consider the asymptotic properties of the ML estimation and show that the second-pass GLS estimator is asymptotically equal to the Gauss-Newton estimator (Gibbons (1982)). This result allows to compute a Likelihood Ratio test (LRT) that involves the quadratic form of pricing errors, the variance-covariance matrix of the errors and the factors. Shanken and Zhou (2007) reformulate this problem in terms of eigenvalues in order to study the distribution of the LRT statistic. MacKinlay and Richardson (1991) and Zhou (1994) relax the assumption of normality of asset returns and provide alternative GMM tests of the CAPM.

1.4 What do we do in this thesis?

We develop an econometric methodology to infer the time-varying behaviour of equity risk premia from large stock returns databases under conditional linear factor models. In contrast to the classical setting, we estimate time-varying risk premia on a large dataset of individual stocks returns, with large cross-sectional and time-series dimensions. Using a large number of individual stocks instead of grouping assets in portfolios helps to avoid the data-snooping bias as described in Lo and MacKinlay (1990). Indeed, aggregation can mask the factor structure of asset returns and lead to misleading result on risk premia. We are not the first to advocate the use of individual stocks (e.g., Litzenberger and Ramaswamy (1979), Berk (2000), Avramov and Chordia (2006), Conrad, Cooper and Kaul (2003), Phalippou (2007), Lewellen, Nagel and Shanken (2010)). Connor and Korajczyk (1988) use a large panel of individual stock returns to extract pervasive latent factors. More recently, Ang, Liu and Schwarz (2008) argue that a lot of efficiency may be lost when only considering portfolios as base assets, instead of individual stocks, to estimate equity risk premia in unconditional models. Compared to Ang, Liu and Schwarz (2008), we consider a modeling framework that is closer to the empirical features of stock returns data, and we provide a more in-depth study of the statistical properties of the estimators.

Our theoretical contributions are threefold. First, we introduce a multi-period economy à la Hansen and Richard (1987) with an approximate factor structure and a continuum of assets. We show that the absence of asymptotic arbitrage opportunities in such an economy implies an empirically testable asset pricing restriction. We formalize the sampling scheme so that the assets in the sample are random draws from an underlying population (Andrews (2005)). This ensures that cross-sectional limits exist and are invariant to reordering of the assets. Such a construction is close to the setting advocated by Al-Najjar (1995, 1998, 1999) in a static framework with exact factor structure. The model accommodates conditional heteroskedasticity as well as weak cross-sectional dependence in the error terms (see Petersen (2008) for stressing the importance of residual dependence when computing standard errors in finance panel data).

Second, we derive a new weighted two-pass cross-sectional estimator of the path over time of the risk premia from large unbalanced panels of excess returns. We relate to the two-pass regression approach that is simple and particularly easy to implement in our framework. Indeed, this approach can be easily extended to accommodate unbalanced characteristics of panel data, i.e., the panel contains missing data (see e.g., Connor and Korajczyk (1987)). This requirement is useful in our framework because, using a dataset of individual stocks returns, we do not observe asset returns for all firms at all dates. Moreover, the two-pass methodology can be applied using any cross-sectional dimension. This characteristic justifies our choice over more efficient, but numerically intractable, estimation methodologies. The first pass consists in computing the time-series OLS estimator in order to estimate the intercept and the factor loadings of the linear factor model for the excess returns. The second pass consists in computing the cross-sectional WLS estimator by regressing the estimated intercept on the estimated factor loadings. The vector of risk premia is the sum of the second pass estimate and the time average of the factors. We study the large sample properties of the estimators using a different asymptotic scheme from the classical theory in order to match the large dimensions of the dataset. Indeed, we study the large sample properties of our estimators applying a simultaneous asymptotics for n and T tending to infinity. From this point of view, our approach is methodologically related to the recent literature developed by Stock and Watson (2002a,b), Bai (2003, 2009), Bai and Ng (2002, 2006), Forni, Hallin, Lippi and Reichlin (2000, 2004, 2005), Pesaran (2006). These authors try to extract information on the unobservable common factors from large panel data. Bai and Ng (2002) introduce a linear factor model that accounts for heteroskedasticity in both the time and cross-section dimensions, and accommodates weak serial and cross-section dependence. They provide estimators for the factor values and the factor loadings. Bai (2003) derives the large-sample properties of these estimators when both n and T are large. In contrast to this literature, we assume observable factors and focus on the estimation of the risk premia. We also relate the results to bias-corrected estimation (Hahn and Kuersteiner (2002), Hahn and Newey (2004), Hahn and Kuersteiner (2011)) accounting for the well-known incidental parameter problem in the panel literature (Neyman and Scott (1948)). We derive all properties for unbalanced panels to avoid the survivorship bias inherent to studies restricted to balanced subsets of available stock return databases (Brown, Goetzmann, Ross (1995)).

Third, we provide a test of the asset pricing restrictions for the unconditional factor model underlying the estimation. The test exploits the asymptotic distribution of the weighted sum of squared residuals of the second-pass cross-sectional regression when n and T go to infinity. A consistent estimator for a large-dimensional sparse covariance matrix is necessary to compute the statistic. We use a thresholded estimator along the lines of Bickel and Levina (2008), El Karoui (2008) and Fan, Liao, and Mincheva (2011). We do not attempt to apply the GRS statistics because of the large cross-sectional dimension. Numerical inversion of the $n \times n$ matrix Ω is too unstable.

The outline of the thesis is as follows. In Chapter 2, we consider an unconditional linear factor model and we illustrate our theoretical contributions in a simple setting. In Chapter 3, we extend all theoretical results to cover conditional linear factor models. The conditioning information set includes instruments common to all assets, e.g., macroeconomic variables, and asset specific instruments, e.g., firm characteristics and stock returns. To make estimation feasible, we assume that the factor loadings are a linear function of the lagged instruments, and risk premia are a linear function of lagged common instruments. Through an appropriate redefinition of the parameters and explanatory variables, the conditional factor model can be rewritten as a Seemingly Unrelated Regression (SUR) model where the regressors are stock-specific. This allows us to adapt the methodology used for the unconditional factor model. In Chapter 4, we provide an empirical analysis on a dataset of U.S. stock returns. We consider the Center for Research in Security Prices (CRSP) database and take the Compustat database to match firm characteristics. The merged dataset comprises about ten thousands stocks with monthly returns from July 1964 to December 2009. We look at factor models popular in the empirical finance literature to explain monthly equity returns. They differ by the choice of the factors. The first model is the CAPM (Sharpe (1964), Lintner (1965)) using market return as the single factor. Then, we consider the three-factor model of Fama and French (1993) based on two additional factors capturing the book-to-market and size effects, and a four-factor extension including a momentum factor (Jegadeesh and Titman (1993), Carhart (1997)). We study both unconditional and conditional factor models. For the conditional versions, we use both macrovariables and firm characteristics as instruments. The estimated paths show that the risk premia are large and volatile in crisis periods, e.g., the oil crisis in 1973-1974, the market crash in October 1987, and the recent financial crisis. Furthermore, the conditional risk premia estimates exhibit large positive and negative strays from unconditional estimates, and follow the macroeconomic cycles. We compare the results obtained from our large dataset of individual stock returns with those obtained with standard datasets of portfolios returns, namely the 25 and 100 Fama-French portfolios. We observe a disagreement between results obtained with portfolios and individual stocks in terms of magnitude, sign and dynamic of the risk premia. The asset pricing restrictions are rejected for a conditional four-factor model capturing market, size, value and momentum effects.

We show our Monte Carlo simulation results in Chapter 5 and robustness checks in Chapter 6. In Appendix A, we gather the technical assumptions and proofs of Propositions and some Lemmas. We use high-level assumptions to get our results and show in Appendix A.4 that we meet all of them under a block cross-sectional dependence structure on the error terms in a serially i.i.d. framework. We place all proofs of technical lemmas in Appendix B. Finally, our approach permits inference for the cost of equity on individual stocks, in a time-varying setting (Fama and French (1997)). We know from standard textbooks in corporate finance that cost of equity = risk free rate + factor loadings $\times$ factor risk premia. It is part of the cost of capital and is a central piece for evaluating investment projects by company managers. Therefore, we also include some empirical results on the cost of equity in Appendix B.2.

Chapter 2

Unconditional factor model

In this chapter, we consider an unconditional linear factor model in order to illustrate the main contributions of the thesis in a simple setting. This covers the CAPM where the single factor is the excess market return.

2.1 Excess return generation and asset pricing restrictions

We start by describing the generating process for the excess returns before examining the implications of absence of arbitrage opportunities in terms of model restrictions. We combine the constructions of Hansen and Richard (1987) and Andrews (2005) to define a multi-period economy with a continuum of assets having strictly stationary and ergodic return processes. We use such a formal construction to guarantee that (i) the economy is invariant to time shifts, so that we can establish all properties by working at $t = 1$, (ii) time series averages converge almost surely to population expectations, (iii) under a suitable sampling mechanism (see the next section), cross-sectional limits exist and are invariant to reordering of the assets, (iv) the derived no-arbitrage restriction is empirically testable. This construction allows reconciling finance and econometric analysis in a coherent framework.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The random vector f admitting values in $\mathbb{R}^K$, and the collection of random variables $\varepsilon(\gamma)$, $\gamma \in [0, 1]$, are defined on this probability space. Moreover, let $\beta = (a, b')'$ be a vector function defined on $[0, 1]$ with values in $\mathbb{R} \times \mathbb{R}^K$. The dynamics is described by the measurable time-shift transformation S mapping Ω into itself. If $\omega \in \Omega$ is the state of the world at time 0, then $S^t(\omega)$ is the state at time t, where S^t denotes the transformation S applied t times successively. Transformation S is assumed to be measure-preserving and ergodic (i.e., any set in $\mathcal{F}$ invariant under S has measure either 1, or 0).

Assumption APR.1 *The excess returns $R_t(\gamma)$ of asset $\gamma \in [0,1]$ at dates $t = 1, 2, ...$ satisfy the unconditional linear factor model:*

$$R_t(\gamma) = a(\gamma) + b(\gamma)' f_t + \varepsilon_t(\gamma), \tag{2.1}$$

where the random variables $\varepsilon_t(\gamma)$ and f_t are defined by $\varepsilon_t(\gamma, \omega) = \varepsilon[\gamma, S^t(\omega)]$ and $f_t(\omega) = f[S^t(\omega)]$.

Assumption APR.1 defines the excess return processes for an economy with a continuum of assets. The index set is the interval $[0,1]$ without loss of generality. Vector f_t gathers the values of the K observable factors at date t, while the intercept $a(\gamma)$ and factor sensitivities $b(\gamma)$ of asset $\gamma \in [0,1]$ are time-invariant. Since transformation S is measure-preserving and ergodic, all processes are strictly stationary and ergodic (Doob (1953)). Let further define $x_t = (1, f_t')'$ which yields the compact formulation:

$$R_t(\gamma) = \beta(\gamma)' x_t + \varepsilon_t(\gamma). \tag{2.2}$$

In order to define the information sets, let $\mathcal{F}_0 \subset \mathcal{F}$ be a sub sigma-field. We assume that random vector f is measurable w.r.t. $\mathcal{F}_0$. Define $\mathcal{F}_t = \{S^{-t}(A), A \in \mathcal{F}_0\}$, $t = 1, 2, ...$, through the inverse mapping S^{-t} and assume that $\mathcal{F}_1$ contains $\mathcal{F}_0$. Then, the filtration $\mathcal{F}_t$, $t = 1, 2, ...$, characterizes the flow of information available to investors.

Let us now introduce supplementary assumptions on factors, factor loadings, and error terms.

Assumption APR.2 *The matrix $\int b(\gamma) b(\gamma)' d\gamma$ is positive definite.*

Assumption APR.2 implies non-degeneracy in the factor loadings across assets.

Assumption APR.3 *For any $\gamma \in [0,1]$, $E[\varepsilon_t(\gamma)|\mathcal{F}_{t-1}] = 0$ and $Cov[\varepsilon_t(\gamma), f_t|\mathcal{F}_{t-1}] = 0$.*

Hence, the error terms have mean zero and are uncorrelated with the factors conditionally on information $\mathcal{F}_{t-1}$. In Assumption APR.4 (i) below, we impose an approximate factor structure for the conditional distribution of the error terms given $\mathcal{F}_{t-1}$ in almost any countable collection of assets. More precisely, for any sequence (γ_i) in $[0,1]$, let $\Sigma_{\varepsilon,t,n}$ denote the $n \times n$ conditional variance-covariance matrix of the error vector

$[\varepsilon_t(\gamma_1), ..., \varepsilon_t(\gamma_n)]'$ given $\mathcal{F}_{t-1}$, for $n \in \mathbb{N}$. Let μ_Γ be the probability measure on the set $\Gamma = [0,1]^{\mathbb{N}}$ of sequences (γ_i) in $[0,1]$ induced by i.i.d. random sampling from a continuous distribution G with support $[0,1]$.

Assumption APR.4 *For any sequence (γ_i) in set $\mathcal{J}$: (i) $eig_{\max}(\Sigma_{\varepsilon,t,n}) = o(n)$, as $n \to \infty$, P-a.s., (ii) $\inf_{n \geq 1} eig_{\min}(\Sigma_{\varepsilon,t,n}) > 0$, P-a.s., where $\mathcal{J} \subset \Gamma$ is such that $\mu_\Gamma(\mathcal{J}) = 1$, and $eig_{\min}(\Sigma_{\varepsilon,t,n})$ and $eig_{\max}(\Sigma_{\varepsilon,t,n})$ denote the smallest and the largest eigenvalues of matrix $\Sigma_{\varepsilon,t,n}$, (iii) $eig_{\min}(V[f_t|\mathcal{F}_{t-1}]) > 0$, P-a.s.*

Assumption APR.4 (i) is weaker than boundedness of the largest eigenvalue, i.e., $\sup_{n\geq 1} eig_{\max}(\Sigma_{\varepsilon,t,n}) < \infty$, P-a.s., as in CR. This is useful for the checks of Appendix A.4 under a block cross-sectional dependence structure. Assumptions APR.4 (ii)-(iii) are mild regularity conditions for the proof of Proposition 1.

Absence of asymptotic arbitrage opportunities generates asset pricing restrictions in large economies (Ross (1976), CR). We define asymptotic arbitrage opportunities in terms of sequences of portfolios p_n, $n \in \mathbb{N}$. Portfolio p_n is defined by the share $\alpha_{0,n}$ invested in the riskfree asset and the shares $\alpha_{i,n}$ invested in the selected risky assets γ_i, for $i = 1, ..., n$. The shares are measurable w.r.t. $\mathcal{F}_0$. Then, $C(p_n) = \sum_{i=0}^{n} \alpha_{i,n}$ is the portfolio cost at $t = 0$, and $p_n = C(p_n)R_0 + \sum_{i=1}^{n} \alpha_{i,n} R_1(\gamma_i)$ is the portfolio payoff at $t = 1$, where R_0 denotes the riskfree gross return measurable w.r.t. $\mathcal{F}_0$. We can work with $t = 1$ because of stationarity.

Assumption APR.5 *There are no asymptotic arbitrage opportunities in the economy, that is, there exists no portfolio sequence (p_n) such that $\lim_{n\to\infty} \mathbb{P}[p_n \geq 0] = 1$ and $\lim_{n\to\infty} \mathbb{P}[C(p_n) \leq 0, p_n > 0] > 0$.*

Assumption APR.5 excludes portfolios that approximate arbitrage opportunities when the number of included assets increases. Arbitrage opportunities are investments with non-negative payoff in each state of the world, and with non-positive cost and positive payoff in some states of the world as in Hansen and Richard (1987), Definition 2.4. Then, Proposition 1 gives the asset pricing restriction.

Proposition 1 *Under Assumptions APR.1-APR.5, there exists a unique vector $\nu \in \mathbb{R}^K$ such that*

$$a(\gamma) = b(\gamma)'\nu, \tag{2.3}$$

for almost all $\gamma \in [0,1]$.

We can rewrite the asset pricing restriction as

$$E\left[R_t(\gamma)\right] = b(\gamma)'\lambda, \tag{2.4}$$

for almost all $\gamma \in [0,1]$, where $\lambda = \nu + E\left[f_t\right]$ is the vector of the risk premia. In the CAPM, we have $K = 1$ and $\nu = 0$. When a factor $f_{k,t}$ is a portfolio excess return, we also have $\nu_k = 0$, $k = 1, ..., K$.

Proposition 1 is already stated by Al-Najjar (1998) Proposition 2 for a strict factor structure in an unconditional economy (static case) with the definition of arbitrage as in CR. We extend his result to an approximate factor structure in a conditional economy (dynamic case) with the definition of arbitrage as in Hansen and Richard (1987). Proposition 1 differs from CR Theorem 3 in terms of the returns generating framework, the definition of asymptotic arbitrage opportunities, and the derived asset pricing restriction. Specifically, we consider a multi-period economy with conditional information as opposed to a single period unconditional economy as in CR. We extend such a setting to time varying risk premia in Chapter 3. We prefer the definition underlying Assumption APR.5 since it corresponds to the definition of arbitrage that is standard in dynamic asset pricing theory (e.g., Duffie (2001)). As pointed out by Hansen and Richard (1987), Ross (1978) has already chosen that type of definition. It also eases the proof based on new arguments. However, in Appendix A.2, we derive the link between the no-arbitrage conditions in Assumptions A.1 i) and ii) of CR, written P-a.s. w.r.t. the conditional information $\mathcal{F}_0$ and for almost every countable collection of assets, and the asset pricing restriction (2.3) valid for the continuum of assets. Hence, we are able to characterize the functions $\beta = (a, b')'$ defined on $[0,1]$ that are compatible with absence of asymptotic arbitrage opportunities under both definitions of arbitrage in the continuum economy. CR derive the pricing restriction $\sum_{i=1}^{\infty}\left(a(\gamma_i) - b(\gamma_i)'\nu\right)^2 < \infty$, for some $\nu \in \mathbb{R}^K$ and for a given sequence (γ_i), while we derive the restriction (2.3), for almost all $\gamma \in [0,1]$. In Appendix A.2, we show that the set of sequences (γ_i) such that $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty}\left(a(\gamma_i) - b(\gamma_i)'\nu\right)^2 < \infty$ has measure 1 under μ_Γ, when the asset pricing restriction (2.3) holds, and measure 0, otherwise. This result is a consequence of the Kolmogorov zero-one law (see e.g. Billingsley (1995)). In other words, validity of the summability condition in CR for a countable collection of assets without validity of the asset pricing restriction (2.3) is an impossible event. From the proofs in Appendix A.2, we also get a reverse implication compared to Proposition 1: when the asset pricing restriction (2.3) does not hold, asymptotic arbitrage in the sense of Assumption APR.5, or of Assumptions A.1 i) and ii) of

CR, exists for μ_Γ-almost any countable collection of assets. The restriction in Proposition 1 is testable with large equity datasets and large sample sizes (Section 2.5). Therefore, we are not affected by the Shanken (1982) critique, namely the problem that finiteness of the sum $\sum_{i=1}^{\infty}\left(a(\gamma_i)-b(\gamma_i)'\nu\right)^2$ for a given countable economy cannot be tested empirically. The next section describes how we get the data from sampling the continuum of assets.

2.2 The sampling scheme

We estimate the risk premia from a sample of observations on returns and factors for n assets and T dates. In available databases, we do not observe asset returns for all firms at all dates. We account for the unbalanced nature of the panel through a collection of indicator variables $I(\gamma)$, $\gamma \in [0,1]$, and define $I_t(\gamma,\omega) = I[\gamma, S^t(\omega)]$. Then $I_t(\gamma) = 1$ if the return of asset γ is observable by the econometrician at date t, and 0 otherwise (Connor and Korajczyk (1987)). To keep the factor structure linear, we assume a missing-at-random design (Rubin (1976)), that is, independence between unobservability and returns generation.

Assumption SC.1 *The random variables $I_t(\gamma)$, $\gamma \in [0,1]$, are independent of $\varepsilon_t(\gamma)$, $\gamma \in [0,1]$, and f_t.*

Another design would require an explicit modeling of the link between the unobservability mechanism and the returns process of the continuum of assets (Heckman (1979)); this would yield a nonlinear factor structure.

Assets are randomly drawn from the population according to a probability distribution G on $[0,1]$. We use a single distribution G in order to avoid the notational burden when working with different distributions on different subintervals of $[0,1]$.

Assumption SC.2 *The random variables γ_i, $i = 1,...,n$, are i.i.d. indices, independent of $\varepsilon_t(\gamma)$, $I_t(\gamma)$, $\gamma \in [0,1]$ and f_t, each with continuous distribution G with support $[0,1]$.*

For any $n, T \in \mathbb{N}$, the excess returns are $R_{i,t} = R_t(\gamma_i)$ and the observability indicators are $I_{i,t} = I_t(\gamma_i)$, for $i = 1,...,n$, and $t = 1,...,T$. The excess return $R_{i,t}$ is observed if and only if $I_{i,t} = 1$. Similarly, let $\beta_i = \beta(\gamma_i) = (a_i, b_i')'$ be the characteristics, $\varepsilon_{i,t} = \varepsilon_t(\gamma_i)$ the error terms, and $\sigma_{ij,t} = E[\varepsilon_{i,t}\varepsilon_{j,t}|x_{\underline{t}}, \gamma_i, \gamma_j]$ the conditional variances and covariances of the assets in the sample, where $x_{\underline{t}} = \{x_t, x_{t-1}, ...\}$. By random sampling, we get a random coefficient panel model (e.g. Hsiao (2003), Chapter 6). The characteristic β_i of asset i is random, and potentially correlated with the error terms $\varepsilon_{i,t}$ and the observability indicators $I_{i,t}$, as well as the conditional variances $\sigma_{ii,t}$, through the index γ_i. If the a_is and b_is were treated as given parameters, and not as realizations of random variables, invoking cross-sectional LLNs and CLTs as in some assumptions and parts of the proofs would have no sense. Moreover, cross-sectional limits would be dependent on the selected ordering of the assets. Instead, our assumptions and results do not rely on a specific ordering of assets. Random elements $(\beta_i', \sigma_{ii,t}, \varepsilon_{i,t}, I_{i,t})'$, $i = 1, ..., n$, are exchangeable (Andrews (2005)). Hence, assets randomly drawn from the population have ex-ante the same features. However, given a specific realization of the indices in the sample, assets have ex-post heterogeneous features.

2.3 Asymptotic properties of risk premium estimation

We consider a two-pass approach (Fama and MacBeth (1973), Black, Jensen and Scholes (1972)) building on Equations (2.1) and (2.3).

First Pass: The first pass consists in computing time-series OLS estimators $\hat{\beta}_i = (\hat{a}_i, \hat{b}_i')' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_i I_{i,t} x_t R_{i,t,}$ for $i = 1, ..., n$, where $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_t x_t'$ and $T_i = \sum_t I_{i,t}$. In available panels, the random sample size T_i for asset i can be small, and the inversion of matrix $\hat{Q}_{x,i}$ can be numerically unstable. This can yield unreliable estimates of β_i. To address this, we introduce a trimming device: $\mathbf{1}_i^{\chi} = \mathbf{1}\left\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}, \tau_{i,T} \leq \chi_{2,T}\right\}$, where $CN\left(\hat{Q}_{x,i}\right) = \sqrt{eig_{\max}\left(\hat{Q}_{x,i}\right) / eig_{\min}\left(\hat{Q}_{x,i}\right)}$ denotes the condition number of matrix $\hat{Q}_{x,i}$, $\tau_{i,T} = T/T_i$, and the two sequences $\chi_{1,T} > 0$ and $\chi_{2,T} > 0$ diverge asymptotically. The first trimming condition $\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}\}$ keeps in the cross-section only assets for which the time series regression is not too badly conditioned. A too large value of $CN\left(\hat{Q}_{x,i}\right)$ indicates multicollinearity problems and ill-conditioning (Belsley, Kuh, and Welsch (2004), Greene (2008)). The second trimming condition $\{\tau_{i,T} \leq \chi_{2,T}\}$ keeps in the cross-section only assets for which the time series is not too short. We use both trimming conditions in the proofs of the asymptotic results.

Second Pass: The second pass consists in computing a cross-sectional estimator of ν by regressing the $\hat{a}_i$s on the $\hat{b}_i$s keeping the non-trimmed assets only. We use a WLS approach. The weights are estimates of $w_i = v_i^{-1}$, where the v_i are the asymptotic variances of the standardized errors $\sqrt{T}\left(\hat{a}_i - \hat{b}_i'\nu\right)$ in the cross-sectional regression for large T. We have $v_i = \tau_i c_\nu' Q_x^{-1} S_{ii} Q_x^{-1} c_\nu$, where $Q_x = E\left[x_t x_t'\right]$,

$S_{ii} = \underset{T\to\infty}{\text{plim}} \frac{1}{T}\sum_t \sigma_{ii,t} x_t x_t' = E\left[\varepsilon_{i,t}^2 x_t x_t' | \gamma_i\right]$, $\tau_i = \underset{T\to\infty}{\text{plim}}\, \tau_{i,T} = E[I_{i,t}|\gamma_i]^{-1}$, and $c_\nu = (1, -\nu')'$. We use the estimates $\hat{v}_i = \tau_{i,T} c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1}$, where $\hat{S}_{ii} = \frac{1}{T_i}\sum_t I_{i,t}\hat{\varepsilon}_{i,t}^2 x_t x_t'$, $\hat{\varepsilon}_{i,t} = R_{i,t} - \hat{\beta}_i' x_t$ and $c_{\hat{\nu}_1} = (1, -\hat{\nu}_1')'$. To estimate c_ν, we use the OLS estimator $\hat{\nu}_1 = \left(\sum_i \mathbf{1}_i^\chi \hat{b}_i \hat{b}_i'\right)^{-1} \sum_i \mathbf{1}_i^\chi \hat{b}_i \hat{a}_i$, i.e., a first-step estimator with unit weights. The WLS estimator is:

$$\hat{\nu} = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{a}_i, \tag{2.5}$$

where $\hat{Q}_b = \frac{1}{n}\sum_i \hat{w}_i \hat{b}_i \hat{b}_i'$ and $\hat{w}_i = \mathbf{1}_i^\chi \hat{v}_i^{-1}$. Weighting accounts for the statistical precision of the first-pass estimates. Under conditional homoskedasticity $\sigma_{ii,t} = \sigma_{ii}$ and a balanced panel $\tau_{i,T} = 1$, we have $v_i = c_\nu' Q_x^{-1} c_\nu \sigma_{ii}$. There, v_i is directly proportional to σ_{ii}, and we can simply pick the weights as $\hat{w}_i = \hat{\sigma}_{ii}^{-1}$, where $\hat{\sigma}_{ii} = \frac{1}{T}\sum_t \hat{\varepsilon}_{i,t}^2$ (Shanken (1992)). The final estimator of the risk premia vector is

$$\hat{\lambda} = \hat{\nu} + \frac{1}{T}\sum_t f_t. \tag{2.6}$$

We can avoid the trimming on the condition number if we substitute $\hat{Q}_x^{-1}$ for $\hat{Q}_{x,i}^{-1}$ in the first-pass estimator definition. However, this increases the asymptotic variance of the bias corrected estimator of ν, and does not extend to the conditional case. Starting from the asset pricing restriction (2.4), another estimator of λ is $\bar{\lambda} = \hat{Q}_b^{-1} \frac{1}{n}\sum_i \hat{w}_i \hat{b}_i \bar{R}_i$, where $\bar{R}_i = \frac{1}{T_i}\sum_t I_{i,t} R_{i,t}$. This estimator is numerically equivalent to $\hat{\lambda}$ in the balanced case, where $I_{i,t} = 1$ for all i and t. In the unbalanced case, it is equal to $\bar{\lambda} = \hat{\nu} + \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i \hat{b}_i \hat{b}_i' \bar{f}_i$, where $\bar{f}_i = \frac{1}{T_i}\sum_t I_{i,t} f_t$. Estimator $\bar{\lambda}$ is often studied by the literature (see, e.g., Shanken (1992), Kandel and Stambaugh (1995), Jagannathan and Wang (1998)), and is also consistent. Estimating $E[f_t]$ with a simple average of the observed factor instead of a weighted average based on estimated betas simplifies the form of the asymptotic distribution in the unbalanced case (see below and Section 2.4). This explains our preference for $\hat{\lambda}$ over $\bar{\lambda}$.

We derive the asymptotic properties under assumptions on the conditional distribution of the error terms.

Assumption A.1 *There exists a positive constant M such that for all n:*

a) $E\left[\varepsilon_{i,t}|\{\varepsilon_{j,\underline{t-1}},\gamma_j, j=1,...,n\}, x_{\underline{t}}\right]=0$, *with* $\varepsilon_{j,\underline{t-1}}=\{\varepsilon_{j,t-1},\varepsilon_{j,t-2},\cdots\}$ *and* $x_{\underline{t}}=\{x_t,x_{t-1},\cdots\}$; *b)* $\frac{1}{M}\leq\sigma_{ii,t}\leq M,\ i=1,...,n$; *c)* $E\left[\frac{1}{n}\sum_{i,j}E\left[\sigma_{ij,t}^2|\gamma_i,\gamma_j\right]^{1/2}\right]\leq M$, *where* $\sigma_{ij,t}=E\left[\varepsilon_{i,t}\varepsilon_{j,t}|x_{\underline{t}},\gamma_i,\gamma_j\right]$.

Assumption A.1 allows for a martingale difference sequence for the error terms (part a)) including potential conditional heteroskedasticity (part b)) as well as weak cross-sectional dependence (part c)). In particular, Assumption A.1 c) is the same as Assumption C.3 in Bai and Ng (2002), except that we have an expectation w.r.t. the random draws of assets. More general error structures are possible but complicate consistent estimation of the asymptotic variances of the estimators (see Section 2.4).

Proposition 2 summarizes consistency of estimators $\hat{\nu}$ and $\hat{\lambda}$ under the double asymptotics $n,T\to\infty$.

Proposition 2 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1 b) and C.1, C.4, C.5, we get a)* $\|\hat{\nu}-\nu\|=o_p(1)$ *and b)* $\left\|\hat{\lambda}-\lambda\right\|=o_p(1)$, *when* $n,T\to\infty$ *such that* $n=O\left(T^{\bar{\gamma}}\right)$ *for* $\bar{\gamma}>0$.

Consistency of the estimators holds under double asymptotics such that the cross-sectional size n grows not faster than a power of the time series size T. For instance, the conditions in Proposition 2 allow for n large w.r.t. T (short panel asymptotics) when $\bar{\gamma}>1$. Shanken (1992) shows consistency of $\hat{\nu}$ and $\hat{\lambda}$ for a fixed n and $T\to\infty$. This consistency does not imply Proposition 2. Shanken (1992) (see also Litzenberger and Ramaswamy (1979)) further shows that we can estimate ν consistently in the second pass with a modified cross-sectional estimator for a fixed T and $n\to\infty$. Since $\lambda=\nu+E[f_t]$, consistent estimation of the risk premia themselves is impossible for a fixed T (see Shanken (1992) for the same point).

Proposition 3 below gives the large-sample distributions under the double asymptotics $n,T\to\infty$. Let us define $\tau_{ij,T}=T/T_{ij}$, where $T_{ij}=\sum_t I_{ij,t}$ and $I_{ij,t}=I_{i,t}I_{j,t}$ for $i,j=1,...,n$. Let us further define $\tau_{ij}=\underset{T\to\infty}{\text{plim}}\,\tau_{ij,T}=E[I_{ij,t}|\gamma_i,\gamma_j]^{-1}$, $S_{ij}=\underset{T\to\infty}{\text{plim}}\,\frac{1}{T}\sum_t\sigma_{ij,t}x_tx_t'=E[\varepsilon_{i,t}\varepsilon_{j,t}x_tx_t'|\gamma_i,\gamma_j]$ and $Q_b=\underset{n\to\infty}{\text{plim}}\,\frac{1}{n}\sum_i w_ib_ib_i'=E[w_ib_ib_i']$. The following assumption describes the CLTs underlying the proof of the distributional properties.

Assumption A.2 *As* $n,T\to\infty$, *a)* $\frac{1}{\sqrt{n}}\sum_i w_i\tau_i\left(Y_{i,T}\otimes b_i\right)\Rightarrow N\left(0,S_b\right)$, *where* $Y_{i,T}=\frac{1}{\sqrt{T}}\sum_t I_{i,t}x_t\varepsilon_{i,t}$ *and* $S_b=\lim_{n\to\infty}E\left[\frac{1}{n}\sum_{i,j}w_iw_j\frac{\tau_i\tau_j}{\tau_{ij}}S_{ij}\otimes b_ib_j'\right]=$ *a.s.* $-\lim_{n\to\infty}\frac{1}{n}\sum_{i,j}w_iw_j\frac{\tau_i\tau_j}{\tau_{ij}}S_{ij}\otimes b_ib_j'$;

b) $\frac{1}{\sqrt{T}}\sum_t \left(f_t - E\left[f_t\right]\right) \Rightarrow N\left(0,\Sigma_f\right)$, *where* $\Sigma_f = \lim_{T\to\infty}\frac{1}{T}\sum_{t,s} Cov\left(f_t, f_s\right)$.

Assumptions A.2a) and b) require the asymptotic normality of cross-sectional and time series averages of scaled error terms, and of time-series averages of factor values, respectively. These CLTs hold under weak serial and cross-sectional dependencies such as temporal mixing and block dependence (see Appendix A.4).

Assumption A.3 *For any* $1 \leq t, s \leq T,\ T \in \mathbb{N}$ *and* $\gamma \in [0,1]$, *we have* $E\left[\varepsilon_t(\gamma)^2\varepsilon_s(\gamma)|x_{\underline{T}}\right] = 0$.

Assumption A.3 is a symmetry condition on the error distribution given the factors. It is used to prove that the sampling variability of the estimated weights $\hat{w}_i$ does not impact the asymptotic distribution of estimator $\hat{\nu}$. Our setting differs from the standard feasible WLS framework since we have to estimate each incidental parameter S_{ii}. We can dispense with Assumption A.3 if we use OLS to estimate parameter ν, i.e., estimator $\hat{\nu}_1$, or if we put a more restrictive condition on the relative rate of n w.r.t. T.

Proposition 3 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.3, and C.1-C.5, we get:*
a) $\sqrt{nT}\left(\hat{\nu} - \nu - \frac{1}{T}\hat{B}_\nu\right) \Rightarrow N\left(0,\Sigma_\nu\right)$, *with* $\Sigma_\nu = a.s.\text{-}\lim_{n\to\infty} Q_b^{-1}\left(\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}}(c_\nu' Q_x^{-1} S_{ij} Q_x^{-1} c_\nu) b_i b_j'\right) Q_b^{-1}$
and the bias term is $\hat{B}_\nu = \hat{Q}_b^{-1}\left(\frac{1}{n}\sum_i \hat{w}_i \tau_{i,T} E_2' \hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1} c_{\hat{\nu}}\right)$, *with* $E_2 = (0 : I_K)'$, $c_{\hat{\nu}} = (1, -\hat{\nu}')'$, *and*
b) $\sqrt{T}\left(\hat{\lambda} - \lambda\right) \Rightarrow N\left(0,\Sigma_f\right)$, *when* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $0 < \bar{\gamma} < 3$.

The asymptotic variance matrix in Proposition 3 can be rewritten as:

$$\Sigma_\nu = a.s.\text{-}\lim_{n\to\infty}\Sigma_{\nu,n}, \qquad \Sigma_{\nu,n} := \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}\frac{1}{n}B_n' W_n V_n W_n B_n\left(\frac{1}{n}B_n' W_n B_n\right)^{-1}, \tag{2.7}$$

where $B_n = (b_1, ..., b_n)'$, $W_n = diag(w_1, ..., w_n)$ and $V_n = [v_{ij}]_{i,j=1,\ldots,n}$ with $v_{ij} = \frac{\tau_i\tau_j}{\tau_{ij}} c_\nu' Q_x^{-1} S_{ij} Q_x^{-1} c_\nu$, which gives $v_{ii} = v_i$. In the homoskedastic and balanced case, we have $c_\nu' Q_x^{-1} c_\nu = 1 + \lambda' V[f_t]^{-1}\lambda$ and $V_n = (1 + \lambda' V[f_t]^{-1}\lambda)\Sigma_{\varepsilon,n}$, where $\Sigma_{\varepsilon,n} = [\sigma_{ij}]_{i,j=1,\ldots,n}$. Then, the asymptotic variance of $\hat{\nu}$ reduces to a.s.-$\lim_{n\to\infty}(1 + \lambda' V[f_t]^{-1}\lambda)\left(\frac{1}{n}B_n' W_n B_n\right)^{-1}\frac{1}{n}B_n' W_n \Sigma_{\varepsilon,n} W_n B_n\left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$. In particular, in the CAPM, we have $K = 1$ and $\nu = 0$, which implies that $\sqrt{\lambda^2/V[f_t]}$ is equal to the slope of the Capital Market Line $\sqrt{E[f_t]^2/V[f_t]}$, i.e., the Sharpe Ratio of the market portfolio.

Proposition 3 shows that the estimator $\hat{\nu}$ has a fast convergence rate $\sqrt{nT}$ and features an asymptotic bias term. Both $\hat{a}_i$ and $\hat{b}_i$ in the definition of $\hat{\nu}$ contain an estimation error; for $\hat{b}_i$, this is the well-known

Error-In-Variable (EIV) problem. The EIV problem does not impede consistency since we let T grow to infinity. However, it induces the bias term $\hat{B}_\nu/T$ which centers the asymptotic distribution of $\hat{\nu}$. The upper bound on the relative expansion rates of n and T in Proposition 3 is $n = O(T^{\bar{\gamma}})$ for $\bar{\gamma} < 3$. The control of first-pass estimation errors uniformly across assets requires that the cross-section dimension n is not too large w.r.t. the time series dimension T.

If we knew the true factor mean, for example $E[f_t] = 0$, and did not need to estimate it, the estimator $\hat{\nu} + E[f_t]$ of the risk premia would have the same fast rate $\sqrt{nT}$ as the estimator of ν, and would inherit its asymptotic distribution. Since we do not know the true factor mean, only the variability of the factor drives the asymptotic distribution of $\hat{\lambda}$, since the estimation error $O_p\left(1/\sqrt{T}\right)$ of the sample average $\frac{1}{T}\sum_t f_t$ dominates the estimation error $O_p\left(1/\sqrt{nT} + 1/T\right)$ of $\hat{\nu}$. This result is an oracle property for $\hat{\lambda}$, namely that its asymptotic distribution is the same irrespective of the knowledge of ν. This property is in sharp difference with the single asymptotics with a fixed n and $T \to \infty$. In the balanced case and with homoskedastic errors, Theorem 1 of Shanken (1992) shows that the rate of convergence of $\hat{\lambda}$ is $\sqrt{T}$ and that its asymptotic variance is $\Sigma_{\lambda,n} = \Sigma_f + \frac{1}{n}(1 + \lambda' V[f_t]^{-1}\lambda)\left(\frac{1}{n}B_n' W_n B_n\right)^{-1}\frac{1}{n}B_n' W_n \Sigma_{\varepsilon,n} W_n B_n \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$, for fixed n and $T \to \infty$. The two components in $\Sigma_{\lambda,n}$ come from estimation of $E[f_t]$ and ν, respectively. In the heteroskedastic setting with fixed n, a slight extension of Theorem 1 in Jagannathan and Wang (1998), or Theorem 3.2 in Jagannathan, Skoulakis, and Wang (2009), to the unbalanced case yields $\Sigma_{\lambda,n} = \Sigma_f + \frac{1}{n}\Sigma_{\nu,n}$, where $\Sigma_{\nu,n}$ is defined in (2.7). Letting $n \to \infty$ gives Σ_f under weak cross-sectional dependence. Thus, exploiting the full cross-section of assets improves efficiency asymptotically, and the positive definite matrix $\Sigma_{\lambda,n} - \Sigma_f$ corresponds to the efficiency gain. Using a large number of assets instead of a small number of portfolios does help to eliminate the contribution coming from estimation of ν.

Proposition 3 suggests exploiting the analytical bias correction $\hat{B}_\nu/T$ and using estimator $\hat{\nu}_B = \hat{\nu} - \frac{1}{T}\hat{B}_\nu$ instead of $\hat{\nu}$. Furthermore, $\hat{\lambda}_B = \hat{\nu}_B + \frac{1}{T}\sum_t f_t$ delivers a bias-free estimator of λ at order $1/T$, which shares the same root-T asymptotic distribution as $\hat{\lambda}$.

Finally, we can relate the results of Proposition 3 to bias-corrected estimation accounting for the well-known incidental parameter problem (Neyman and Scott (1948)) in the panel literature (see Lancaster (2000) for a review). We can write model (2.1) under restriction (2.3) as $R_{i,t} = b_i'(f_t + \nu) + \varepsilon_{i,t}$. In the likelihood setting of Hahn and Newey (2004) (see also Hahn and Kuersteiner (2002)), the b_is correspond to the individidual fixed effects and ν to the common parameter of interest. Available results on the fixed-effects approach tell us: (i) the Maximum Likelihood (ML) estimator of ν is inconsistent if n goes to infinity while T is held fixed, (ii) the ML estimator of ν is asymptotically biased even if T grows at the same rate as n, (iii) an analytical bias correction may yield an estimator of ν that is root-(nT) asymptotically normal and centered at the truth if T grows faster than $n^{1/3}$. The two-pass estimators $\hat{\nu}$ and $\hat{\nu}_B$ exhibit the properties (i)-(iii) as expected by analogy with unbiased estimation in large panels. This clear link with the incidental parameter literature highlights another advantage of working with ν in the second pass regression. Chamberlain (1992) considers a general random coefficient model nesting Model (1) under restriction (3). He establishes asymptotic normality of an estimator of ν for fixed T and balanced panel data. His estimator does not admit a closed-form and requires a numerical optimization. This leads to computational difficulties in the conditional extension of Chapter 3. This also makes the study of his estimator under double asymptotics and cross-sectional dependence challenging. Recent advances on the incidental parameter problem in random coefficient models for fixed T are Arellano and Bonhomme (2012) and Bonhomme (2012).

2.4 Confidence intervals

We can use Proposition 3 to build confidence intervals by means of consistent estimation of the asymptotic variances. We can check with these intervals whether the risk of a given factor $f_{k,t}$ is not remunerated, i.e., $\lambda_k = 0$, or the restriction $\nu_k = 0$ holds when the factor is traded. We estimate Σ_f by a standard HAC estimator $\hat{\Sigma}_f$ such as in Newey and West (1994) or Andrews and Monahan (1992). Hence, the construction of confidence intervals with valid asymptotic coverage for components of $\hat{\lambda}$ is straightforward. On the contrary, getting a HAC estimator for $\bar{\Sigma}_f$ appearing in the asymptotic distribution of $\bar{\lambda}$ is not obvious in the unbalanced case.

The construction of confidence intervals for the components of $\hat{\nu}$ is more difficult. Indeed, Σ_ν involves a limiting double sum over S_{ij} scaled by n and not n^2. A naive approach consists in replacing S_{ij} by any consistent estimator such as $\hat{S}_{ij} = \frac{1}{T_{ij}} \sum_t I_{ij,t} \hat{\varepsilon}_{i,t} \hat{\varepsilon}_{j,t} x_t x_t'$, but this does not work here. To handle this, we rely on recent proposals in the statistical literature on consistent estimation of large-dimensional sparse covariance matrices by thresholding (Bickel and Levina (2008), El Karoui (2008)). Fan, Liao, and Mincheva (2011) focus on the estimation of the variance-covariance matrix of the errors in large balanced panel with nonrandom coefficients.

The idea is to assume sparse contributions of the S_{ij}s to the double sum. Then, we only have to account for sufficiently large contributions in the estimation, i.e., contributions larger than a threshold vanishing asymptotically. Thresholding permits an estimation invariant to asset permutations; the absence of any natural cross-sectional ordering among the matrices S_{ij} motivates this choice of estimator. In the following assumption, we use the notion of sparsity suggested by Bickel and Levina (2008) adapted to our framework with random coefficients.

Assumption A.4 *There exist constants* $q, \delta \in [0,1)$ *such that* $\max_i \sum_j \|S_{ij}\|^q = O_p\left(n^\delta\right)$.

Assumption A.4 tells us that we can neglect most cross-asset contributions $\|S_{ij}\|$. As sparsity increases, we can choose coefficients q and δ closer to zero. Assumption A.4 does not impose sparsity of the covariance matrix of the returns themselves. Assumption A.1 c) is also a sparsity condition, which ensures that the limit matrix Σ_ν is well-defined when combined with Assumption C.4. We meet both sparsity assumptions, as well as the approximate factor structure Assumption APR.4 (i), under weak cross-sectional dependence between the error terms, for instance, under a block dependence structure (see Appendix A.4).

As in Bickel and Levina (2008), let us introduce the thresholded estimator $\tilde{S}_{ij} = \hat{S}_{ij}\mathbf{1}\left\{\left\|\hat{S}_{ij}\right\| \geq \kappa\right\}$ of S_{ij}, which we refer to as $\hat{S}_{ij}$ thresholded at $\kappa = \kappa_{n,T}$. We can derive an asymptotically valid confidence interval for the components of $\hat{\nu}$ from the next proposition giving a feasible asymptotic normality result.

Proposition 4 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.4,C.1-C.5, we have* $\tilde{\Sigma}_\nu^{-1/2}\sqrt{nT}\left(\hat{\nu} - \frac{1}{T}\hat{B}_\nu - \nu\right) \Rightarrow N\left(0, I_K\right)$ *with* $\tilde{\Sigma}_\nu = \hat{Q}_b^{-1}\left[\frac{1}{n}\sum_{i,j}\hat{w}_i\hat{w}_j\frac{\tau_{i,T}\tau_{j,T}}{\tau_{ij,T}}(c_{\hat{\nu}}'\hat{Q}_x^{-1}\tilde{S}_{ij}\hat{Q}_x^{-1}c_{\hat{\nu}})\hat{b}_i\hat{b}_j'\right]\hat{Q}_b^{-1}$, *when* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $0 < \bar{\gamma} < \min\left\{3, \eta\frac{1-q}{2\delta}\right\}$, *and* $\kappa = M\sqrt{\frac{\log n}{T^\eta}}$ *for a constant* $M > 0$ *and* $\eta \in (0,1]$ *as in Assumption C.1.*

In Assumption C.1, we define constant $\eta \in (0,1]$ which is related to the time series dependence of processes $(\varepsilon_{i,t})$ and (x_t). We have $\eta = 1$, when $(\varepsilon_{i,t})$ and (x_t) are serially i.i.d. as in Appendix A.4 and Bickel and Levina (2008). The stronger the time series dependence (smaller η) and the lower the sparsity (q and δ closer to 1), the more restrictive the condition on the relative rate $\bar{\gamma}$. We cannot guarantee the matrix made of thresholded blocks $\tilde{S}_{ij}$ to be semi definite positive (sdp). However, we expect that the double summation on i and j makes $\tilde{\Sigma}_\nu$ sdp in empirical applications. In case it is not, El Karoui (2008) discusses a few solutions based on shrinkage.

2.5 Tests of asset pricing restrictions

The null hypothesis underlying the asset pricing restriction (2.3) is

$$\mathcal{H}_0: \text{ there exists } \nu \in \mathbb{R}^K \text{ such that } a(\gamma) = b(\gamma)'\nu, \quad \text{for almost all } \gamma \in [0,1].$$

This null hypothesis is written on the continuum of assets. Under $\mathcal{H}_0$, we have $E\left[(a_i - b_i'\nu)^2\right] = 0$. Since we estimate ν via the WLS cross-sectional regression of the estimates $\hat{a}_i$ on the estimates $\hat{b}_i$, we suggest a test based on the weighted sum of squared residuals SSR of the cross-sectional regression. The weighted SSR is $\hat{Q}_e = \frac{1}{n}\sum_i \hat{w}_i \hat{e}_i^2$, with $\hat{e}_i = c_{\hat{\nu}}'\hat{\beta}_i$, which is an empirical counterpart of $E\left[w_i\left(a_i - b_i'\nu\right)^2\right]$.

Let us define $S_{ii,T} = \frac{1}{T}\sum_t I_{i,t}\sigma_{ii,t}x_t x_t'$, and introduce the commutation matrix $W_{m,n}$ of order $mn \times mn$ such that $W_{m,n} vec\,[A] = vec\,[A']$ for any matrix $A \in \mathbb{R}^{m\times n}$, where the vector operator $vec\,[\cdot]$ stacks the elements of an $m \times n$ matrix as a $mn \times 1$ vector. If $m = n$, we write W_n instead $W_{n,n}$. For two $(K+1)\times(K+1)$ matrices A and B, equality $W_{K+1}(A \otimes B) = (B \otimes A)\,W_{K+1}$ also holds (see Chapter 3 of Magnus and Neudecker (2007, MN) for other properties).

Assumption A.5 *For $n, T \to \infty$, we have $\frac{1}{\sqrt{n}}\sum_i w_i \tau_i^2\left(Y_{i,T} \otimes Y_{i,T} - vec\,[S_{ii,T}]\right) \Rightarrow N(0, \Omega)$, where the asymptotic variance matrix is:*

$$\begin{aligned}
\Omega &= \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left[S_{ij}\otimes S_{ij} + \left(S_{ij}\otimes S_{ij}\right)W_{K+1}\right]\right] \\
&= a.s.\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left[S_{ij}\otimes S_{ij} + \left(S_{ij}\otimes S_{ij}\right)W_{K+1}\right].
\end{aligned}$$

Assumption A.5 is a high-level CLT condition. We can prove this assumption under primitive conditions on the time series and cross-sectional dependence. For instance, we prove in Appendix A.4 that Assumption A.5 holds under a cross-sectional block dependence structure for the errors. Intuitively, the expression of the variance-covariance matrix Ω is related to the result that, for random $(K+1)\times 1$ vectors Y_1 and Y_2 which are jointly normal with covariance matrix S, we have $Cov\left(Y_1 \otimes Y_1, Y_2 \otimes Y_2\right) = S \otimes S + \left(S \otimes S\right) W_{K+1}$.

Let us now introduce the following statistic $\hat{\xi}_{nT} = T\sqrt{n}\left(\hat{Q}_e - \frac{1}{T}\hat{B}_\xi\right)$, where the recentering term simplifies to $\hat{B}_\xi = 1$ thanks to the weighting scheme. Under the null hypothesis $\mathcal{H}_0$, we prove that $\hat{\xi}_{nT} = \left(vec\left[\hat{Q}_x^{-1} c_{\hat{\nu}} c_{\hat{\nu}}' \hat{Q}_x^{-1}\right]\right)' \frac{1}{\sqrt{n}} \sum_i w_i \tau_i^2 \left(Y_{iT} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right) + o_p(1)$, which implies $\hat{\xi}_{nT} \Rightarrow N\left(0, \Sigma_\xi\right)$, where $\Sigma_\xi = 2 \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j v_{ij}^2\right] = 2$ a.s.- $\lim_{n\to\infty} \frac{1}{n}\sum_{i,j} w_i w_j v_{ij}^2$ as $n, T \to \infty$ (see Appendix A.2.5). Then, a feasible testing procedure exploits the consistent estimator $\tilde{\Sigma}_\xi = 2\frac{1}{n}\sum_{i,j} \hat{w}_i \hat{w}_j \tilde{v}_{ij}^2$ of the asymptotic variance Σ_ξ, where $\tilde{v}_{ij} = \frac{\tau_{i,T}\tau_{j,T}}{\tau_{ij,T}} c_{\hat{\nu}}' \hat{Q}_x^{-1} \tilde{S}_{ij} \hat{Q}_x^{-1} c_{\hat{\nu}}$.

Proposition 5 *Under $\mathcal{H}_0$, and Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.5 and C.1-C.5, we have $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT} \Rightarrow N(0,1)$, as $n, T \to \infty$ such that $n = O\left(T^{\bar{\gamma}}\right)$ for $0 < \bar{\gamma} < \min\left\{2, \eta\frac{1-q}{2\delta}\right\}$.*

In the homoskedastic case, the asymptotic variance of $\hat{\xi}_{nT}$ reduces to $\Sigma_\xi = 2$a.s.- $\lim_{n\to\infty} \frac{1}{n}\sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}^2} \frac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}}$. For fixed n, we can rely on the test statistic $T\hat{Q}_e$, which is asymptotically distributed as $\frac{1}{n}\sum_j eig_j \chi_j^2$ for $j = 1, \ldots, (n-K)$, where the χ_j^2 are independent chi-square variables with 1 degree of freedom, and the coefficients eig_j are the non-zero eigenvalues of matrix $V_n^{1/2}(W_n - W_n B_n (B_n' W_n B_n)^{-1} B_n' W_n) V_n^{1/2}$ (see Kan, Robotti and Shanken (2012)). By letting n grow, the sum of chi-square variables converges to a Gaussian variable after recentering and rescaling, which yields heuristically the result of Proposition 5. The condition on the relative expansion rate of n and T for the distributional result on the test statistic in Proposition 5 is more restrictive than the condition for feasible asymptotic normality of the estimators in Proposition 4.

The alternative hypothesis is

$$\mathcal{H}_1: \quad \inf_{\nu \in \mathbb{R}^K} E\left[\left(a_i - b_i'\nu\right)^2\right] > 0.$$

Let us define the pseudo-true value $\nu_\infty = \arg\inf_{\nu \in \mathbb{R}^K} Q_\infty^w(\nu)$, where $Q_\infty^w(\nu) = E\left[w_i\left(a_i - b_i'\nu\right)^2\right]$ (White (1982), Gourieroux et al. (1984)) and population errors $e_i = a_i - b_i'\nu_\infty = c_{\nu_\infty}'\beta_i$, $i = 1, ..., n$, for all n. In the next proposition, we prove consistency of the test, namely that the statistic $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT}$ diverges to $+\infty$

under the alternative hypothesis $\mathcal{H}_1$ for large n and T. The test of the null $\mathcal{H}_0$ against the alternative $\mathcal{H}_1$ is a one-sided test. We also give the asymptotic distribution of estimators $\hat{\nu}$ and $\hat{\lambda}$ under $\mathcal{H}_1$.

Proposition 6 *Under $\mathcal{H}_1$ and Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.5 and C.1-C.5, we have: a)* $\sqrt{n}\left(\hat{\nu} - \frac{1}{T}\hat{B}_{\nu_\infty} - \nu_\infty\right) \Rightarrow N\left(0, \Sigma_{\nu_\infty}\right)$, *where* $\hat{B}_{\nu_\infty} = \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i \tau_{i,T} E_2' \hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1} c_{\hat{\nu}}$ *and* $\Sigma_{\nu_\infty} = Q_b^{-1} E[w_i^2 e_i^2 b_i b_i'] Q_b^{-1}$, *and b)* $\sqrt{T}\left(\hat{\lambda} - \lambda_\infty\right) \Rightarrow N\left(0, \Sigma_f\right)$, *where* $\lambda_\infty = \nu_\infty + E\left[f_t\right]$, *as* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $1 < \bar{\gamma} < 3$*; c)* $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT} \xrightarrow{p} +\infty$, *as* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $0 < \bar{\gamma} < \min\left\{2,\ \eta\frac{1-q}{2\delta}\right\}$.

Under the alternative hypothesis $\mathcal{H}_1$, the convergence rate of $\hat{\nu}$ is slower than under $\mathcal{H}_0$, while the convergence rate of $\hat{\lambda}$ remains the same. The asymptotic distribution of the bias-adjusted estimator $\hat{\nu} - \frac{1}{T}\hat{B}_{\nu_\infty}$ is the same as the one got from a cross-sectional regression of a_i on b_i. The condition $\bar{\gamma} > 1$ in Propositions 6 a) and b) ensures that cross-sectional estimation of ν has asymptotically no impact on the estimation of λ.

To study the local asymptotic power, we can adopt the local alternative $\mathcal{H}_{1,nT} : \inf_{\nu \in \mathbb{R}^K} Q_\infty^w(\nu) = \frac{\psi}{\sqrt{nT}} > 0$, for a constant $\psi > 0$. Then we can show that $\hat{\xi}_{nT} \Rightarrow N(\psi, \Sigma_\xi)$, and the test is locally asymptotically powerful. Pesaran and Yamagata (2008) consider a similar local analysis for a test of slope homogeneity in large panels.

Finally, we can derive a test for the null hypothesis when the factors come from tradable assets, i.e., are portfolio excess returns:

$$\mathcal{H}_0 : a(\gamma) = 0 \text{ for almost all } \gamma \in [0,1] \quad \Leftrightarrow \quad E[a_i^2] = 0,$$

against the alternative hypothesis

$$\mathcal{H}_1 : E\left[a_i^2\right] > 0.$$

We only have to substitute $\hat{a}_i$ for $\hat{e}_i$, and $E_1 = (1, 0')'$ for $c_{\hat{\nu}}$ in Proposition 5. This gives an extension of Gibbons, Ross and Shanken (1989) with double asymptotics. Implementing the original Gibbons, Ross and Shanken (1989) test, which uses a weighting matrix corresponding to an inverted estimated $n \times n$ covariance matrix, becomes quickly problematic. We expect to compensate the potential loss of power induced by a diagonal weighting via the larger number of restrictions. Our Monte Carlo simulations show that the test exhibits good power properties against both risk-based and non risk-based alternatives (e.g. MacKinlay

(1995)) already for a thousand assets with a time series dimension similar to the one in the empirical analysis.

Chapter 3

Conditional factor model

In this chapter, we extend the setting of Chapter 2 to conditional specifications in order to model possibly time-varying risk premia (see Connor and Korajczyk (1989) for an intertemporal competitive equilibrium version of the APT yielding time-varying risk premia and Ludvigson (2011) for a discussion within scaled consumption-based models). We do not follow rolling short-window regression approaches to account for time-variation (Fama and French (1997), Lewellen and Nagel (2006)) since we favor a structural econometric framework to conduct formal inference in large cross-sectional equity datasets. A five-year window of monthly data yields a very short time-series panel for which asymptotics with fixed T and large n are better suited, but keeping T fixed impedes consistent estimation of the risk premia as already mentioned in the previous chapter.

3.1 Excess return generation and asset pricing restrictions

The following assumptions are the analogues of Assumptions APR.1 and APR.2, and Proposition 7 is the analogue of Proposition 1.

Assumption APR.6 *The excess returns $R_t(\gamma)$ of asset $\gamma \in [0,1]$ at dates $t = 1, 2, ...$ satisfy the conditional linear factor model:*

$$R_t(\gamma) = a_t(\gamma) + b_t(\gamma)' f_t + \varepsilon_t(\gamma), \tag{3.1}$$

where $a_t(\gamma, \omega) = a[\gamma, S^{t-1}(\omega)]$ and $b_t(\gamma, \omega) = b[\gamma, S^{t-1}(\omega)]$, for any $\omega \in \Omega$ and $\gamma \in [0,1]$, and random *variable $a(\gamma)$ and random vector $b(\gamma)$, for $\gamma \in [0,1]$, are $\mathcal{F}_0$-measurable.*

The intercept $a_t(\gamma)$ and factor sensitivity $b_t(\gamma)$ of asset $\gamma \in [0,1]$ at time t are $\mathcal{F}_{t-1}$-measurable, where the information set $\mathcal{F}_t$ is defined by $\mathcal{F}_t = \left\{S^{-t}(A), A \in \mathcal{F}_0\right\}$ for $\mathcal{F}_0 \in \mathcal{F}$, as in Chapter 2.

Assumption APR.7 *The matrix $\int b(\gamma)b(\gamma)'d\gamma$ is positive definite, P-a.s..*

Since transformation S is measure preserving, Assumption APR.7 implies that the matrix $\int b_t(\gamma)b_t(\gamma)'d\gamma$ is positive definite, P-a.s., for any date $t = 1, 2, \ldots$.

Proposition 7 *Under Assumptions APR.3-APR.7, for any date $t = 1, 2, \ldots$ there exists a unique random vector $\nu_t \in \mathbb{R}^K$ such that ν_t is $\mathcal{F}_{t-1}$-measurable and:*

$$a_t(\gamma) = b_t(\gamma)'\nu_t, \tag{3.2}$$

P-a.s. and for almost all $\gamma \in [0,1]$.

We can rewrite the asset pricing restriction as

$$E\left[R_t(\gamma)|\mathcal{F}_{t-1}\right] = b_t(\gamma)'\lambda_t, \tag{3.3}$$

for almost all $\gamma \in [0,1]$, where $\lambda_t = \nu_t + E\left[f_t|\mathcal{F}_{t-1}\right]$ is the vector of the conditional risk premia.

To have a workable version of Equations (3.1) and (3.2), we further specify the conditioning information and how coefficients depend on it. The conditioning information is such that instruments $Z \in \mathbb{R}^p$ and $Z(\gamma) \in \mathbb{R}^q$, for $\gamma \in [0,1]$, are $\mathcal{F}_0$-measurable. Then, the information $\mathcal{F}_{t-1}$ contains $Z_{\underline{t-1}}$ and $Z_{\underline{t-1}}(\gamma)$, for $\gamma \in [0,1]$, where we define $Z_t(\omega) = Z[S^t(\omega)]$ and $Z_t(\gamma,\omega) = Z[\gamma, S^t(\omega)]$. The lagged instruments Z_{t-1} are common to all stocks. They may include the constant and past observations of the factors and some additional variables such as macroeconomic variables. The lagged instruments $Z_{t-1}(\gamma)$ are specific to stock γ. They may include past observations of firm characteristics and stock returns. To end up with a linear regression model, we specify that the vector of factor sensitivities $b_t(\gamma)$ is a linear function of lagged instruments Z_{t-1} (Shanken (1990), Ferson and Harvey (1991)) and $Z_{t-1}(\gamma)$ (Avramov and Chordia (2006)): $b_t(\gamma) = B(\gamma)Z_{t-1} + C(\gamma)Z_{t-1}(\gamma)$, where $B(\gamma) \in \mathbb{R}^{K\times p}$ and $C(\gamma) \in \mathbb{R}^{K\times q}$, for any $\gamma \in [0,1]$

and $t = 1, 2, \ldots$. We can account for nonlinearities by including powers of some explanatory variables among the lagged instruments. We also specify that the vector of risk premia is a linear function of lagged instruments Z_{t-1} (Cochrane (1996), Jagannathan and Wang (1996)): $\lambda_t = \Lambda Z_{t-1}$, where $\Lambda \in \mathbb{R}^{K\times p}$, for any t. Furthermore, we assume that the conditional expectation of Z_t given the information $\mathcal{F}_{t-1}$ depends on Z_{t-1} only and is linear, as, for instance, in an exogeneous Vector Autoregressive (VAR) model of order 1. Since f_t is a subvector of Z_t, then $E\left[f_t|\mathcal{F}_{t-1}\right] = FZ_{t-1}$, where $F \in \mathbb{R}^{K\times p}$, for any t. Under these functional specifications the asset pricing restriction (3.2) implies that the intercept $a_t(\gamma)$ is a quadratic form in lagged instruments Z_{t-1} and $Z_{t-1}(\gamma)$, namely:

$$a_t(\gamma) = Z_{t-1}'B(\gamma)'\left(\Lambda - F\right)Z_{t-1} + Z_{t-1}(\gamma)'C(\gamma)'\left(\Lambda - F\right)Z_{t-1}. \tag{3.4}$$

This shows that assuming a priori linearity of $a_t(\gamma)$ in the lagged instruments Z_{t-1} and $Z_{t-1}(\gamma)$ is in general not compatible with linearity of $b_t(\gamma)$ and $E\left[f_t|Z_{t-1}\right]$.

The sampling scheme is the same as in Section 2.2, and we use the same type of notation, for example $b_{i,t} = b_t(\gamma_i)$, $B_i = B(\gamma_i)$, $C_i = C(\gamma_i)$ and $Z_{i,t-1} = Z_{t-1}(\gamma_i)$. In particular, we allow for potential correlation between parameters B_i, C_i and asset specific instruments $Z_{i,t-1}$ via the random index γ_i. Then, the conditional factor model (3.1) with asset pricing restriction (3.4) written for the sample observations becomes

$$R_{i,t} = Z_{t-1}'B_i'\left(\Lambda - F\right)Z_{t-1} + Z_{i,t-1}'C_i'\left(\Lambda - F\right)Z_{t-1} + Z_{t-1}'B_i'f_t + Z_{i,t-1}'C_i'f_t + \varepsilon_{i,t}, \tag{3.5}$$

which is nonlinear in the parameters Λ, F, B_i, and C_i. In order to implement the two-pass methodology in a conditional context, we rewrite model (3.5) as a model that is linear in transformed parameters and new regressors. The regressors include $x_{2,i,t} = \left(f_t' \otimes Z_{t-1}', f_t' \otimes Z_{i,t-1}'\right)' \in \mathbb{R}^{d_2}$ with $d_2 = K(p+q)$. The first components with common instruments take the interpretation of scaled factors (Cochrane (2005)), while the second components do not since they depend on i. The regressors also include the predetermined variables $x_{1,i,t} = \left(vech\left[X_t\right]', Z_{t-1}' \otimes Z_{i,t-1}'\right)' \in \mathbb{R}^{d_1}$ with $d_1 = p(p+1)/2 + pq$, where the symmetric matrix $X_t = [X_{t,k,l}] \in \mathbb{R}^{p\times p}$ is such that $X_{t,k,l} = Z_{t-1,k}^2$, if $k = l$, and $X_{t,k,l} = 2Z_{t-1,k}Z_{t-1,l}$, otherwise, $k, l = 1, \ldots, p$. The vector-half operator $vech\left[\cdot\right]$ stacks the lower elements of a $p \times p$ matrix as a $p\left(p+1\right)/2 \times 1$ vector (see Chapter 2 in Magnus and Neudecker (2007) for properties of this matrix tool). To parallel the analysis of the unconditional case, we can express model (3.5) as in (2.2) through appropriate redefinitions of the regressors and loadings (see Appendix A.3):

$$R_{i,t} = \beta_i' x_{i,t} + \varepsilon_{i,t}, \tag{3.6}$$

where $x_{i,t} = \left(x_{1,i,t}', x_{2,i,t}'\right)'$ has dimension $d = d_1 + d_2$, and $\beta_i = \left(\beta_{1,i}', \beta_{2,i}'\right)'$ is such that

$$\beta_{1,i} = \Psi \beta_{2,i}, \qquad \beta_{2,i} = \left(vec\left[B_i'\right]', vec\left[C_i'\right]'\right)', \tag{3.7}$$

$$\Psi = \begin{pmatrix} \frac{1}{2} D_p^+ \left[(\Lambda - F)' \otimes I_p + I_p \otimes (\Lambda - F)' W_{p,K}\right] & 0 \\ 0 & (\Lambda - F)' \otimes I_q \end{pmatrix}.$$

The matrix D_p^+ is the $p(p+1)/2 \times p^2$ Moore-Penrose inverse of the duplication matrix D_p, such that $vech\,[A] = D_p^+ vec\,[A]$ for any $A \in \mathbb{R}^{p \times p}$ (see Chapter 3 in Magnus and Neudecker (2007)). When $Z_t = 1$ and $Z_{i,t} = 0$, we have $p = 1$ and $q = 0$, and model (3.6) reduces to model (2.2).

In (3.7), the $d_1 \times 1$ vector $\beta_{1,i}$ is a linear transformation of the $d_2 \times 1$ vector $\beta_{2,i}$. This clarifies that the asset pricing restriction (3.4) implies a constraint on the distribution of random vector β_i via its support. The coefficients of the linear transformation depend on matrix $\Lambda - F$. For the purpose of estimating the loading coefficients of the risk premia in matrix Λ, we rewrite the parameter restrictions as (see Appendix A.3):

$$\beta_{1,i} = \beta_{3,i}\nu, \; \nu = vec\left[\Lambda' - F'\right], \; \beta_{3,i} = \left(\left[D_p^+ \left(B_i' \otimes I_p\right)\right]', \left[W_{p,q}\left(C_i' \otimes I_p\right)\right]'\right)'. \tag{3.8}$$

Furthermore, we can relate the $d_1 \times Kp$ matrix $\beta_{3,i}$ to the vector $\beta_{2,i}$ (see Appendix A.3):

$$vec\left[\beta_{3,i}'\right] = J_a \beta_{2,i}, \tag{3.9}$$

where the $d_1 pK \times d_2$ block-diagonal matrix of constants J_a is given by $J_a = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix}$ with diagonal blocks $J_{11} = W_{p(p+1)/2,pK}\left(I_K \otimes \left[\left(I_p \otimes D_p^+\right)\left(W_p \otimes I_p\right)\left(I_p \otimes vec\,[I_p]\right)\right]\right)$ and $J_{22} = W_{pq,pK}\left(I_K \otimes \left[\left(I_p \otimes W_{p,q}\right)\left(W_{p,q} \otimes I_p\right)\left(I_q \otimes vec\,[I_p]\right)\right]\right)$. The link (3.9) is instrumental in deriving the asymptotic results. The parameters $\beta_{1,i}$ and $\beta_{2,i}$ correspond to the parameters a_i and b_i of the unconditional case, in which the matrix J_a is equal to I_K. Equations (3.8) and (3.9) in the conditional setting are the counterparts of restriction (2.3) in the unconditional setting.

3.2 Asymptotic properties of time-varying risk premium estimation

We consider a two-pass approach building on Equations (3.6) and (3.8).

First Pass: The first pass consists in computing time-series OLS estimators $\hat{\beta}_i = (\hat{\beta}_{1,i}', \hat{\beta}_{2,i}')' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} R_{i,t}$, for $i = 1, ..., n$, where $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} x_{i,t}'$. We use the same trimming device as in Chapter 2.

Second Pass: The second pass consists in computing a cross-sectional estimator of ν by regressing the $\hat{\beta}_{1,i}$ on the $\hat{\beta}_{3,i}$ keeping non-trimmed assets only. We use a WLS approach. The weights are estimates of $w_i = (diag\left[v_i\right])^{-1}$, where the v_i are the asymptotic variances of the standardized errors $\sqrt{T}\left(\hat{\beta}_{1,i} - \hat{\beta}_{3,i}\nu\right)$ in the cross-sectional regression for large T. We have $v_i = \tau_i C_\nu' Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} C_\nu$, where $Q_{x,i} = E\left[x_{i,t} x_{i,t}' | \gamma_i\right]$, $S_{ii} = \underset{T\to\infty}{\text{plim}} \frac{1}{T} \sum_t \sigma_{ii,t} x_{i,t} x_{i,t}' = E\left[\varepsilon_{i,t}^2 x_{i,t} x_{i,t}' | \gamma_i\right]$, $\sigma_{ii,t} = E\left[\varepsilon_{i,t}^2 | x_{i,\underline{t}}, \gamma_i\right]$, and $C_\nu = \left(E_1' - \left(I_{d_1} \otimes \nu'\right) J_a E_2'\right)'$, with $E_1 = (I_{d_1} : 0_{d_1 \times d_2})'$, $E_2 = (0_{d_2 \times d_1} : I_{d_2})'$. We use the estimates $\hat{v}_i = \tau_{i,T} C_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1}$, where $\hat{S}_{ii} = \frac{1}{T_i} \sum_t I_{i,t} \hat{\varepsilon}_{i,t}^2 x_{i,t} x_{i,t}'$, $\hat{\varepsilon}_{i,t} = R_{i,t} - \hat{\beta}_i' x_{i,t}$ and $C_{\hat{\nu}_1} = \left(E_1' - \left(I_{d_1} \otimes \hat{\nu}_1'\right) J_a E_2'\right)'$. To estimate C_ν, we use the OLS estimator $\hat{\nu}_1 = \left(\sum_i \mathbf{1}_i^\chi \hat{\beta}_{3,i}' \hat{\beta}_{3,i}\right)^{-1} \sum_i \mathbf{1}_i^\chi \hat{\beta}_{3,i}' \hat{\beta}_{1,i}$, i.e., a first-step estimator with unit weights. The WLS estimator is:

$$\hat{\nu} = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}_{3,i}' \hat{w}_i \hat{\beta}_{1,i}, \tag{3.10}$$

where $\hat{Q}_{\beta_3} = \frac{1}{n} \sum_i \hat{\beta}_{3,i}' \hat{w}_i \hat{\beta}_{3,i}$ and $\hat{w}_i = \mathbf{1}_i^\chi \left(diag\left[\hat{v}_i\right]\right)^{-1}$. The final estimator of the risk premia is $\hat{\lambda}_t = \hat{\Lambda} Z_{t-1}$, where we deduce $\hat{\Lambda}$ from the relationship $vec\left[\hat{\Lambda}'\right] = \hat{\nu} + vec\left[\hat{F}'\right]$ with the estimator $\hat{F}$ obtained by a SUR regression of factors f_t on lagged instruments Z_{t-1}: $\hat{F} = \sum_t f_t Z_{t-1}' \left(\sum_t Z_{t-1} Z_{t-1}'\right)^{-1}$.

The next assumption is similar to Assumption A.1.

Assumption B.1 *There exists a positive constant M such that for all n, T:*
a) $E\left[\varepsilon_{i,t} | \{\varepsilon_{j,\underline{t-1}}, x_{j,\underline{t}}, \gamma_j, j = 1, ..., n\}\right] = 0$, *with* $x_{j,\underline{t}} = \{x_{j,t}, x_{j,t-1}, \cdots\}$; *b)* $\frac{1}{M} \leq \sigma_{ii,t} \leq M,\ i = 1, ..., n$;
c) $E\left[\frac{1}{n} \sum_{i,j} E\left[\left|\sigma_{ij,t}\right|^2 | \gamma_i, \gamma_j\right]^{1/2}\right] \leq M$, *where* $\sigma_{ij,t} = E\left[\varepsilon_{i,t} \varepsilon_{j,t} | x_{i,\underline{t}}, x_{j,\underline{t}}, \gamma_i, \gamma_j\right]$.

Proposition 8 summarizes consistency of estimators $\hat{\nu}$ and $\hat{\Lambda}$ under the double asymptotics $n, T \to \infty$. It extends Proposition 2 to the conditional case.

Proposition 8 *Under Assumptions APR.3-APR.7,SC.1-SC.2,B.1b) and C.1,C.4-C.6, we get a)* $\|\hat{\nu} - \nu\| = o_p(1)$, *b)* $\left\|\hat{\Lambda} - \Lambda\right\| = o_p(1)$, *when* $n, T \to \infty$ *such that* $n = O(T^{\bar{\gamma}})$ *for* $\bar{\gamma} > 0$.

Part b) implies $\sup_t \left\|\hat{\lambda}_t - \lambda_t\right\| = o_p(1)$ under boundedness of process Z_t (Assumption C.4 written for the conditional model).

Proposition 9 below gives the large-sample distributions under the double asymptotics $n, T \to \infty$. It extends Proposition 3 to the conditional case through adequate use of selection matrices. The following assumptions are similar to Assumptions A.2 and A.3. We make use of $Q_{\beta_3} = E\left[\beta'_{3,i} w_i \beta_{3,i}\right]$, $Q_z = E\left[Z_t Z'_t\right]$, $S_{ij} = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_t \sigma_{ij,t} x_{i,t} x'_{j,t} = E[\varepsilon_{i,t}\varepsilon_{j,t} x_{i,t} x'_{j,t} | \gamma_i, \gamma_j]$ and $S_{Q,ij} = Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1}$, otherwise, we keep the same notations as in Chapter 2.

Assumption B.2 *As* $n, T \to \infty$, *a)* $\frac{1}{\sqrt{n}}\sum_i \tau_i \left[(Q_{x,i}^{-1} Y_{i,T}) \otimes v_{3,i}\right] \Rightarrow N(0, S_{\nu_3})$, *where* $Y_{i,T} = \frac{1}{\sqrt{T}}\sum_t I_{i,t} x_{i,t} \varepsilon_{i,t}$, $v_{3,i} = vec[\beta'_{3,i} w_i]$ *and* $S_{v_3} = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} S_{Q,ij} \otimes v_{3,i} v'_{3,j}\right] =$ *a.s.-* $\lim_{n\to\infty} \frac{1}{n}\sum_{i,j} \frac{\tau_i\tau_j}{\tau_{ij}} [S_{Q,ij} \otimes v_{3,i} v'_{3,j}]$*;*
b) $\frac{1}{\sqrt{T}}\sum_t u_t \otimes Z_{t-1} \Rightarrow N(0, \Sigma_u)$, *where* $\Sigma_u = E\left[u_t u'_t \otimes Z_{t-1} Z'_{t-1}\right]$ *and* $u_t = f_t - F Z_{t-1}$.

Assumption B.3 *For any* $1 \le t, s \le T$, $T \in \mathbb{N}$ *and* $\gamma \in [0,1]$, *we have* $E\left[\varepsilon_t(\gamma)^2 \varepsilon_s(\gamma) | Z_{\underline{T}}, Z_{\underline{T}}(\gamma)\right] = 0$.

Proposition 9 *Under Assumptions APR.3-APR.7,SC.1-SC.2,B.1-B.3 and C.1-C.6, we have a)* $\sqrt{nT}\left(\hat{\nu} - \nu - \frac{1}{T}\hat{B}_\nu\right) \Rightarrow N(0, \Sigma_\nu)$ *where* $\hat{B}_\nu = \hat{Q}_{\beta_3}^{-1} J_b \frac{1}{n}\sum_i \tau_{i,T} vec\left[E'_2 \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i\right]$ *and* $\Sigma_\nu = \left(vec\left[C'_\nu\right] \otimes Q_{\beta_3}^{-1}\right)' S_{v_3} \left(vec\left[C'_\nu\right] \otimes Q_{\beta_3}^{-1}\right)$ *with* $J_b = \left(vec\left[I_{d_1}\right]' \otimes I_{Kp}\right)(I_{d_1} \otimes J_a)$ *and* $C_{\hat{\nu}} = (E'_1 - (I_{d_1} \otimes \hat{\nu}') J_a E'_2)'$*; b)* $\sqrt{T} vec\left[\hat{\Lambda}' - \Lambda'\right] \Rightarrow N(0, \Sigma_\Lambda)$ *where* $\Sigma_\Lambda = \left(I_K \otimes Q_z^{-1}\right) \Sigma_u \left(I_K \otimes Q_z^{-1}\right)$, *when* $n, T \to \infty$ *such that* $n = O(T^{\bar{\gamma}})$ *for* $0 < \bar{\gamma} < 3$.

Since $\lambda_t = \Lambda Z_{t-1} = \left(Z'_{t-1} \otimes I_K\right) W_{p,K} vec\left[\Lambda'\right]$, part b) implies conditionally on Z_{t-1} that $\sqrt{T}\left(\hat{\lambda}_t - \lambda_t\right) \Rightarrow N\left(0, \left(Z'_{t-1} \otimes I_K\right) W_{p,K} \Sigma_\Lambda W_{K,p} \left(Z_{t-1} \otimes I_K\right)\right)$.

We can use Proposition 9 to build confidence intervals. It suffices to replace the unknown quantities Q_x, Q_z, Q_{β_3}, Σ_u, and ν by their empirical counterparts. For matrix S_{v_3}, we use the thresholded estimator $\tilde{S}_{ij}$ as in Section 2.4. Then, we can extend Proposition 4 to the conditional case under Assumptions B.1-B.3, A.4 and C.1-C.6.

3.3 Tests of conditional asset pricing restrictions

Since the equations in (3.8) correspond to the asset pricing restriction (2.3), the null hypothesis of correct specification of the conditional model is

$$\mathcal{H}_0: \text{ there exists } \nu \in \mathbb{R}^{pK} \text{ such that } \beta_1(\gamma) = \beta_3(\gamma)\nu, \text{ for almost all } \gamma \in [0,1],$$

where $\beta_1(\gamma)$ and $\beta_3(\gamma)$ are defined as $\beta_{1,i}$ and $\beta_{3,i}$ in Equations (3.7) and (3.8) replacing $B(\gamma)$ and $C(\gamma)$ for B_i and C_i. Under $\mathcal{H}_0$, we have $E\left[(\beta_{1,i} - \beta_{3,i}\nu)'(\beta_{1,i} - \beta_{3,i}\nu)\right] = 0$. The alternative hypothesis is

$$\mathcal{H}_1: \quad \inf_{\nu \in \mathbb{R}^{pK}} E\left[(\beta_{1,i} - \beta_{3,i}\nu)'(\beta_{1,i} - \beta_{3,i}\nu)\right] > 0.$$

As in Section 2.5, we build the SSR $\hat{Q}_e = \frac{1}{n}\sum_i \hat{e}_i'\hat{w}_i\hat{e}_i$, with $\hat{e}_i = \hat{\beta}_{1,i} - \hat{\beta}_{3,i}\hat{\nu} = C_{\hat{\nu}}'\hat{\beta}_i$ and the statistic $\hat{\xi}_{nT} = T\sqrt{n}\left(\hat{Q}_e - \frac{1}{T}\hat{B}_\xi\right)$, where $\hat{B}_\xi = d_1$.

Assumption B.4 *For $n, T \to \infty$, we have $\frac{1}{\sqrt{n}}\sum_i \tau_i^2 \left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)(Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right])\right] \otimes vec[w_i]$ $\Rightarrow N(0, \Omega)$, where the asymptotic variance matrix is:*

$$\begin{aligned}\Omega &= \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j}\frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left[S_{Q,ij} \otimes S_{Q,ij} + (S_{Q,ij} \otimes S_{Q,ij})W_d\right] \otimes \left(vec[w_i]vec[w_j]'\right)\right] \\ &= \textit{a.s.-}\lim_{n\to\infty}\frac{1}{n}\sum_{i,j}\frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left[S_{Q,ij} \otimes S_{Q,ij} + (S_{Q,ij} \otimes S_{Q,ij})W_d\right] \otimes \left(vec[w_i]vec[w_j]'\right).\end{aligned}$$

Proposition 10 *Under $\mathcal{H}_0$ and Assumptions APR.3-APR.7, SC.1-SC.2, B.1-B.4, A.4 and C.1-C.6, we have $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT} \Rightarrow N(0,1)$, where $\tilde{\Sigma}_\xi = 2\frac{1}{n}\sum_{i,j}\frac{\tau_{i,T}^2\tau_{j,T}^2}{\tau_{ij,T}^2} tr\left[\hat{w}_i\left(C_{\hat{\nu}}'\hat{Q}_{x,i}^{-1}\tilde{S}_{ij}\hat{Q}_{x,j}^{-1}C_{\hat{\nu}}\right)\hat{w}_j\left(C_{\hat{\nu}}'\hat{Q}_{x,j}^{-1}\tilde{S}_{ji}\hat{Q}_{x,i}^{-1}C_{\hat{\nu}}\right)\right]$ as $n, T \to \infty$ such that $n = O\left(T^{\bar{\gamma}}\right)$ for $0 < \bar{\gamma} < \min\left\{2, \eta\frac{1-q}{2\delta}\right\}$.*

Under $\mathcal{H}_1$, we have $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT} \xrightarrow{p} +\infty$, as in Proposition 6.

As in Section 2.5, the null hypothesis when the factors are tradable assets becomes:

$$\mathcal{H}_0: \quad \beta_1(\gamma) = 0 \text{ for almost all } \gamma \in [0,1],$$

against the alternative hypothesis

$$\mathcal{H}_1 : \quad E\left[\beta_{1,i}'\beta_{1,i}\right] > 0.$$

We only have to substitute $\hat{Q}_a = \frac{1}{n}\sum_i \hat{\beta}_{1,i}'\hat{w}_i\hat{\beta}_{1,i}$ for $\hat{Q}_e$, and $E_1 = (I_{d_1} : 0_{d_1 \times d_2})'$ for $C_{\hat{\nu}}$. This gives an extension of Gibbons, Ross and Shanken (1989) to the conditional case with double asymptotics. The implementation of the original Gibbons, Ross and Shanken (1989) test is unfeasible here because of the large number nd_1 of restrictions; each $\beta_{1,i}$ is of dimension $d_1 \times 1$, and the estimated covariance matrix to invert is of dimension $nd_1 \times nd_1$.

Chapter 4

Empirical results

4.1 Asset pricing model and data description

Our baseline asset pricing model is a four-factor model with $f_t = (r_{m,t}, r_{smb,t},\ r_{hml,t}, r_{mom,t})'$ where $r_{m,t}$ is the month t excess return on CRSP NYSE/AMEX/Nasdaq value-weighted market portfolio over the risk free rate, and $r_{smb,t}$, $r_{hml,t}$ and $r_{mom,t}$ are the month t returns on zero-investment factor-mimicking portfolios for size, book-to-market, and momentum (see Fama and French (1993), Jegadeesh and Titman (1993), Carhart (1997)). We proxy the risk free rate with the monthly 30-day T-bill beginning-of-month yield. To account for time-varying alphas, betas and risk premia, we use a conditional specification based on two common variables and a firm-level variable. We take the instruments $Z_t = (1,\ Z_t^{*\prime})'$, where bivariate vector Z_t^* includes the term spread, proxied by the difference between yields on 10-year Treasury and three-month T-bill, and the default spread, proxied by the yield difference between Moody's Baa-rated and Aaa-rated corporate bonds. We take a scalar $Z_{i,t}$ corresponding to the book-to-market equity of firm i. We refer to Avramov and Chordia (2006) for convincing theoretical and empirical arguments in favor of the chosen conditional specification. The vector $x_{i,t}$ has dimension $d = 25$, and parsimony explains why we have not included e.g. the size of firm i as an additional stock specific instrument. We report robustness checks with other conditional specifications in the supplementary materials.

We compute the firm characteristics from Compustat as in the appendix of Fama and French (2008). The CRSP database provides the monthly stock returns data and we exclude financial firms (Standard Industrial Classification Codes between 6000 and 6999) as in Fama and French (2008). The dataset after matching

CRSP and Compustat contents comprises $n = 9,936$ stocks, and covers the period from July 1964 to December 2009 with $T = 546$ months. For comparison purposes with a standard methodology for small n, we consider the 25 and 100 Fama-French (FF) portfolios as base assets. We have downloaded the time series of factors, portfolio returns, and portfolio characteristics from the website of Kenneth French.

4.2 Estimation results

We first present unconditional estimates before looking at the path of the time-varying estimates. We use $\chi_{1,T} = 15$ as advocated by Greene (2008), together with $\chi_{2,T} = 546/12$ for the unconditional estimation and $\chi_{2,T} = 546/60$ for the conditional estimation. In the results reported for each model, we denote by n^{χ} the dimension of the cross-section after trimming. We compute confidence intervals with a data-driven threshold selected by cross-validation as in Bickel and Levina (2008). Table 4.1 gathers the estimated annual risk premia, with the corresponding confidence intervals at 95% level, for the following unconditional models: the four-factor model, the Fama-French model, and the CAPM. For the Fama-French model and the CAPM, the trimming level $\chi_{1,T}$ is not binding when $\chi_{2,T} = 546/12$. In Table 4.2, we display the estimates of the components of ν. For individual stocks, we use bias-corrected estimates for λ and ν. For portfolios, we use asymptotics for fixed n and $T \to \infty$. The estimated risk premia for the market factor are of the same magnitude and all positive across the three universes of assets and the three models. For the four-factor model and the individual stocks, the size factor is positively remunerated (2.86%) and it is not significantly different from zero. The value factor commands a significant negative reward (-4.60%). Phalippou (2007) obtains a similar growth premium for portfolios built on stocks with a high institutional ownership. The momentum factor is largely remunerated (7.16%) and significantly different from zero. For the 25 and 100 FF portfolios, we observe that the size factor is not significantly positively remunerated while the value factor is significantly positively remunerated (4.81% and 5.11%). The momentum factor bears a significant positive reward (34.03% and 17.29%). The large, but imprecise, estimate for the momentum premium when $n = 25$ and $n = 100$ comes from the estimate for ν_{mom} (25.40% and 8.66%) that is much larger and less accurate than the estimates for ν_m, ν_{smb} and ν_{hml} (0.85%, -0.26%, 0.03%, and 0.55%, 0.01%, 0.33%). Moreover, while the estimates of ν_m, ν_{smb} and ν_{hml} are statistically not significant for portfolios, the estimates of ν_m and ν_{hml} are statistically different from zero for individual stocks. In particular, the estimate of ν_{hml} is large and negative, which explains the negative estimate on the value premium displayed in Table 4.1. The size, value and momentum factors are tradable in theory. In practice, their implementation faces transaction costs due to rebalancing and short selling. A non zero ν might capture these market imperfections (Cremers, Petajisto, and Zitzewitz (2010)).

A potential explanation of the discrepancies revealed in Tables 4.1 and 4.2 between individual stocks and portfolios is the much larger heterogeneity of the factor loadings for the former. As already discussed in Lewellen, Nagel and Shanken (2010), the portfolio betas are all concentrated in the middle of the cross-sectional distribution obtained from the individual stocks. Creating portfolios distorts information by shrinking the dispersion of betas. The estimation results for the momentum factor exemplify the problems related to a small number of portfolios exhibiting a tight factor structure. For λ_m, λ_{smb}, and λ_{hml}, we obtain similar inferential results when we consider the Fama-French model. Our point estimates for λ_m, λ_{smb} and λ_{hml}, for large n agree with Ang, Liu and Schwarz (2008). Our point estimates and confidence intervals for λ_m, λ_{smb} and λ_{hml}, agree with the results reported by Shanken and Zhou (2007) for the 25 portfolios.

Let us now consider the conditional four-factor specification. Figure 4.1 plots the estimated time-varying path of the four risk premia from the individual stocks. For comparison purpose, we also plot the unconditional estimates and the average lambda over time. A well-known bias coming from market-timing and volatility-timing (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth, Carlson, Fisher and Simutin (2011)) explains the discrepancy between the unconditional estimate and the average over time. After trimming, we compute the risk premia on $n^{\chi} = 3,900$ individual assets in the four-factor model. The risk premia for the market, size and value factors feature a counter-cyclical pattern. Indeed, these risk premia increase during economic contractions and decrease during economic booms. Gomes, Kogan and Zhang (2003) and Zhang (2005) construct equilibrium models exhibiting a counter-cyclical behavior in size and book-to-market effects. On the contrary, the risk premium for the momentum factor is pro-cyclical. Furthermore, conditional estimates of the value premium are often negative and take positive values mostly in recessions. The conditional estimates of the size premium are most of the time slightly positive.

Figure 4.2 plots the estimated time-varying path of the four risk premia from the 25 portfolios. We also plot the unconditional estimates and the average lambda over time. The discrepancy between the unconditional estimate and the averages over time is also observed for $n = 25$. The conditional point estimates for $\lambda_{mom,t}$ are typically smaller than the unconditional estimate in Table 4.1. Finally, by comparing Figures

4.1 and 4.2, we observe that the patterns of risk premia look similar except for the book-to-market factor. Indeed, the risk premium for the value effect estimated from the 25 portfolios is pro-cyclical, contradicting the counter-cyclical behavior predicted by finance theory. By comparing Figures 4.2 and 4.3, we observe that increasing the number of portfolios to 100 does not help in reconciling the discrepancy.

4.3 Results on testing the asset pricing restrictions

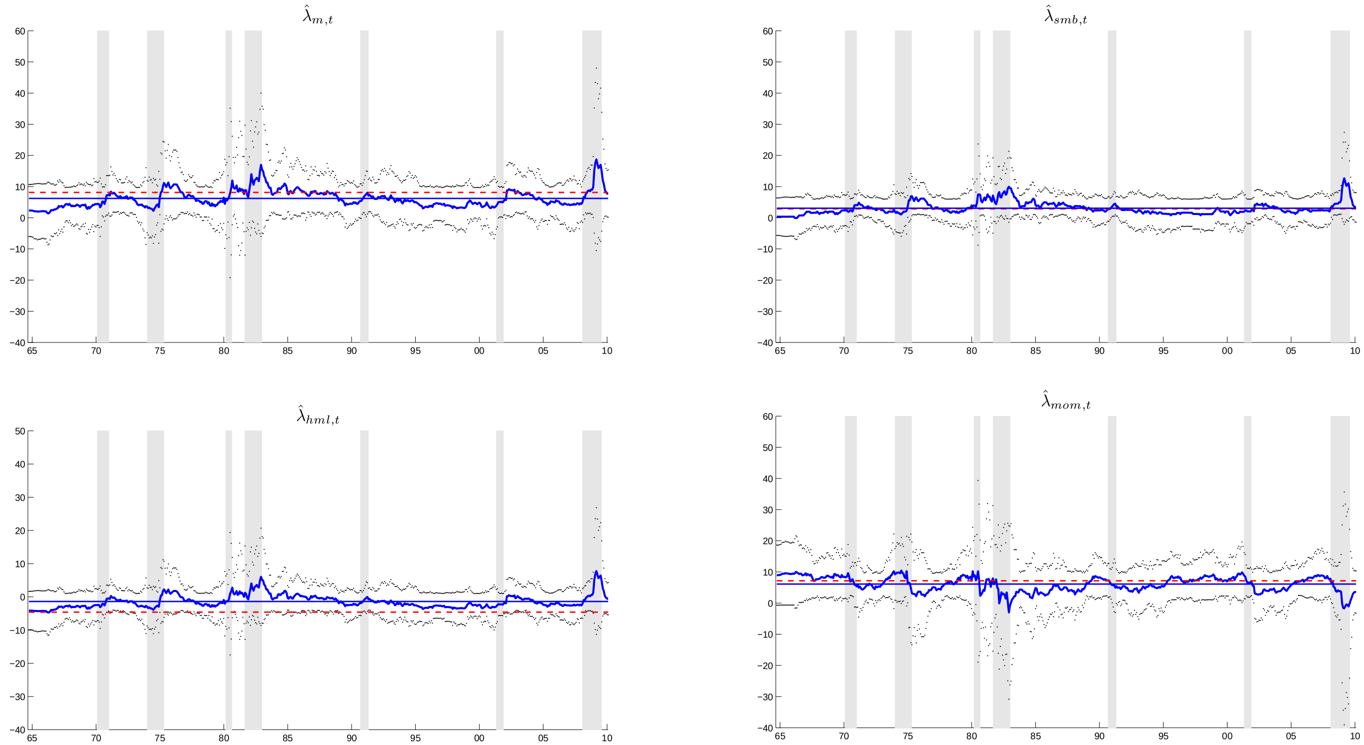
As already discussed in Lewellen, Nagel and Shanken (2010), the 25 FF portfolios have four-factor market and momentum betas close to one and zero, respectively. For the 100 FF portfolios, the dispersion around one and zero is slightly larger. As depicted in Figure 1 by Lewellen, Nagel and Shanken (2010), this empirical concentration implies that it is easy to get artificially large estimates $\hat{\rho}^2$ of the cross-sectional R^2 for three- and four-factor models. On the contrary, the observed heterogeneity in the betas coming from the individual stocks impedes this. This suggests that it is much less easy to find factors that explain the cross-sectional variation of expected excess returns on individual stocks than on portfolios. Reporting large $\hat{\rho}^2$, or small SSR $\hat{Q}_e$, when n is large, is much more impressive than when n is small.

Table 4.3 gathers the results for the tests of the asset pricing restrictions in unconditional factor models. As already mentioned, when n is large, we prefer working with test statistics based on the SSR $\hat{Q}_e$ instead of $\hat{\rho}^2$ since the population R^2 is not well-defined with tradable factors under the null hypothesis of well-specification (its denominator is zero). For the individual stocks, we compute the test statistics $\tilde{\Sigma}_{\xi}^{-1/2}\hat{\xi}_{nT}$ based on $\hat{Q}_e$ and $\hat{Q}_a$ as well as their associated one-side p-value. Our Monte Carlo simulations show that we need to set a stronger trimming level $\chi_{2,T}$ to compute the test statistic than to estimate the risk premium. We use $\chi_{2,T} = 546/240$. For the 25 and 100 FF portfolios, we compute weighted test statistics (Gibbons, Ross and Shanken (1989)) as well as their associated p-values. For individual stocks, the test statistics reject both null hypotheses $\mathcal{H}_0 : \ a(\gamma) = b(\gamma)'\nu$ and $\mathcal{H}_0 : \ a(\gamma) = 0$ for the three specifications at 5% level. Instead, the null hypothesis $\mathcal{H}_0 : \ a(\gamma) = b(\gamma)'\nu$ is not rejected for the four-factor specification at 1% level. Similar conclusions are obtained when using the two sets of Fama-French portfolios as base assets. Table 4.4 gathers the results for tests of the asset pricing restrictions in conditional specifications. Contrary to the unconditional case, we do not report the values of the weighted test statistics (Gibbons, Ross and Shanken (1989)) computed for portfolios because of the numerical instability in the inversion of the covariance matrix. The latter has dimension $2,500 \times 2,500$ for the conditional four-factor specification with the 100FF portfolios. Instead, we report the values of the test statistics $T\hat{Q}_e$ and $T\hat{Q}_a$. For individual stocks, the test statistics reject both null hypotheses $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma)\nu$ and $\mathcal{H}_0 : \beta_1(\gamma) = 0$ for the three specifications at 5% level, but not for the conditional CAPM at 1% level. For portfolios, the two null hypotheses are not rejected under the conditional CAPM even at 5% level.

For individual stocks, the rejection of the asset pricing restriction using a conditional multi-factor specification (at 1% level), and the non rejection under an unconditional specification, might seem counterintuitive. Indeed, for a given choice of the factors and instruments, the set of unconditional specifications satisfying the no-arbitrage restriction $a(\gamma) = b(\gamma)'\nu$, is a strict subset of the collection of conditional specifications with $a_t(\gamma) = b_t(\gamma)'\nu_t$. However, what we are testing here is whether the projection of the DGP on a given conditional or unconditional factor specification is compatible with no-arbitrage. The set of unconditional factor models is included in the set of conditional factor models, and it may well be the case that the projection of the DGP on the former set satisfies the no-arbitrage restrictions, while the projection on the latter does not. Therefore, the results in Tables 4.3 and 4.4 for individual stocks are not incompatible with each other. A similar argument might explain why in Table 4.4 we fail to reject the asset pricing restriction $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma)\nu$ under the conditional CAPM (at level 1% for individual assets, and 5% for portfolios), while this restriction is rejected under the three- and four-factor specifications.

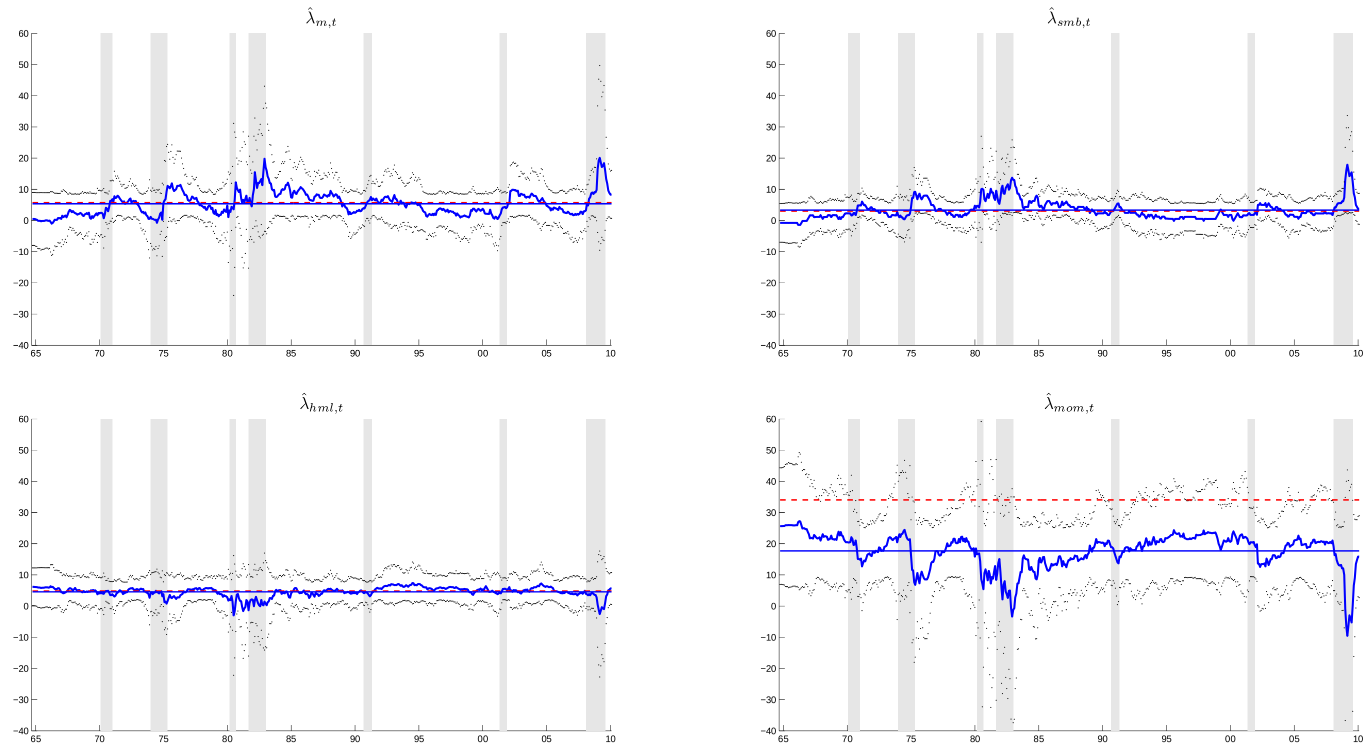
The analysis of the validity of the asset pricing restrictions could be completed by an analysis of correct specification of the different conditional and unconditional factor models. A specification test would assess whether the proposed set of linear factors captures the systematic risk component in equity returns, and clearly differs from the test of the no-arbitrage restrictions introduced above. Developing a test of correct specification of conditional factor models with an unbalanced panel and double asymptotics is beyond the scope of the thesis. We leave this interesting topic for future research.

Figure 4.1: Path of estimated annualized risk premia with $n = 9,936$



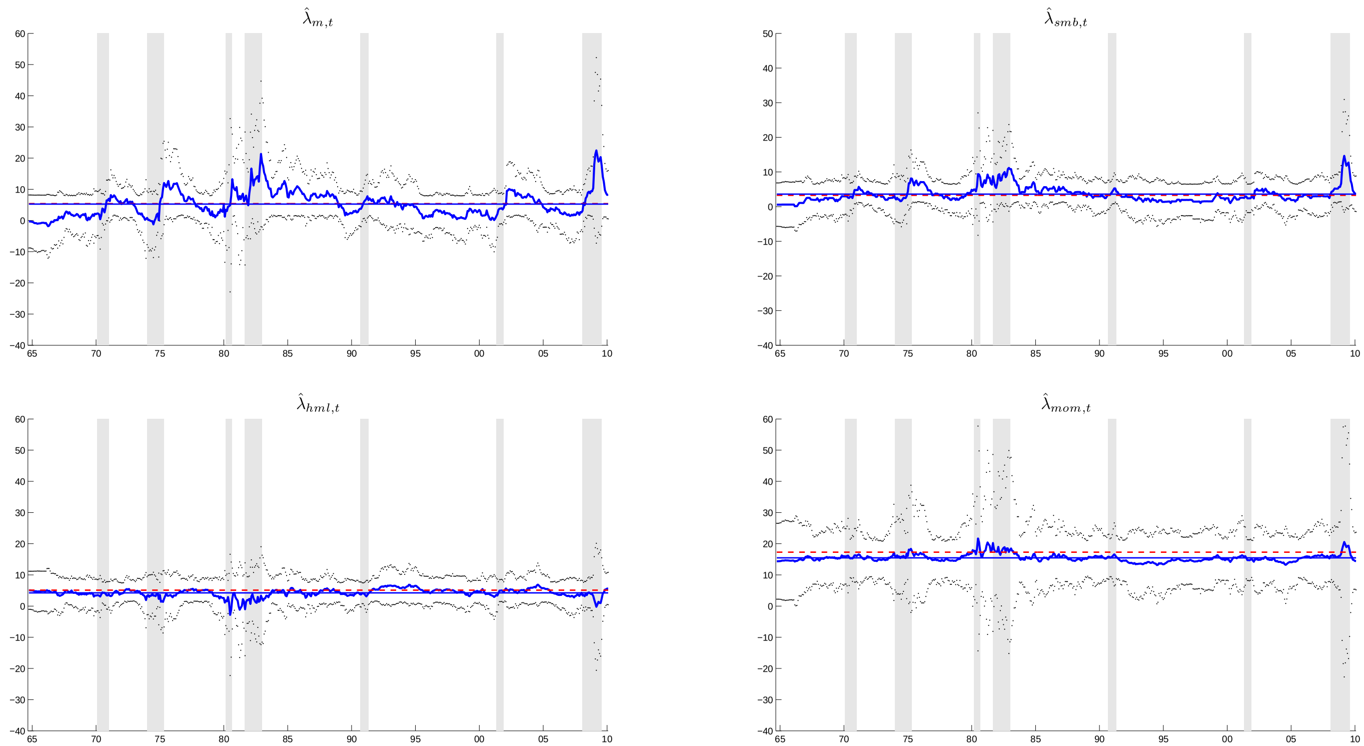
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% probability level. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks as base assets ($n = 9,936$ and $n^{\chi} = 3,900$). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER). The recessions start at the peak of a business cycle and end at the trough.

Figure 4.2: Path of estimated annualized risk premia with $n = 25$



The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% probability level. We use the returns of the 25 Fama-French portfolios. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER).

Figure 4.3: Path of estimated annualized risk premia with $n = 100$



The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% probability level. We use the returns of the 100 Fama-French portfolios. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER).

Table 4.1: Estimated annualized risk premia for the unconditional models

	Stocks ($n = 9,936$)		Portfolios ($n = 25$)		Portfolios ($n = 100$)	
	bias corrected estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval
			Four-factor model			
($n^{\chi} = 9,902$)						
λ_m	8.14	(3.26, 13.02)	5.70	(0.73, 10.67)	5.41	(0.42, 10.39)
λ_{smb}	2.86	(-0.50, 6.22)	3.02	(-0.48, 6.51)	3.28	(-0.27, 6.83)
λ_{hml}	-4.60	(-8.06, -1.14)	4.81	(1.21, 8.41)	5.11	(1.52, 8.71)
λ_{mom}	7.16	(2.56, 11.76)	34.03	(9.98, 58.07)	17.29	(8.55, 26.03)
			Fama-French model			
($n^{\chi} = 9,904$)						
λ_m	7.77	(2.89, 12.65)	5.04	(0.11, 9.97)	4.88	(-0.08, 9.83)
λ_{smb}	2.64	(-0.72, 5.99)	3.00	(-0.42, 6.42)	3.35	(-0.13, 6.83)
λ_{hml}	-5.18	(-8.65, -1.72)	5.20	(1.66, 8.74)	5.20	(1.63, 8.77)
			CAPM			
($n^{\chi} = 9,904$)						
λ_m	7.42	(2.54, 12.31)	6.98	(1.93, 12.02)	7.16	(2.06, 12.25)

The table contains the estimated annualized risk premia for the market (λ_m), size (λ_{smb}), book-to-market (λ_{hml}) and momentum (λ_{mom}) factors. We report the bias corrected estimates $\hat{\lambda}_B$ of λ for individual stocks ($n = 9,936$). In order to build the confidence intervals for $n = 9,936$, we use the HAC estimator $\hat{\Sigma}_f$ defined in Section 2.4. When we consider 25 and 100 portfolios as base assets, we compute an estimate of the variance-covariance matrix $\Sigma_{\lambda,n}$ defined in Section 2.4.

Table 4.2: Estimated annualized ν for the unconditional models

	Stocks ($n = 9,936$)		Portfolios ($n = 25$)		Portfolios ($n = 100$)	
	bias corrected estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval
Four-factor model						
($n^{\chi} = 9,902$)						
ν_m	3.29	(2.88, 3.69)	0.85	(-0.10, 1.79)	0.55	(-0.46, 1.57)
ν_{smb}	-0.41	(-0.95, 0.13)	-0.26	(-1.24, 0.72)	0.01	(-1.14, 1.16)
ν_{hml}	-9.38	(-10.12, -8.64)	0.03	(-0.95, 1.01)	0.33	(-0.63, 1.30)
ν_{mom}	-1.47	(-2.86, -0.08)	25.40	(1.80, 49.00)	8.66	(1.23, 16.10)
Fama-French model						
($n^{\chi} = 9,904$)						
ν_m	2.92	(2.48, 3.35)	0.18	(-0.51, 0.87)	0.02	(-0.84, 0.88)
ν_{smb}	-0.63	(-1.11, -0.15)	-0.27	(-0.93, 0.40)	0.08	(-0.85, 1.01)
ν_{hml}	-9.96	(-10.62, -9.31)	0.41	(-0.32, 1.15)	0.42	(-0.44, 1.28)
CAPM						
($n^{\chi} = 9,904$)						
ν_m	2.57	(2.17, 2.97)	2.12	(0.85, 3.40)	2.30	(0.84, 3.77)

The table contains the annualized estimates of the components of vector ν for the market (ν_m), size (ν_{smb}), book-to-market (ν_{hml}) and momentum (ν_{mom}) factors. We report the bias corrected estimates $\hat{\nu}_B$ of ν for individual stocks ($n = 9,936$). In order to build the confidence intervals, we compute $\hat{\Sigma}_{\nu}$ in Proposition 4 for $n = 9,936$. When we consider 25 and 100 portfolios as base assets, we compute an estimate of the variance-covariance matrix $\Sigma_{\nu,n}$ defined in Section 2.4.

Table 4.3: Test results for asset pricing restrictions in the unconditional models

	Test of the null hypothesis $\mathcal{H}_0: \ a(\gamma) = b(\gamma)' \nu$			Test of the null hypothesis $\mathcal{H}_0: \ a(\gamma) = 0$		
	Stocks	Portfolios ($n = 25$)	Portfolios ($n = 100$)	Stocks	Portfolios ($n = 25$)	Portfolios ($n = 100$)
Four-factor model						
	$(n^{\chi} = 1,400)$			$(n^{\chi} = 1,400)$		
Test statistic	2.0088	35.2231	253.2575	19.1803	74.9100	263.3395
p-value	0.0223	0.0267	0.0000	0.0000	0.0000	0.0000
Fama-French model						
	$(n^{\chi} = 1,400)$			$(n^{\chi} = 1,400)$		
Test statistic	2.9593	83.6846	253.9652	28.0328	87.3767	270.7899
p-value	0.0015	0.0000	0.0000	0.0000	0.0000	0.0000
CAPM						
	$(n^{\chi} = 1,400)$			$(n^{\chi} = 1,400)$		
Test statistic	8.2576	110.8368	276.3679	11.5882	111.6735	278.3949
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

We compute the statistics $\tilde{\Sigma}_{\xi}^{-1/2}\hat{\xi}_{nT}$ based on $\hat{Q}_e$ and $\hat{Q}_a$ defined in Proposition 5 for the individual stocks to test the null hypotheses $\mathcal{H}_0 : \ a(\gamma) = b(\gamma)' \nu$ and $\mathcal{H}_0 : \ a(\gamma) = 0$, respectively. The trimming levels are $\chi_{1,T} = 15$ and $\chi_{2,T} = 546/240$. For $n = 25$ and $n = 100$, we compute the weighted statistics $T\hat{e}'\hat{V}^{-1}\hat{e}$ and $T\hat{a}'\hat{V}_a^{-1}\hat{a}$ (Gibbons, Ross and Shanken (1989)), where $\hat{e}$ and $\hat{a}$ are $n \times 1$ vectors with elements $\hat{e}_i$ and $\hat{a}_i$, and $\hat{V} = (\hat{v}_{ij})$ and $\hat{V}_a = (\hat{v}_{a,ij})$ are $n \times n$ matrices with elements $\hat{v}_{ij} = \hat{c}_{\nu}'\hat{Q}_x^{-1}\hat{S}_{ij}\hat{Q}_x^{-1}\hat{c}_{\nu}$, and $\hat{v}_{a,ij} = E_1'\hat{Q}_x^{-1}\hat{S}_{ij}\hat{Q}_x^{-1}E_1$. The table reports the p-values of the statistics.

Table 4.4: Test results for the asset pricing restrictions in the conditional models

	Test of the null hypothesis $\mathcal{H}_0:\ \beta_1(\gamma)=\beta_3(\gamma)\nu$			Test of the null hypothesis $\mathcal{H}_0:\ \beta_1(\gamma)=0$		
	Stocks	Portfolios ($n=25$)	Portfolios ($n=100$)	Stocks	Portfolios ($n=25$)	Portfolios ($n=100$)
			Four-factor model			
	$(n^\chi=1,373)$			$(n^\chi=1,373)$		
Test statistic	3.2514	13.4815	11.6389	3.8683	14.3080	12.2098
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
			Fama-French model			
	$(n^\chi=1,393)$			$(n^\chi=1,393)$		
Test statistic	3.1253	15.7895	12.8938	3.8136	15.9038	13.0349
p-value	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000
			CAPM			
	$(n^\chi=1,395)$			$(n^\chi=1,395)$		
Test statistic	1.7322	9.2934	9.5153	1.7381	9.6680	9.8007
p-value	0.0416	0.2076	0.1825	0.0411	0.0000	0.0000

We compute the statistics $\tilde{\Sigma}_{\xi}^{-1/2}\hat{\xi}_{nT}$ based on $\hat{Q}_e$ and $\hat{Q}_a$ defined in Proposition 5 for the individual stocks to test the null hypotheses $\mathcal{H}_0:\ \beta_1(\gamma)=\beta_3(\gamma)\nu$ and $\mathcal{H}_0:\ \beta_1(\gamma)=0$, respectively. The trimming levels are $\chi_{1,T}=15$ and $\chi_{2,T}=546/240$. For $n=25$ and $n=100$, we compute the test statistics $T\hat{Q}_e$ and $T\hat{Q}_a$. The table reports the p-values of the statistics.

Chapter 5

Monte-Carlo experiments

In this chapter, we perform simulation exercises on balanced and unbalanced panels in order to study the properties of our estimation and testing approaches. We pay particular attention to the interaction between panel dimensions n and T in finite samples since we face conditions like $n = o(T^3)$ for inference with $\hat{\nu}$, and $n = o(T^2)$ for inference with $\hat{Q}_e$ and $\hat{Q}_a$, in the theoretical results. The simulation design mimics the empirical features of our data. The balanced case serves as benchmark to understand when T is not sufficiently large w.r.t. n to apply the theory. The unbalanced case shows that we can exploit the guidelines found for the balanced case when we substitute the average of the sample sizes of the individual assets, i.e., a kind of operative sample size, for T. To summarize our Monte Carlo findings, we do not face any finite sample distortions for the inference with $\hat{\nu}$ when $n = 1,000$ and $T = 150$, and with $\hat{Q}_e$ and $\hat{Q}_a$ when $n = 1,000$ and $T = 350$. In light of these results, we do not expect to face significant inference bias in our empirical application.

5.1 Balanced panel

We simulate S datasets of excess returns from an unconditional one-factor model (CAPM), we estimate the parameter ν, and compute the test statistics. A simulated dataset includes: a vector of intercepts $a^s \in \mathbb{R}^n$, a vector of factor loadings $b^s \in \mathbb{R}^n$, and a variance-covariance matrix $\Omega^s \in \mathbb{R}^{n \times n}$. At each simulation $s = 1, ..., S$, we randomly draw $n \leq 9,904$ assets from the empirical sample that comprises $9,904$ individual stocks. The assets are listed by industrial sectors. We use the classification proposed by Ferson and Harvey

(1999). The vector b^s is composed by the estimated factor loadings for the n randomly chosen assets. At each simulation, we build a block diagonal matrix Ω^s with blocks matching industrial sectors. The n elements of the main diagonal of Ω^s correspond to the variances of the estimated residuals of the individual assets. The off-diagonal elements of Ω^s are covariances computed by fixing correlations within a block equal to the average correlation of the industrial sector computed from the $9,904 \times 9,904$ thresholded variance-covariance matrix of estimated residuals. Hence we get a setting in line with the block dependence case developed in Appendix A.4.

In order to study the size and power properties of our procedure, we set the values of the intercepts a_i^s according to four data generating processes:

DGP1: The true parameter is $\nu_0 = 0.00\%$ and the a_i^s are generated under the null hypothesis $\mathcal{H}_0 : \ a_i^s = 0$;

DGP2: The true parameter is the empirical estimate of ν, $\nu_0 = 2.57\%$, and the a_i^s are generated under the null hypothesis $\mathcal{H}_0 : \ a_i^s = b_i^s \nu_0$;

DGP3: The a_i^s are generated under the alternative hypothesis $\mathcal{H}_a : \ a_i^s = (0.5 b_i^s + 0.5)\, \nu_0$, where $\nu_0 = 2.57\%$;

DGP4: The a_i^s are generated under the three-factor alternative hypothesis: $\mathcal{H}_a : \ a_i^s = b_{i,(3)}^{s\prime} \nu_{0,(3)}$ where $b_{i,(3)}^s \in \mathbb{R}^3$ and $\nu_{0,(3)} = [2.92\%, \ -0.63\%, \ -9.96\%]'$ are estimates for the Fama-French model on the CRSP dataset.

DGP1 and DGP2 match two different null hypotheses. The null hypothesis for DGP1 assumes that the factor comes from a tradable asset, and for DGP2 that it does not. DGP3 and DGP4 match two different alternative hypotheses as suggested by MacKinlay (1995). DGP3 is a “non risk-based alternative”. It represents a deviation from CAPM, which is unrelated to risk: we take the one-factor model calibrated on the data with intercepts deviating from the no arbitrage restriction. DGP4 is a “risk-based alternative”. It represents a deviation from CAPM, which comes from missing risk factors: we take intercepts from a three-factor model calibrated on the data, and then we estimate a one-factor model.

Let us define the simulated excess returns $R_{i,t}^s$ of asset i at time t as follows

$$R_{i,t}^s = a_i^s + b_i^s f_t + \varepsilon_{i,t}^s, \text{ for } i = 1, ..., n, \text{ and } t = 1, ..., T, \tag{5.1}$$

where f_t is the market excess return and $\varepsilon_{i,t}^s$ is the error term. The $n \times 1$ error vectors $\boldsymbol{\varepsilon}_t^s$ are independent across time and Gaussian with mean zero and variance-covariance matrix Ω^s. We apply our estimation approach on every simulated dataset of excess returns. We estimate the parameter ν and we compute the statistics described in Section 2.5. Since the panel is balanced, we do not need to fix $\chi_{2,T}$. We only use $\chi_{1,T} = 15$. However, this trimming level does not affect the number of assets n in the simulations. In order to compute the thresholded estimator of the variance-covariance matrix of $\hat{\nu}$, namely $\tilde{\Sigma}_\nu$ (see Proposition 4 in the paper), and the thresholded variance estimator $\tilde{\Sigma}_\xi$ for the test statistics, we fix the parameter M equal to 0.0780, that is used in the empirical application. We define the parameter M using a cross-validation method as proposed in Bickel and Levina (2008). We build random subsamples from the CRSP sample. For each subsample, we minimize a risk function that exploits the difference between a thresholded variance-covariance matrix and a target variance-covariance matrix (see Bickel and Levina (2008) for details).

In order to understand how our estimation approach works for different finite samples, we perform exercises combining different values of the cross-sectional dimension n and the time dimension T. Table 5.1 reports estimation results for estimator $\hat{\nu}$, and for the bias-adjusted estimator $\hat{\nu}_B$, under DGP 1 and 2. The results include the bias of both estimators, the variance and the Root Mean Square Error (RMSE) of estimator $\hat{\nu}_B$, and the coverage of the 95% confidence interval for parameter ν based on Proposition 4. The bias of estimator $\hat{\nu}$ is decreasing in absolute value with time series size T and is rather stable w.r.t. cross-sectional size n. The analytical bias correction is rather effective, and the bias of estimator $\hat{\nu}_B$ is small. For instance, for sample sizes $T = 150$ and $n = 1000$, under DGP 2 the bias of estimator $\hat{\nu}_B$ is equal to -0.03, which in absolute value is about 1% of the true value of the parameter $\nu = 2.57$. The variance of estimator $\hat{\nu}_B$ is decreasing w.r.t. both time-series and cross-sectional sample sizes T and n. These features reflect the large sample distribution of the estimators derived in Proposition 3. The coverage of the confidence intervals is close to the nominal level 95% across the considered designs and sample sizes.

In Table 5.2, we display the rejection rates for the test of the null hypothesis $\nu = 0$ (tradable factor). This null hypothesis is satisfied in DGP 1, and the rejection rates are rather close to the nominal size 5% of the test, with a slight overrejection. In DGP 2, parameter ν is different from zero, and the test features a power equal to 100%.

Tables 5.3 and 5.4 report the results for the tests of the null hypotheses $\mathcal{H}_0 : a(\gamma) = 0$ and $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$, respectively. The test statistics are based on $\hat{Q}_a$ and $\hat{Q}_e$ as defined in Proposition 5. DGP 1 satisfies the null hypothesis for both tests. For $T = 150$, we observe an oversize, that is increasing w.r.t. cross-sectional size n. The time series dimension $T = 150$ is likely too small compared to cross-sectional size $n = 1000$ and this combination does not reflect the condition $n = o(T^2)$ for the validity of the asymptotic Gaussian approximation of the statistics. For $T = 500$ instead, the rejection rates of the tests are quite close to the nominal size. DGP 2 satisfies the null hypothesis of the test based on $\hat{Q}_e$, but corresponds to an alternative hypothesis for the test based on $\hat{Q}_a$. The former statistic features a similar behaviour as under DGP 1, while the power of the latter statistic is increasing w.r.t. n. Finally, the power of both statistics under the "non risk-based"' and "risk-based" alternatives in DGP 3 and DGP 4 is very large, with rejection rates close to 100% for all considered combinations of sample sizes n and T.

5.2 Unbalanced panel

Let us repeat similar exercises as in the previous section, but with unbalanced characteristics for the simulated datasets. We introduce these characteristics through a matrix of observability indicators $I^s \in \mathbb{R}^{n\times T}$. The matrix gathers the indicator vectors for the n randomly chosen assets. We fix the maximal sample size $T = 546$ as in the empirical application. In the unbalanced setting, the excess returns $R^s_{i,t}$ of asset i at time t is:

$$R^s_{i,t} = a^s_i + b^s_i f_t + \varepsilon^s_{i,t}, \text{ if } I^s_{i,t} = 1, \text{ for } i = 1, ..., n, \text{ and } t = 1, ..., T, \tag{5.2}$$

where $I^s_{i,t}$ is the observability indicator of asset i at time t.

In Tables 5.5 and 5.6, we provide the operative cross-sectional and time-series sample sizes in the Monte-Carlo repetitions for trimming $\chi_{1,T} = 15$ and four different levels of trimming $\chi_{2,T}$. More precisely, in Table 5.5 we report the average number $\bar{n}^\chi$ of retained assets across simulations, as well as the minimum $\min(n^\chi)$ and the maximum $\max(n^\chi)$ across simulations. For the lowest level of trimming $\chi_{2,T} = T/12$, all assets are kept in all simulations, while for the level of trimming $\chi_{2,T} = T/60$ on average we keep about two thirds of the assets. In Table 5.6, we report the average across assets of the $\bar{T}_i$, that are the average time-series size T_i for asset i across simulations, as well as the min and the max of the $\bar{T}_i$. Since the distribution of T_i for an asset i is right-skewed, we also report the average across assets of the median T_i. For trimming level $\chi_{2,T} = T/60$, the average mean time-series size is about 180 months, while the average median time-series size is 140 months.

In Table 5.7, we display the results for estimators $\hat{\nu}$ and $\hat{\nu}_B$. The bias adjustment reduces substantially the bias for estimation of parameter ν. For trimming level $\chi_{2,T} = T/60$, the coverage of the confidence interval is close to the nominal size 95% for all considered cross-sectional sizes, while for $\chi_{2,T} = T/12$ the coverage deteriorates with increasing cross-sectional size. In comparison with Table 5.1, the bias and variance of estimator $\hat{\nu}_B$ are larger than the ones obtained in the balanced case with time-series size $T = 500$. However, for trimming level $\chi_{2,T} = T/60$, the results are similar to the ones with $T = 150$ in Table 5. In fact, this time-series size of the balanced panel reflects the operative sample sizes for that trimming level observed in Table 5.6. Similar comments apply for Table 5.8, where we report the results for the test of the hypothesis $\nu = 0$. For trimming level $\chi_{2,T} = T/60$, the size of the test is close to the nominal level 5% under DGP 1, and the the power is 100% under DGP 2.

In Tables 5.9 and 5.10, we display the results for the tests based on $\hat{Q}_a$ and $\hat{Q}_e$, respectively. For trimming level $\chi_{2,T} = T/120$, we observe an oversize, that increases with the cross-sectional dimension. We get a similar behaviour with more severe oversize with lower trimming levels (not reported). We expect these findings from the results in the previous section. Indeed, for trimming level $\chi_{2,T} = T/120$, the operative time-series sample size in Table 10 is around 200 months, and in Tables 5.3 and 5.4, for a balanced panel with $T = 150$, the statistics are oversized. For trimming level $\chi_{2,T} = T/240$ with operative size of about 350 months, the oversize of the statistics is moderate. Finally, the power of the statistics is very large also in the unbalanced case, and close to 100%.

Table 5.1: Estimation of ν, balanced case

$T = 150$	DGP 1				DGP 2			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Bias($\hat{\nu}$)	-0.0742	-0.0567	-0.0585	-0.0586	-0.1630	-0.1472	-0.1484	-0.1493
Bias($\hat{\nu}_B$)	-0.0244	-0.0063	-0.0082	-0.0083	-0.0319	-0.0156	-0.0169	-0.0178
Var($\hat{\nu}_B$)	0.1167	0.0394	0.0179	0.0121	0.1140	0.0401	0.0189	0.0121
RMSE($\hat{\nu}_B$)	0.3423	0.1985	0.1340	0.1102	0.3390	0.2007	0.1383	0.1114
Coverage	0.9320	0.9290	0.9350	0.9370	0.9370	0.9290	0.9320	0.9360
$T = 500$	DGP 1				DGP 2			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Bias ($\hat{\nu}$)	-0.0587	-0.0640	-0.0687	-0.0654	-0.0847	-0.0926	-0.0972	-0.0937
Bias($\hat{\nu}_B$)	-0.0002	-0.0063	-0.0110	-0.0077	-0.0025	-0.0074	-0.0120	-0.0085
Var($\hat{\nu}_B$)	0.0343	0.0113	0.0060	0.0040	0.0341	0.0114	0.0061	0.0041
RMSE($\hat{\nu}_B$)	0.1851	0.1066	0.0781	0.0634	0.1846	0.1068	0.0788	0.0642
Coverage	0.9370	0.9340	0.9370	0.9390	0.9430	0.9370	0.9360	0.9320

Table 5.2: Test of $\nu = 0$, balanced case

$T = 150$	DGP 1				DGP 2			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Rejection rate	0.0680	0.0710	0.0650	0.0630	1.0000	1.0000	1.0000	1.0000
$T = 500$	DGP 1				DGP 2			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Rejection rate	0.0630	0.0660	0.0630	0.0610	1.0000	1.0000	1.0000	1.0000

Table 5.3: Test of the null hypothesis $\mathcal{H}_0 :\ a(\gamma) = 0$, balanced case

$T = 150$	**DGP 1**			**DGP 2**			**DGP 3**			**DGP 4**		
n	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500
Size/Power	0.1180	0.1400	0.1500	0.3850	0.5720	0.7170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$T = 500$	**DGP 1**			**DGP 2**			**DGP 3**			**DGP 4**		
n	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500
Size/Power	0.0730	0.0610	0.0740	0.9240	0.9920	0.9970	0.9990	1.0000	1.0000	0.9990	1.0000	1.0000

Table 5.4: Test of the null hypothesis $\mathcal{H}_0 :\ a(\gamma) = b(\gamma)' \nu$, balanced case

$T = 150$	**DGP 1**			**DGP 2**			**DGP 3**			**DGP 4**		
n	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500
Size/Power	0.1110	0.1340	0.1460	0.1070	0.1360	0.1420	0.9970	1.0000	1.0000	1.0000	1.0000	1.0000
$T = 500$	**DGP 1**			**DGP 2**			**DGP 3**			**DGP 4**		
n	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500	500	1,000	1,500
Size/Power	0.0710	0.0570	0.0730	0.0730	0.0690	0.0750	0.9990	1.0000	1.0000	0.9990	1.0000	1.0000

Table 5.5: Operative cross-sectional sample size

trimming level	$\chi_{2,T} = \frac{T}{12}$				$\chi_{2,T} = \frac{T}{60}$			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
$\bar{n}^{\chi}$	1,000	3,000	6,000	9,000	660	2,000	4,000	6,000
$\min(n^{\chi})$	1,000	3,000	6,000	9,000	600	1,900	3,900	5,900
$\max(n^{\chi})$	1,000	3,000	6,000	9,000	700	2,100	4,100	6,100
trimming level	$\chi_{2,T} = \frac{T}{120}$				$\chi_{2,T} = \frac{T}{240}$			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
$\bar{n}^{\chi}$	400	1,250	2,400	3,600	140	430	850	1,250
$\min(n^{\chi})$	350	1,100	2,300	3,500	100	370	800	1,200
$\max(n^{\chi})$	440	1,300	2,500	3,650	170	470	900	1,300

Table 5.6: Operative time-series sample size

trimming level	$\chi_{2,T} = \frac{T}{12}$	$\chi_{2,T} = \frac{T}{60}$	$\chi_{2,T} = \frac{T}{120}$	$\chi_{2,T} = \frac{T}{240}$
$\text{mean}(\bar{T}_i)$	130	180	240	360
$\min(\bar{T}_i)$	110	160	210	350
$\max(\bar{T}_i)$	140	190	260	380
$\text{mean}(\text{median}(T_i))$	90	140	197	330

Table 5.7: Estimation of ν, unbalanced case

trimming level: $\chi_{2,T} = \frac{T}{12}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Bias($\hat{\nu}$)	-0.3059	-0.3119	-0.3047	-0.3021	-0.4211	-0.4324	-0.4202	-0.4201
Bias($\hat{\nu}_B$)	-0.0893	-0.0954	-0.0880	-0.0854	-0.1127	-0.1233	-0.1113	-0.1113
Var($\hat{\nu}_B$)	0.1207	0.0409	0.0214	0.0124	0.1222	0.0405	0.0218	0.0124
RMSE($\hat{\nu}_B$)	0.3586	0.2235	0.1706	0.1402	0.3671	0.2360	0.1848	0.1574
Coverage	0.9230	0.9010	0.8740	0.8750	0.9180	0.8880	0.8410	0.8320
trimming level: $\chi_{2,T} = \frac{T}{60}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Bias($\hat{\nu}$)	-0.1703	-0.1738	-0.1675	-0.1596	-0.2454	-0.2478	-0.0411	-0.2329
Bias($\hat{\nu}_B$)	-0.0349	-0.0381	-0.0318	-0.0238	-0.0453	-0.0474	-0.0411	-0.0325
Var($\hat{\nu}_B$)	0.1294	0.0436	0.0231	0.0141	0.1281	0.0438	0.0232	0.0144
RMSE($\hat{\nu}_B$)	0.3613	0.2122	0.1551	0.1212	0.3606	0.2145	0.1578	0.1241
Coverage	0.9360	0.9310	0.9240	0.9350	0.9430	0.9310	0.9200	0.9300

Table 5.8: Test of $\nu = 0$, unbalanced case

trimming level: $\chi_{2,T} = \frac{T}{12}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Rejection rate	0.0770	0.0990	0.1260	0.1250	1.0000	1.0000	1.0000	1.0000
trimming level: $\chi_{2,T} = \frac{T}{60}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Rejection rate	0.0640	0.0690	0.0760	0.0650	1.0000	1.0000	1.0000	1.0000

Table 5.9: Test of the null hypothesis $\mathcal{H}_0: \ \beta_1(\gamma) = 0$, unbalanced case

trimming level: $\chi_{2,T} = \frac{T}{120}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	0.1180	0.1710	0.2420	0.3030	0.6010	0.9410	0.9980	1.000
	DGP 3				**DGP 4**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	1.0000	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000	1.0000
trimming level: $\chi_{2,T} = \frac{T}{240}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	0.0880	0.0860	0.1020	0.1310	0.5320	0.8730	0.9920	1.0000
	DGP 3				**DGP 4**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	1.0000	1.0000	1.0000	1.0000	0.9740	1.0000	1.0000	1.0000

Table 5.10: Test of the null hypothesis $\mathcal{H}_0: \ \beta_1(\gamma) = \beta_3(\gamma)\,\nu$, unbalanced case

trimming level: $\chi_{2,T} = \frac{T}{120}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	0.1130	0.1670	0.2370	0.3010	0.0940	0.2190	0.2590	0.3740
	DGP 3				**DGP 4**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	1.0000	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000	1.0000
trimming level: $\chi_{2,T} = \frac{T}{240}$								
	DGP 1				**DGP 2**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	0.0800	0.0790	0.1000	0.1290	0.0790	0.0870	0.1080	0.1440
	DGP 3				**DGP 4**			
n	1,000	3,000	6,000	9,000	1,000	3,000	6,000	9,000
Size/Power	0.9990	1.0000	1.0000	1.0000	0.9690	1.0000	1.0000	1.0000

Chapter 6

Robustness checks

In this chapter, we perform several checks to evaluate the robustness of the empirical results reported in Chapter 4. In particular, we estimate the paths of the time-varying risk premia and we compute the test statistics by:

a. Assuming several asset pricing models as baseline specification;

b. Using several sets of asset-specific instruments $Z_{i,t-1}$;

c. Using several sets of common instruments Z_{t-1};

d. Assuming that the time-varying betas $b_{i,t}$ depend only on the asset-specific instruments.

In Table 6.1, we provide the details of the conditional specifications for the four exercises. We use the following abbreviations. For common instruments, we denote by ts_t the term spread, ds_t the default spread, and $divY_t$ the dividend yield. The dividend yield is provided by CRSP. For asset-specific instruments, we denote by $mc_{i,t}$ the market capitalization, $bm_{i,t}$ the book-to-market, and $ind_{i,t}$ the return of the corresponding industry portfolio. For each exercise, when not explicitly indicated in Table 6.1, the specification is the four-factor model, the vector of common instruments is $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$ and the asset-specific instrument is the scalar $Z_{i,t-1} = bm_{i,t-1}$. Table 6.1 reports the operative trimmed population of individual stocks and the number of regressors in the first-pass time series regression for each exercise that we implement. Indeed, the population of individual stocks changes depending on the asset pricing model (Exercise a) as an effect of the trimming conditions: the number of assets decreases as the number K of factors increases.

Moreover, by using the four-factor model as baseline and modifying the sets of instruments, the number of assets decreases as the number of regressors in the first pass increases (see Exercise c) .

We first present conditional estimates of risk premia by using several asset pricing models as baseline (Exercise a). Panel A of Figure 6.1 compares the estimated time-varying paths of market risk premia when we assume the four-factor model (shown in Chapter 4) and the CAPM. Panel B compares the estimates $\hat{\lambda}_{m,t}$ for the four-factor model and the Fama-French model. The paths look very similar. The discrepancy between the estimates of the CAPM and the four-factor model is explained by the three factors (size, value and momentum factor) that we introduce in the four-factor model. Figure 6.2 plots the estimated time-varying paths of risk premia for the size and value factors computed on the four-factor model and on the Fama-French model. The risk premium for the size factor is very similar for the two models. The value risk premium for the Fama-French model takes slightly smaller values than that for the four-factor model and it exhibits a counter-cyclical path. Overall, the conditional estimates of the risk premia are stable with respect to the asset pricing model that is assumed for the excess returns.

Figures 6.3 and 6.4 plot the estimates of the risk premia by adopting several sets of asset-specific instruments $Z_{i,t-1}$ (Exercise b). We do not modify the set of common instruments Z_{t-1} compared to Chapter 4. In Figure 6.3, we get the estimates by setting the scalar $Z_{i,t-1}$ equal to the market capitalization of firm i. In Figure 6.4, we set $Z_{i,t-1}$ equal to the monthly returns of the industry portfolio for the industry asset i belongs to. We use the 48 Fama-French industry portfolios. The risk premia paths look very similar to the results in Chapter 4. The results for the tests of the asset pricing restrictions for the conditional specifications in Exercise b are reported in Table 6.2, upper panel. The test statistics reject the null hypotheses at 5% level.

The time-varying paths of the risk premia showed in Figures 6.5 and 6.6 are computed by modifying the set of common instruments $Z_{t-1} = \left[1,\ Z_{t-1}^{*\prime}\right]'$ (Exercise c). In Figure 6.5, Z_t^* is a bivariate vector that includes the default spread and the dividend yield. The paths of the risk premia for market, value and momentum factors look similar to the results in Chapter 4. However, the risk premium for the size factor features a very stable pattern that does not correspond to the unconditional estimate. In Figure 6.6, vector Z_t^* includes the term spread, the default spread, and the dividend yield. The paths of the risk premia look similar to the results in Chapter 4. Introducing the dividend yield increases the discrepancy between the unconditional estimates and the average over time of conditional estimates for the size and momentum factors w.r.t. the results shown in Figure 4.1. On the contrary, this discrepancy is smaller for the value premium. Moreover, the risk premium of the momentum factor takes larger values than that in Figure 4.1. We also notice that including the dividend yield among the common instruments decreases the number of stocks after trimming. The test statistics reject the null hypothesis at 5% level (see Table 6.2), middle panel.

Finally, we consider conditional specifications in which the time-varying betas are linear functions of asset specific instruments $Z_{i,t-1}$ only (Exercise d). The risk premia are modelled via common instruments $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$ as usual. In Figure 6.7, $Z_{i,t-1}$ is a bivariate vector that includes the constant and the book-to-market equity of firm i. In Figure 6.8, vector $Z_{i,t-1}$ includes the constant and the return of the industry portfolio as asset-specific instrument. The paths of the risk premia for the four factors in Figure 6.7 look more volatile w.r.t. the paths in Figure 4.1. The risk premia for market, size and value factors in Figure 6.8 look similar to the results in Chapter 4. The risk premium for the momentum factor features a less stable pattern, albeit its confidence intervals look similar to that in Figure 4.1. In Table 6.2, lower panel, the test statistic does not reject the asset pricing restrinction $\mathcal{H}_0 :\ \beta_1(\gamma) = \beta_3(\gamma)\,\nu$ for the conditional specification with time-varying betas depending on book-to-market equity.

Table 6.1: Operative cross-sectional sample size (n^{χ}), number of factors (K) and instruments (q and p) and first-pass regressors (d) in the four exercises of robustness checks

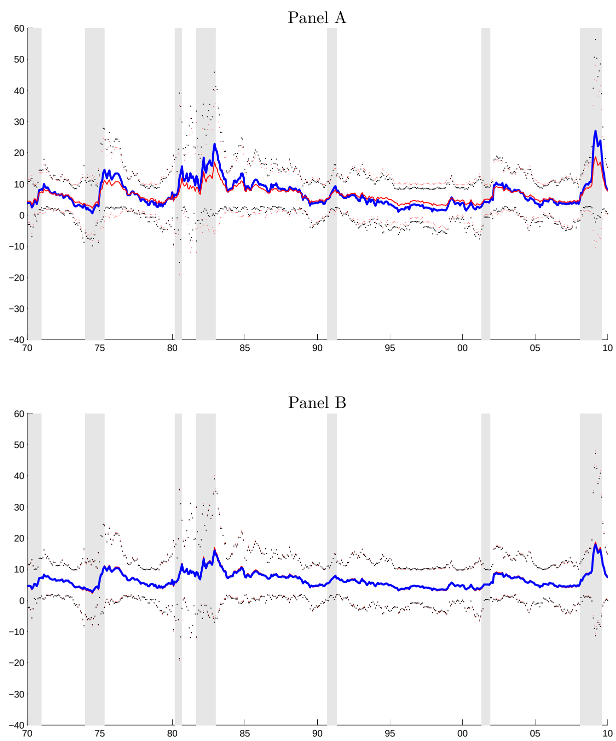
	n^{χ}	K	p	q	d		n^{χ}	K	p	q	d
Exercise a.						Exercise c.					
CAPM	5,225	1	3	1	13	$Z_{t-1} = [1, ds_{t-1}, divY_{t-1}]'$	1,107	4	3	1	25
Fama-French model	4,545	3	3	1	21	$Z_{t-1} = [1, ds_{t-1}, ts_{t-1}, divY_{t-1}]'$	667	4	4	1	34
Exercise b.						Exercise d.					
$Z_{i,t-1} = mc_{i,t-1}$	3,835	4	3	1	25	$Z_{i,t-1} = [1, bm_{i,t-1}]'$	6,208	4	3	2	8
$Z_{i,t-1} = ind_{i,t-1}$	4,748	4	3	1	25	$Z_{i,t-1} = [1, ind_{i,t-1}]'$	6,430	4	3	2	8

Table 6.2: Test results for asset pricing restrictions

	Test of the null hypothesis $\mathcal{H}_0: \beta_1(\gamma) = \beta_3(\gamma)\nu$		Test of the null hypothesis $\mathcal{H}_0: \beta_1(\gamma) = 0$	
Exercise b.				
	$Z_{i,t-1} = mc_{i,t-1}$	$Z_{i,t-1} = ind_{i,t-1}$	$Z_{i,t-1} = mc_{i,t-1}$	$Z_{i,t-1} = ind_{i,t-1}$
	$(n^\chi = 3,835)$	$(n^\chi = 4,748)$	$(n^\chi = 3,835)$	$(n^\chi = 4,748)$
Test statistic	8.0493	5.7601	8.7126	6.4357
p-value	0.0000	0.0000	0.0000	0.0000
Exercise c.				
	$Z_{t-1} = [1, ds_{t-1}, divY_{t-1}]'$	$Z_{t-1} = [1, ds_{t-1}, ts_{t-1}, divY_{t-1}]'$	$Z_{t-1} = [1, ds_{t-1}, divY_{t-1}]'$	$Z_{t-1} = [1, ds_{t-1}, ts_{t-1}, divY_{t-1}]'$
	$(n^\chi = 1,107)$	$(n^\chi = 667)$	$(n^\chi = 1,107)$	$(n^\chi = 667)$
Test statistic	2.7954	2.2463	3.6335	2.8434
p-value	0.0026	0.0123	0.0000	0.0022
Exercise d.				
	$Z_{i,t-1} = [1, bm_{i,t-1}]'$	$Z_{i,t-1} = [1, ind_{i,t-1}]'$	$Z_{i,t-1} = [1, bm_{i,t-1}]'$	$Z_{i,t-1} = [1, ind_{i,t-1}]'$
	$(n^\chi = 6,208)$	$(n^\chi = 6,430)$	$(n^\chi = 6,208)$	$(n^\chi = 6,430)$
Test statistic	1.3394	5.4079	3.5772	8.3430
p-value	0.0902	0.0000	0.0000	0.0000

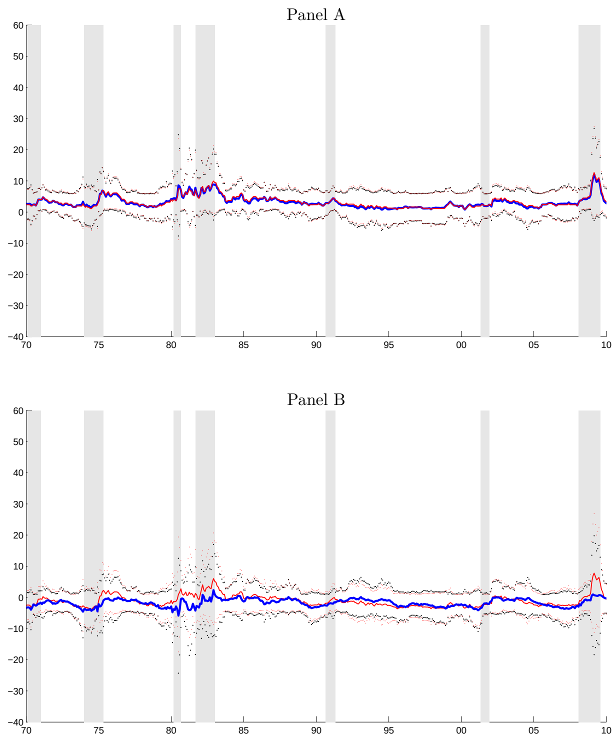
We compute the statistics $\tilde{\Sigma}_\xi^{-1/2}\hat{\xi}_{nT}$ based on $\hat{Q}_e$ and $\hat{Q}_a$ defined in Proposition 5 for n^χ individual stocks to test the null hypotheses $\mathcal{H}_0: \beta_1(\gamma) = \beta_3(\gamma)\nu$ and $\mathcal{H}_0: \beta_1(\gamma) = 0$. The table reports the statistics and their p-values when we use several sets of asset-specific instruments $Z_{i,t-1}$ (Exercise b) and common instruments Z_{t-1} (Exercise c), and when time-varying betas are functions of the asset-specific instruments only (Exercise d).

Figure 6.1: Path of estimated annualized risk premia for the market factor



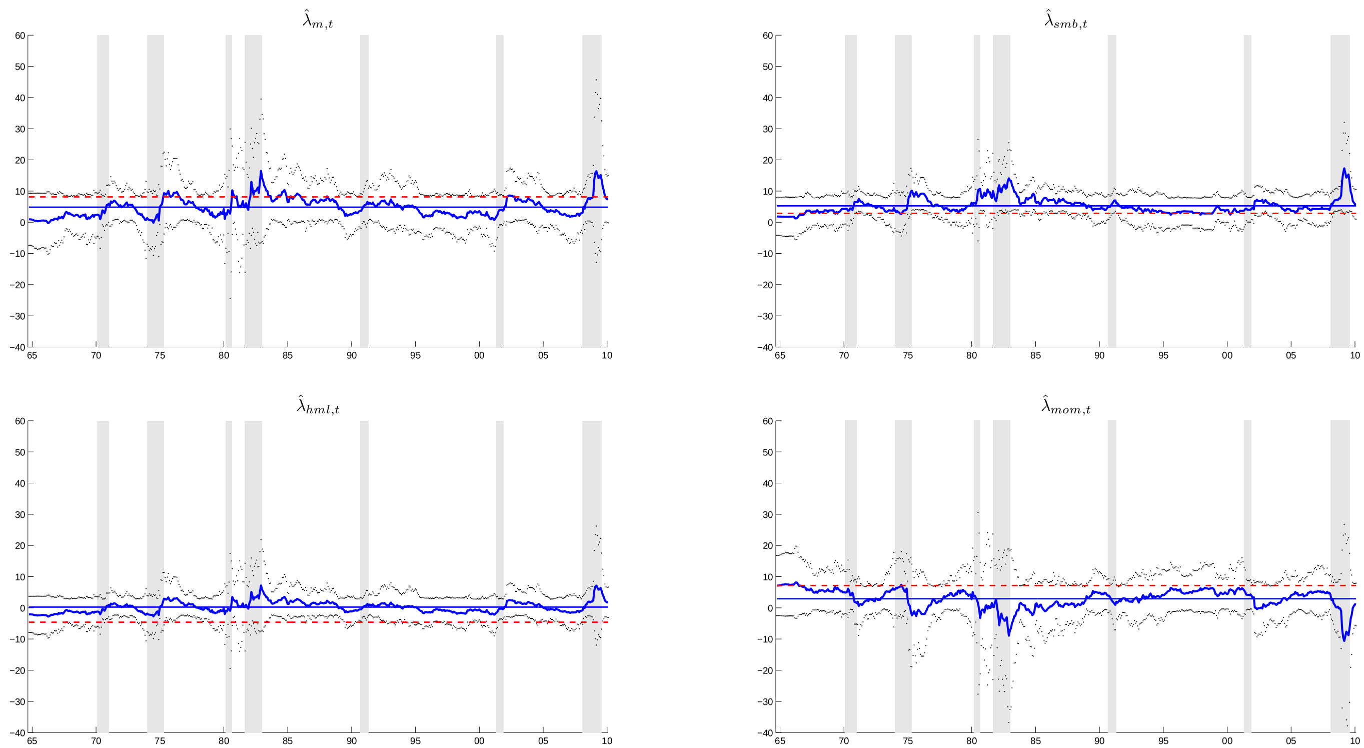
Panel A plots the paths of estimated annualized market risk premia $\hat{\lambda}_{m,t}$ computed by using the four-factor model (thin red line) and the CAPM (thick blue line). Panel B plot the paths of market risk premia $\hat{\lambda}_{m,t}$ estimated by assuming the four-factor model (thin red line) and the Fama-French model (thick blue line). The pointwise confidence intervals at 95% level are also displayed. The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.2: Path of estimated annualized risk premia for the size and value factors



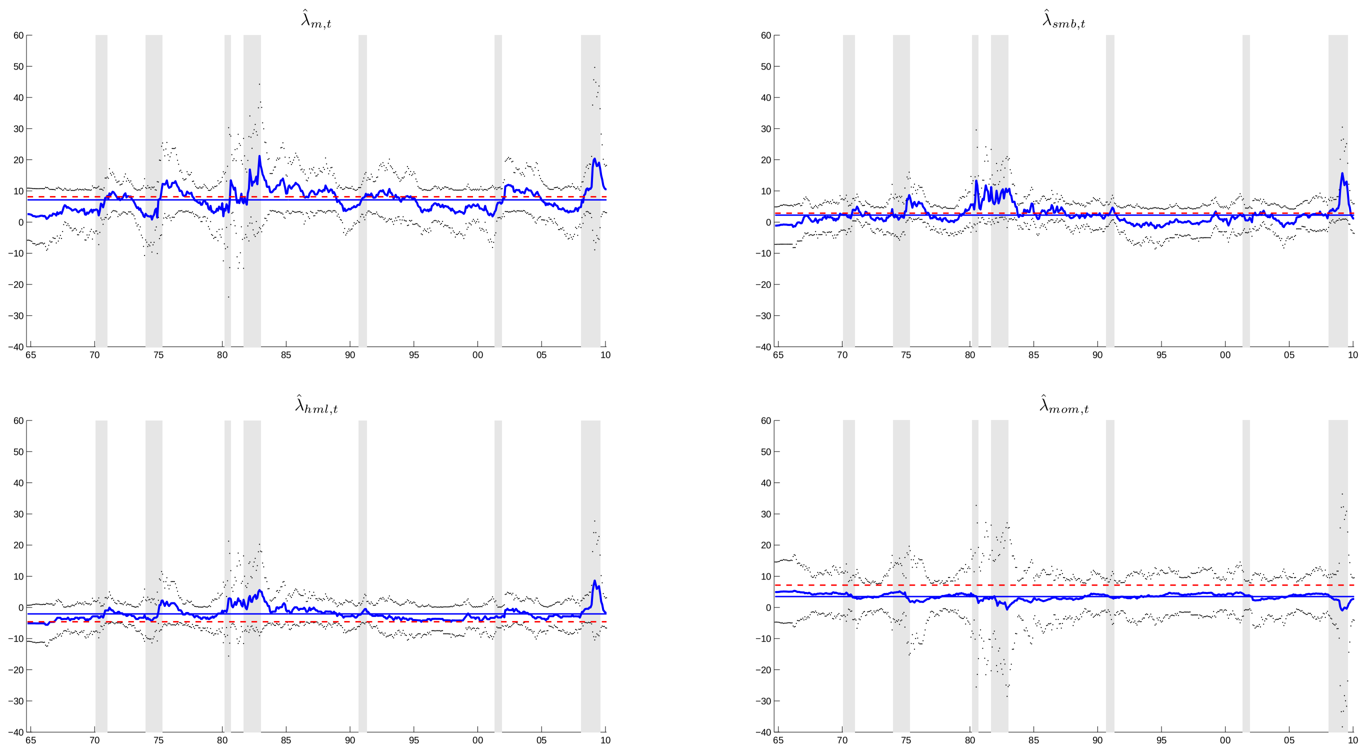
The figure plots the paths of estimated annualized risk premia $\hat{\lambda}_{smb,t}$ (Panel A) and $\hat{\lambda}_{hml,t}$ (Panel B) computed by using the four-factor model (thin red line) and the Fama-French model (thick blue line). The pointwise confidence intervals at 95% level are also displayed. The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.3: Path of estimated annualized risk premia computed using $Z_{i,t-1} = mc_{i,t-1}$



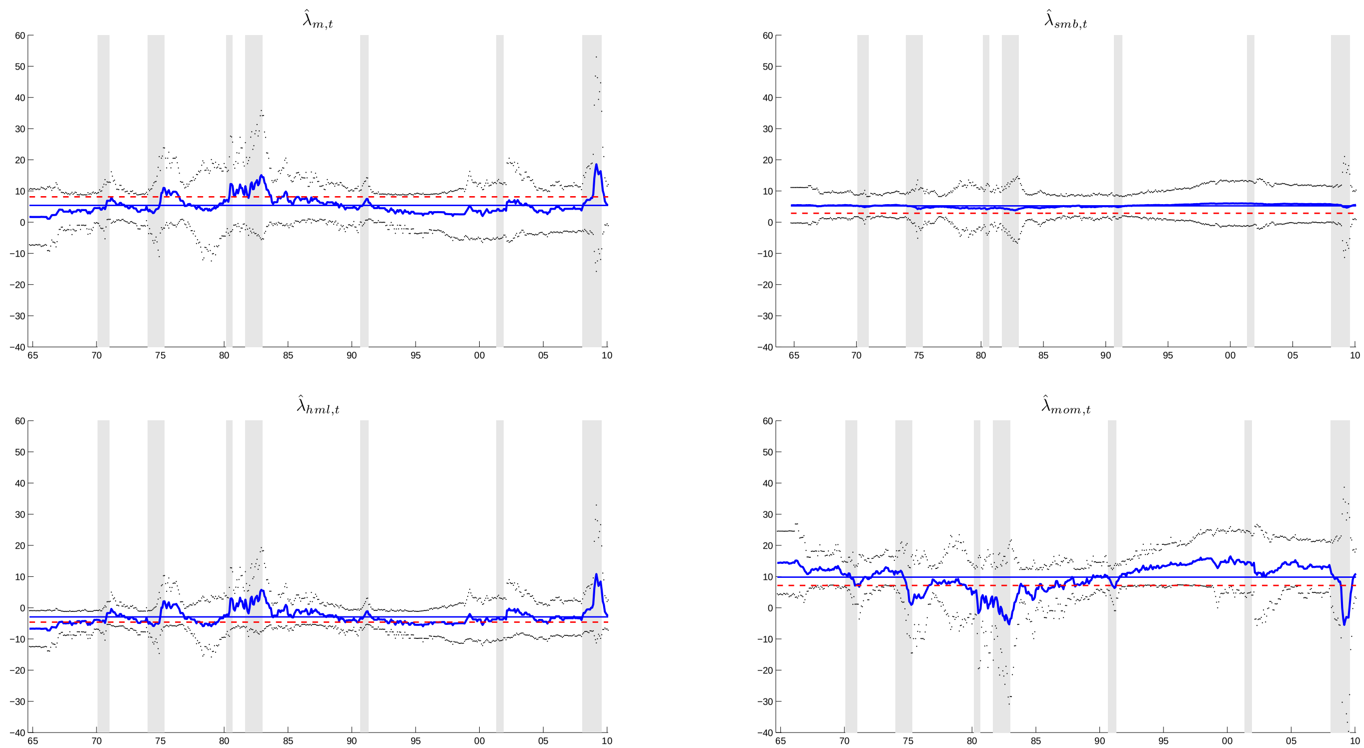
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when market capitalization is used as asset-specific instrument. The vector of common instruments is $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$. We also display the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks as base assets ($n = 9,936$ and $n^{\chi} = 3,835$). The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.4: Path of estimated annualized risk premia computed using $Z_{i,t-1} = ind_{i,t-1}$



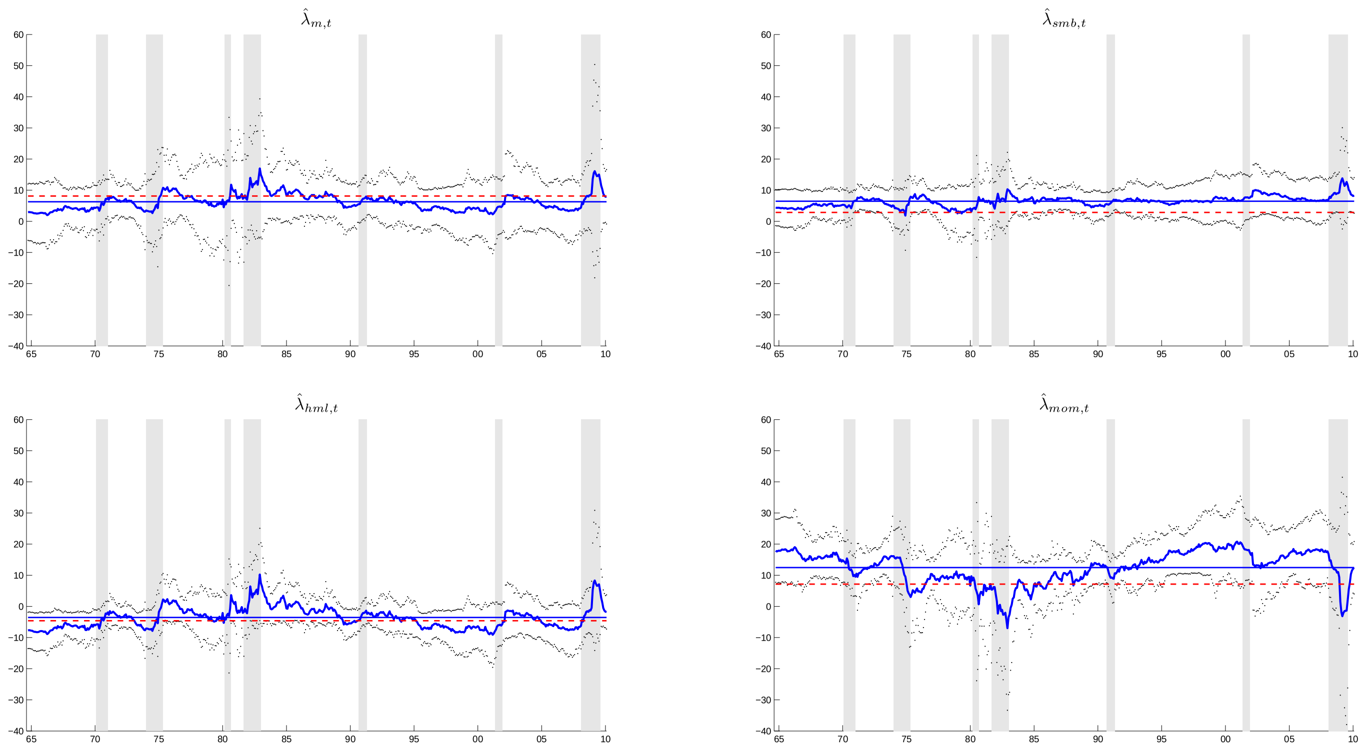
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when the returns of industry portfolios are used as asset-specific instrument. The vector of common instruments is $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$. We also display the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks as base assets ($n = 9,936$ and $n^{\chi} = 4,748$). The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.5: Path of estimated annualized risk premia computed using $Z_{t-1} = [1,\ ds_{t-1},\ divY_{t-1}]'$



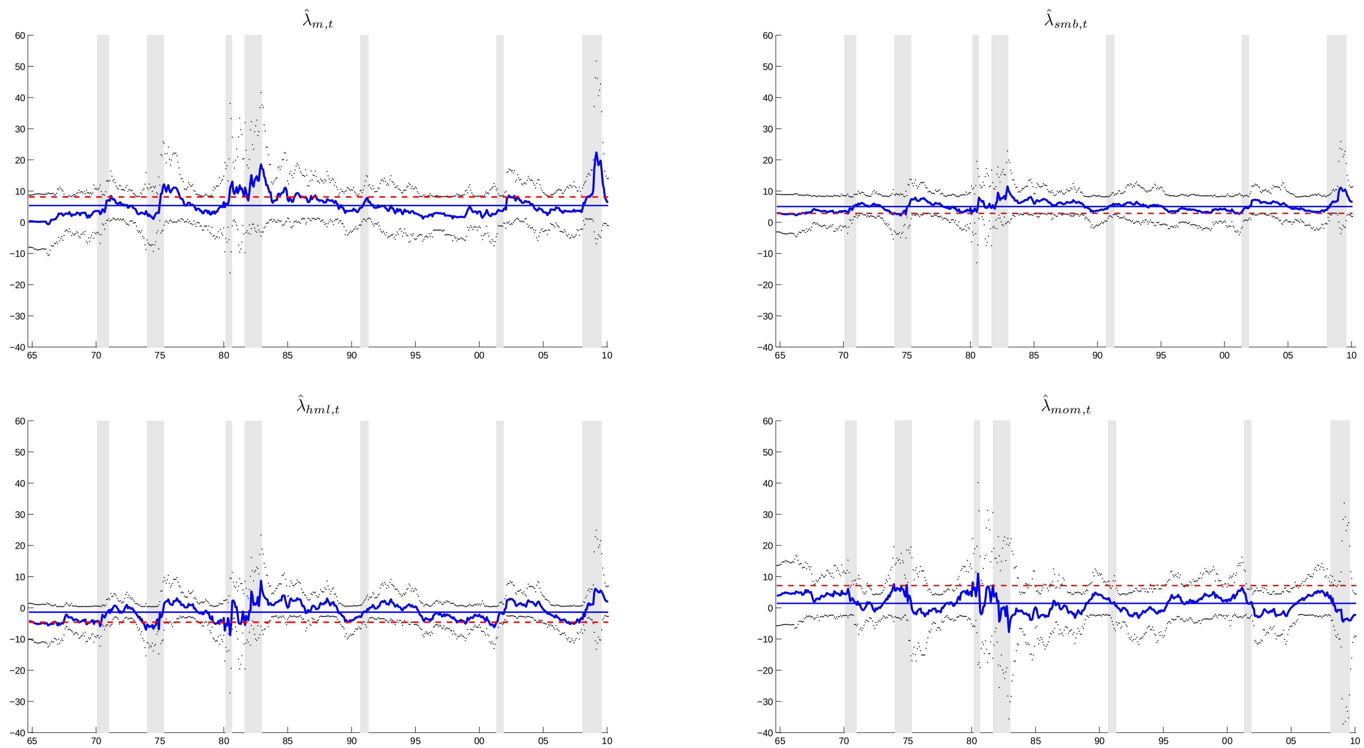
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when default spread and dividend yield are used as common instruments. The stock specific instrument is book-to-market equity. We also display the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks as base assets ($n = 9,936$ and $n^{\chi} = 1,107$). The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.6: Path of estimated annualized risk premia computed using $Z_{t-1} = [1,\ ds_{t-1},\ ts_{t-1},\ divY_{t-1}]'$



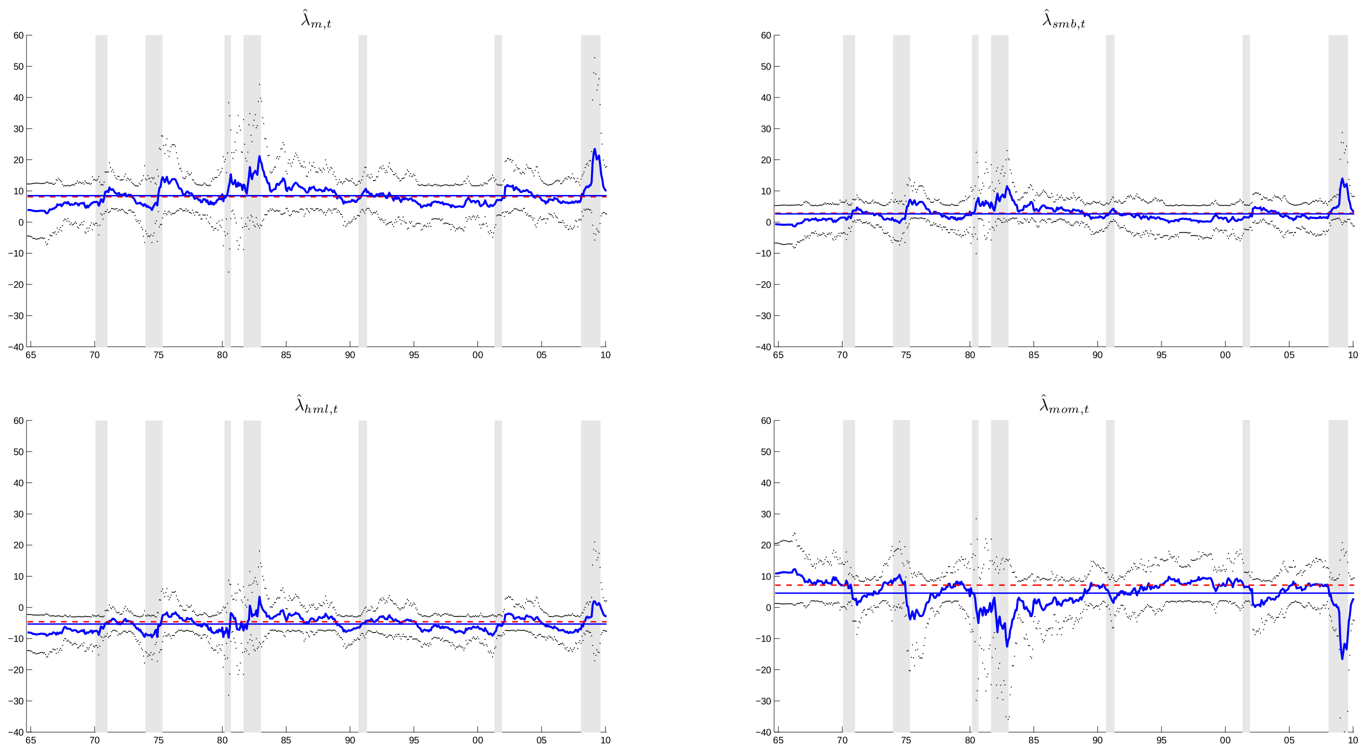
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when default spread, term spread and dividend yield are used as common instruments. The stock specific instrument is book-to-market equity. We also display the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks as base assets ($n = 9,936$ and $n^{\chi} = 667$). The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.7: Path of estimated annualized risk premia with time-varying betas modelled via $Z_{i,t-1} = [1,\ bm_{i,t-1}]'$



The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when time-varying betas are linear functions of the book-to-market instrument only. The risk premia vector involves the common instruments $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks ($n = 9,936$ and $n^{\chi} = 6,208$) as base assets. The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Figure 6.8: Path of estimated annualized risk premia with time-varying betas modelled via $Z_{i,t-1} = [1,\ ind_{i,t-1}]'$



The figure plots the path of estimated annualized risk premia $\hat{\lambda}_{m,t}$, $\hat{\lambda}_{smb,t}$, $\hat{\lambda}_{hml,t}$ and $\hat{\lambda}_{mom,t}$ and their pointwise confidence intervals at 95% level when time-varying betas are linear functions of industry portfolio returns. The risk premia vector involves the common instruments $Z_{t-1} = [1,\ ts_{t-1},\ ds_{t-1}]'$. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks ($n = 9,936$ and $n^{\chi} = 6,430$) as base assets. The vertical shaded areas denote recessions determined by the National Bureau of Economic Reasearch (NBER).

Chapter 7

Conclusions

The purpose of this thesis is to infer the path of risk premia from a large unbalanced panel of individual stock returns. First, we provide theoretical contributions that concern the finance and the econometric theory assuming an unconditional factor model for the excess return of an asset. Then, we extend the setting of unconditional models to conditional linear factor models in order to capture time-varying factor loadings and time-varying risk premia.

From the point of view of the finance theory, we derive an empirically testable no-arbitrage restriction in a multi-period conditional economy with a continuum of assets and an approximate factor structure. In this setting, our model accounts for conditional heteroskedasticity and weak cross-sectional dependence in the error terms.

For the econometric contribution, we provide a new two-pass cross-sectional estimation approach that allows us to estimate time-varying risk premia implied by conditional linear asset pricing model. We study the large sample properties of the risk premia estimator under a double asymptotics in cross-sectional and time-series dimensions. We also address testing for asset pricing restrictions induced by the no-arbitrage assumption in large economies.

We provide an empirical analysis on returns for about ten thousands US stocks from July 1964 to December 2009. We look at three factor models popular in the empirical finance literature to explain monthly equity returns: the CAPM, the three-factor Fama-French model, and the four-factor model that includes a momentum factor. We present unconditional and conditional estimates for the dataset of individual stocks and for 25 and 100 Fama-French portfolios. For individual stocks, we get that the conditional risk premia are large and volatile in crisis periods. They exhibit large positive and negative strays from unconditional estimates, follow the macroeconomic cycles, and do not match risk premia estimates on standard sets of portfolios.

This thesis is a first methodological step towards the econometric analysis of large-scale equity datasets. We believe that the empirical results are interesting and encouraging. We conclude by mentioning two open issues that we could not address in this work. The first issue concerns the identification of systematic risk factors for individual stocks. In our empirical application, we consider the Fama-French three factor model as the benchmark specification. This choice allows us to compare our empirical results with the results available in the empirical finance literature using portfolio returns as base assets. However, the Fama-French factors could explain only a part of the systematic risk of asset returns. A possibility is to identify latent common factors by applying the asymptotic principal components (APC) method proposed by Connor and Korajczyk (1986, 1987, 1988) (see also Stock and Watson (2002 a, b) and Bai and Ng (2002)). The second issue is about the potential misspecification of the factor model. In the thesis, we assume that the linear factor models are correctly specified. However, the one or three-factors specifications usually considered for portfolios could be misspecified when we consider individual stocks as base assets. Under the maintained assumption of a linear factor structure, misspecification can be due to an incorrect number of common factors, or an incorrect selection of the observable factors. We could theoretically investigate the large sample distributions of the estimators and test statistics under a misspecified factor model (see Kan, Robotti and Shanken (2012)) accounting for the unbalanced characteristic of the dataset. Moreover, we could propose a test of misspecification. These topics are interesting avenues for future research.

Appendix A: Proofs of Propositions

A.1 Regularity conditions

In this Appendix, we list and comment the additional assumptions used to derive the large sample properties of the estimators and test statistics. For unconditional models, we use Assumptions C.1-C.5 below with $x_t = (1, f_t')'$.

Assumption C.1 *There exist constants $\eta, \bar{\eta} \in (0,1]$ and $C_1, C_2, C_3, C_4 > 0$ such that for all $\delta > 0$ and $T \in \mathbb{N}$ we have:*

a) $\mathbb{P}\left[\left\|\frac{1}{T}\sum_t \left(x_t x_t' - E\left[x_t x_t'\right]\right)\right\| \geq \delta\right] \leq C_1 T \exp\left\{-C_2 \delta^2 T^{\eta}\right\} + C_3 \delta^{-1} \exp\left\{-C_4 T^{\bar{\eta}}\right\}$.

Furthermore, for all $\delta > 0$, $T \in \mathbb{N}$, and $1 \leq k, l, m \leq K+1$, the same upper bound holds for:

b) $\sup_{\gamma \in [0,1]} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_t(\gamma)\left(x_t x_t' - E\left[x_t x_t'\right]\right)\right\| \geq \delta\right]$;

c) $\sup_{\gamma \in [0,1]} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_t(\gamma) x_t \varepsilon_t(\gamma)\right\| \geq \delta\right]$;

d) $\sup_{\gamma, \gamma' \in [0,1]} \mathbb{P}\left[\left|\frac{1}{T}\sum_t (I_t(\gamma) I_t(\gamma') - E[I_t(\gamma) I_t(\gamma')])\right| \geq \delta\right]$;

e) $\sup_{\gamma, \gamma' \in [0,1]} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_t(\gamma) I_t(\gamma')\left(\varepsilon_t(\gamma)\varepsilon_t(\gamma') x_t x_t' - E\left[\varepsilon_t(\gamma)\varepsilon_t(\gamma') x_t x_t'\right]\right)\right\| \geq \delta\right]$;

f) $\sup_{\gamma, \gamma' \in [0,1]} \mathbb{P}\left[\left|\frac{1}{T}\sum_t I_t(\gamma) I_t(\gamma') x_{t,k} x_{t,l} x_{t,m} \varepsilon_t(\gamma)\right| \geq \delta\right]$.

Assumption C.2 *There exists a constant $M > 0$ such that for all $T \in \mathbb{N}$ we have:*

$\sup_{\gamma \in [0,1]} E\left[\frac{1}{T}\sum_{t_1, t_2, t_3} |cov(\varepsilon_{t_1}^2(\gamma), \varepsilon_{t_2}(\gamma)\varepsilon_{t_3}(\gamma) | x_{\underline{T}})|\right] \leq M$.

Assumption C.3 *There exists a constant $M > 0$ such that for all $n, T \in \mathbb{N}$ we have:*

a) $E\left[\frac{1}{nT}\sum_{i,j}\sum_{t_1,t_2} E\left[\left|cov\left(\varepsilon_{i,t_1}^2, \varepsilon_{j,t_2}^2 | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M.$

b) $E\left[\frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4} E\left[\left|cov\left(\varepsilon_{i,t_1}\varepsilon_{i,t_2}, \varepsilon_{j,t_3}\varepsilon_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M;$

c) $E\left[\frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4} E\left[\left|cov\left(\eta_{i,t_1}\varepsilon_{i,t_2}, \eta_{j,t_3}\varepsilon_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M$*, where* $\eta_{i,t} := \varepsilon_{i,t}^2 - \sigma_{ii,t}$*;*

d) $E\left[\frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4} E\left[\left|cov\left(\eta_{i,t_1}\eta_{i,t_2}, \eta_{j,t_3}\eta_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M;$

e) $E\left[\frac{1}{nT^3}\sum_{i,j}\sum_{t_1,\dots,t_6} E\left[\left|cov\left(\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{i,t_3}, \varepsilon_{j,t_4}\varepsilon_{j,t_5}\varepsilon_{j,t_6} | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M;$

f) $E\left[\frac{1}{nT^3}\sum_{i,j}\sum_{t_1,\dots,t_6} E\left[\left|cov\left(\eta_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{i,t_3}, \eta_{j,t_4}\varepsilon_{j,t_5}\varepsilon_{j,t_6} | x_{\underline{T}}, \gamma_i, \gamma_j\right)\right|^2 |\gamma_i, \gamma_j\right]^{\frac{1}{2}}\right] \leq M.$

Assumption C.4 *a) There exists a constant $M > 0$ such that $\|x_t\| \leq M$, P-a.s.. Moreover, b)* $\sup_{\gamma\in[0,1]} \|\beta(\gamma)\| < \infty$ *and c)* $\inf_{\gamma\in[0,1]} E[I_t(\gamma)] > 0$.

Assumption C.5 *The trimming constants satisfy $\chi_{1,T} = O\left((\log T)^{\kappa_1}\right)$ and $\chi_{2,T} = O\left((\log T)^{\kappa_2}\right)$, with κ_1, $\kappa_2 > 0$.*

Assumptions C.1 and C.2 restrict the serial dependence of the factors and the individual processes of observability indicators and error terms. Specifically, Assumption C.1 a) gives an upper bound for large-deviation probabilities of the sample average of random matrices $x_t x_t'$. It implies that the first two sample moments of the factor vector converge in probability to the corresponding population moments at a rate $O_p(T^{-\eta/2}(\log T)^c)$, for some $c > 0$. Assumptions C.1 b)-f) give similar upper bounds for large-deviation probabilities of sample averages of processes involving factors, observability indicators and error terms, uniformly w.r.t. $\gamma \in [0,1]$. We use these assumptions to prove the convergence of time series averages uniformly across assets. Assumption C.2 involves conditional covariances of products of error terms. Assumptions C.1 and C.2 are satisfied e.g. when the factors and the individual processes of observability indicators and error terms feature mixing serial dependence, with mixing coefficients uniformly bounded w.r.t. $\gamma \in [0,1]$ (see e.g. Bosq (1998), Theorems 1.3 and 1.4). Assumptions C.3 a)-f) restrict both serial and cross-sectional dependence of the error terms. They involve conditional covariances between products of error terms $\varepsilon_{i,t}$ and innovations $\eta_{i,t} = \varepsilon_{i,t}^2 - \sigma_{ii,t}$ for different assets and dates. These assumptions can be satisfied under weak serial and cross-sectional dependence of the errors, such as temporal mixing and block dependence across assets. Assumptions C.4 a) and b) require uniform upper bounds on factor values, factor loadings and intercepts. Assumption C.4 c) implies that asymptotically the fraction of the time period in which an asset return is observed is bounded away from zero uniformly across assets. Assumptions C.4 a)-c) ease the proofs. Assumption C.5 gives an upper bound on the divergence rate of the trimming constants. The slow logarithmic divergence rate allows to control the first-pass estimation error in the second-pass regression.

For conditional models, we use Assumptions C.1-C.5 with x_t replaced by the extended vector of common and firm-specific regressors as defined in Section 3.1. More precisely, for Assumption C.1a) we replace x_t by $x_t(\gamma) := \left(vech(X_t)', Z_{t-1}' \otimes Z_{t-1}(\gamma)', f_t' \otimes Z_{t-1}', f_t' \otimes Z_{t-1}(\gamma)'\right)'$, and require the bound to be valid uniformly w.r.t. $\gamma \in [0,1]$. For Assumptions C.1 b)-f) we replace x_t by $x_t(\gamma)$. For Assumptions C.2 and C.3 we replace $x_{\underline{T}}$ by $x_{\underline{T}}(\gamma)$, and by $x_{\underline{T}}(\gamma_i), x_{\underline{T}}(\gamma_j)$, respectively. For Assumption C.4a) we replace the bound on $\|x_t\|$ with bounds on $\|Z_t\|$, and on $\|Z_t(\gamma)\|$ uniformly w.r.t. $\gamma \in [0,1]$. Furthermore, we use:

Assumption C.6 *There exists a constant $M > 0$ such that $\left\|E\left[u_t u_t' | Z_{t-1}\right]\right\| \leq M$ for all t, where $u_t = f_t - E[f_t|\mathcal{F}_{t-1}]$.*

Assumption C.6 requires a bounded conditional variance-covariance matrix for the linear innovation u_t associated with the factor process. We use this assumption to prove that we can consistently estimate matrix F of the coefficients of the linear projection of factor f_t on variables Z_{t-1} by a SUR regression.

A.2 Unconditional factor model

A.2.1 Proof of Proposition 1 and link with Chamberlain and Rothschild (1983)

To ease notations, we assume w.l.o.g. that the continuous distribution G is uniform on $[0,1]$. For a given countable collection of assets $\gamma_1, \gamma_2, ...$ in $[0,1]$, let $\mu_n = A_n + B_n E[f_1|\mathcal{F}_0]$ and $\Sigma_n = B_n V[f_1|\mathcal{F}_0] B_n' + \Sigma_{\varepsilon,1,n}$, for $n \in \mathbb{N}$, be the mean vector and the covariance matrix of asset excess returns $\left(R_1(\gamma_1), ..., R_1(\gamma_n)\right)'$ conditional on $\mathcal{F}_0$, where $A_n = [a(\gamma_1), ..., a(\gamma_n)]'$, and $B_n = [b(\gamma_1), ..., b(\gamma_n)]'$. Let $e_n = \mu_n - B_n \left(B_n' B_n\right)^{-1} B_n' \mu_n = A_n - B_n \left(B_n' B_n\right)^{-1} B_n' A_n$ be the residual of the orthogonal projection of μ_n (and A_n) onto the columns of B_n. Furthermore, let $\mathcal{P}_n$ denote the set of portfolios p_n that invest in the risk-free asset and risky assets $\gamma_1, ..., \gamma_n$, for $n \in \mathbb{N}$, with portfolio shares measurable w.r.t. $\mathcal{F}_0$, and let $\mathcal{P}$ denote the set of portfolio sequences (p_n), with $p_n \in \mathcal{P}_n$. For portfolio $p_n \in \mathcal{P}_n$, the cost, the conditional expected return, and the conditional variance are given by $C(p_n) = \alpha_{0,n} + \alpha_n^{'}\iota_n$, $E[p_n|\mathcal{F}_0] = R_0 C(p_n) + \alpha_n^{'}\mu_n$, and $V[p_n|\mathcal{F}_0] = \alpha_n^{'}\Sigma_n\alpha_n$, where $\iota_n = (1, ..., 1)^{'}$ and $\alpha_n = (\alpha_{1,n}, ..., \alpha_{n,n})^{'}$. Moreover, let $\rho = \sup_p E[p|\mathcal{F}_0]/V[p|\mathcal{F}_0]^{1/2}$, where the sup is w.r.t. portfolios $p \in \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ with $C(p) = 0$ and $p \neq 0$, be the maximal Sharpe ratio of zero-cost portfolios. For expository purpose, we do not make explicit the dependence of μ_n, Σ_n, e_n, $\mathcal{P}_n$, and ρ on the collection of assets (γ_i).

The statement of Proposition 1 is proved by contradiction. Suppose that $\inf_{\nu\in\mathbb{R}^K} \int [a(\gamma) - b(\gamma)'\nu]^2 d\gamma = \int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma > 0$, where $\nu_\infty = \left(\int b(\gamma)\, b(\gamma)'\, d\gamma\right)^{-1} \int b(\gamma)a(\gamma)d\gamma$. By the strong LLN and Assumption APR.2, we have that:

$$\frac{1}{n}\|e_n\|^2 = \inf_{\nu\in\mathbb{R}^K} \frac{1}{n}\sum_{i=1}^{n}[a(\gamma_i) - b(\gamma_i)'\nu]^2 \to \int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma, \tag{a.1}$$

as $n \to \infty$, for any sequence (γ_i) in a set $\mathcal{J}_1 \subset \Gamma$, with measure $\mu_\Gamma(\mathcal{J}_1) = 1$. Let us now show that an asymptotic arbitrage portfolio exists based on any sequence in $\mathcal{J}_1 \bigcap \mathcal{J}$, where set $\mathcal{J}$ is defined in Assumption APR.4 (i). Define the portfolio sequence (q_n) with investments $\alpha_n = \frac{1}{\|e_n\|^2} e_n$ and $\alpha_{0,n} = -\iota_n'\alpha_n$. This static portfolio has zero cost, i.e., $C(q_n) = 0$, while $E[q_n|\mathcal{F}_0] = 1$ and $V[q_n|\mathcal{F}_0] \leq eig_{\max}(\Sigma_{\varepsilon,1,n})\|e_n\|^{-2}$. Moreover, we have $V[q_n|\mathcal{F}_0] = E\left[(q_n - E[q_n|\mathcal{F}_0])^2 |\mathcal{F}_0\right] \geq E\left[(q_n - E[q_n|\mathcal{F}_0])^2 |\mathcal{F}_0, q_n \leq 0\right] P[q_n \leq 0|\mathcal{F}_0] \geq P[q_n \leq 0|\mathcal{F}_0]$. Hence, we get: $P[q_n > 0|\mathcal{F}_0] \geq 1 - V[q_n|\mathcal{F}_0] \geq 1 - eig_{\max}(\Sigma_{\varepsilon,1,n})\|e_n\|^{-2}$. Thus, by using $eig_{\max}(\Sigma_{\varepsilon,1,n}) = o(n)$ from Assumption APR.4 (i) and $\|e_n\|^{-2} = O(1/n)$ from Equation (a.1), we get $P[q_n > 0|\mathcal{F}_0] \to 1$, P-a.s.. By using the Law of Iterated Expectation and the Lebesgue dominated convergence theorem, $P[q_n > 0] \to 1$. Hence, portfolio (q_n) is an asymptotic arbitrage opportunity. Since asymptotic arbitrage portfolios are ruled out by Assumption APR.5, it follows that we must have $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma = 0$, that is, $a(\gamma) = b(\gamma)'\nu$, for $\nu = \nu_\infty$ and almost all $\gamma \in [0, 1]$. Such vector ν is unique by Assumption APR.2, and Proposition 1 follows.

Let us now establish the link between the no-arbitrage conditions and asset pricing restrictions in CR on the one hand, and the asset pricing restriction (2.3) in the other hand. Let $\mathcal{J}^* \subset \Gamma$ be the set of countable collections of assets (γ_i) such that $\mathbb{P}$ [Conditions (i) and (ii) hold for any static portfolio sequence (p_n) in $\mathcal{P}] = 1$, where Conditions (i) and (ii) are: (i) If $V[p_n|\mathcal{F}_0] \to 0$ and $C(p_n) \to 0$, then $E[p_n|\mathcal{F}_0] \to 0$; (ii) If $V[p_n|\mathcal{F}_0] \to 0$, $C(p_n) \to 1$ and $E[p_n|\mathcal{F}_0] \to \delta$, then $\delta \geq 0$. Condition (i) means that, if the conditional variability and cost vanish, so does the conditional expected return. Condition (ii) means that, if the conditional variability vanishes and the cost is positive, the conditional expected return is non-negative. They correspond to Conditions A.1 (i) and (ii) in CR written conditionally on $\mathcal{F}_0$ and for a given countable collection of assets (γ_i). Hence, the set $\mathcal{J}^*$ is the set permitting no asymptotic arbitrage opportunities in the sense of CR almost surely (see also Chamberlain (1983)).

Proposition APR: *Under Assumptions APR.1-APR.4, either* $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = \mu_\Gamma(\mathcal{J}^*) = 1$, *or* $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = \mu_\Gamma(\mathcal{J}^*) = 0$. *The former case occurs if, and only if, the asset pricing restriction (2.3) holds.*

The fact that $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right)$ is either $= 1$, or $= 0$, is a consequence of the Kolmogorov zero-one law (e.g., Billingsley (1995)). Indeed, $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$ if, and only if, $\inf_{\nu \in \mathbb{R}^K} \sum_{i=n}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$, for any $n \in \mathbb{N}$. Thus, the zero-one law applies since the event $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$ belongs to the tail sigma-field $\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(\gamma_i, i = n, n+1, ...)$, and the variables γ_i are i.i.d. under measure μ_Γ.

Proof of Proposition APR: The proof involves four steps.

STEP 1: If $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) > 0$, then the asset pricing restriction (2.3) holds. This step is proved by contradiction. Suppose that the asset pricing restriction (2.3) does not hold, and thus $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma > 0$. Then, we get $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = 0$, by the convergence in (a.1).

STEP 2: If the asset pricing restriction (2.3) holds, then $\mu_\Gamma\left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = 1$. Indeed, $\mu_\Gamma\left(\sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 = 0\right) = 1$, if the asset pricing restriction (2.3) holds for some vector $\nu \in \mathbb{R}^K$.

STEP 3: If $\mu_\Gamma(\mathcal{J}^*) > 0$, then the asset pricing restriction (2.3) holds. By following the same arguments as in

CR on p. 1295-1296, we have $\rho^2 \geq \mu_n' \Sigma_{\varepsilon,1,n}^{-1} \mu_n$ and $\Sigma_{\varepsilon,1,n}^{-1} \geq eig_{\max}(\Sigma_{\varepsilon,1,n})^{-1} [I_n - B_n(B_n'B_n)^{-1}B_n']$, for any (γ_i) in $\mathcal{J}^*$. Thus, we get: $\rho^2 eig_{\max}(\Sigma_{\varepsilon,1,n}) \geq \mu_n' \left(I_n - B_n(B_n'B_n)^{-1}B_n'\right) \mu_n = \min_{\lambda \in \mathbb{R}^K} \|\mu_n - B_n\lambda\|^2 = \min_{\nu \in \mathbb{R}^K} \|A_n - B_n\nu\|^2 = \min_{\nu \in \mathbb{R}^K} \sum_{i=1}^n [a(\gamma_i) - b(\gamma_i)'\nu]^2$, for any $n \in \mathbb{N}$, P-a.s.. Hence, we deduce

$$\min_{\nu \in \mathbb{R}^K} \frac{1}{n} \sum_{i=1}^n [a(\gamma_i) - b(\gamma_i)'\nu]^2 \leq \rho^2 \frac{1}{n} eig_{\max}(\Sigma_{\varepsilon,1,n}), \tag{a.2}$$

for any n, P-a.s., and for any sequence (γ_i) in $\mathcal{J}^*$. Moreover, $\rho < \infty$, P-a.s., by the same arguments as in CR, Corollary 1, and by using that the condition in CR, footnote 6, is implied by our Assumption APR.4 (ii). Then, by the convergence in (a.1), the LHS of Inequality (a.2) converges to $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma$, for μ_Γ-almost every sequence (γ_i) in $\mathcal{J}^*$. From Assumption APR.4 (i), the RHS is $o(1)$, P-a.s., for μ_Γ-almost every sequence (γ_i) in Γ. Since $\mu_\Gamma(\mathcal{J}^*) > 0$, it follows that $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma = 0$, i.e., $a(\gamma) = b(\gamma)'\nu$, for $\nu = \nu_\infty$ and almost all $\gamma \in [0,1]$.

STEP 4: If the asset pricing restriction (2.3) holds, then $\mu_\Gamma(\mathcal{J}^*) = 1$. If (2.3) holds, it follows that $e_n = 0$ and $\mu_n = B_n(B_n'B_n)^{-1}B_n'\mu_n$, for all n, for μ_Γ-almost all sequences (γ_i). Then, we get $E[p_n|\mathcal{F}_0] = R_0 C(p_n) + \alpha_n' B_n(B_n'B_n/n)^{-1}B_n'\mu_n/n$. Moreover, we have: $V[p_n|\mathcal{F}_0] = (B_n'\alpha_n)'V[f_1|\mathcal{F}_0](B_n'\alpha_n) + \alpha_n'\Sigma_{\varepsilon,1,n}\alpha_n \geq eig_{\min}(V[f_1|\mathcal{F}_0]) \left\|B_n'\alpha_n\right\|^2$, where $eig_{\min}(V[f_1|\mathcal{F}_0]) > 0$, P-a.s. (Assumption APR.4 (iii)). Since $B_n'B_n/n$ converges to a positive definite matrix and $B_n'\mu_n/n$ is bounded, for μ_Γ-almost any sequence (γ_i), Conditions (i) and (ii) in the definition of set $\mathcal{J}^*$ follow, for μ_Γ-almost any sequence (γ_i), that is, $\mu_\Gamma(\mathcal{J}^*) = 1$.

A.2.2 Proof of Proposition 2

a) Consistency of $\hat{\nu}$. From Equation (2.5) and the asset pricing restriction (2.3), we have:

$$\hat{\nu} - \nu = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i c_\nu' \left(\hat{\beta}_i - \beta_i\right). \tag{a.3}$$

The consistency of $\hat{\nu}$ follows from the next Lemma, which is proved in Section A.2.2 c) below. The notation $I_{n,T} = O_{p,log}(a_{n,T})$ means that $I_{n,T}/a_{n,T}$ is bounded in probability by some power of the logarithmic term $\log(T)$ as $n, T \to \infty$.

Lemma 1 *Under Assumptions A.1 b), SC.1-SC.2, C.1, C.4 and C.5, we have:*

(i) $\sup_i \mathbf{1}_i^{\chi} \|\hat{\beta}_i - \beta_i\| = O_{p,log}\left(T^{-\eta/2}\right)$*; (ii)* $\sup_i w_i = O(1)$*; (iii)* $\frac{1}{n}\sum_i |\hat{w}_i - w_i| = o_p(1)$*; (iv)* $\hat{Q}_b - Q_b = o_p(1)$*, when* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $\bar{\gamma} > 0$.

b) Consistency of $\hat{\lambda}$. By Assumption C.1a), we have $\frac{1}{T}\sum_t f_t - E\left[f_t\right] = o_p\left(1\right)$, and thus $\left\|\hat{\lambda} - \lambda\right\| \leq \|\hat{\nu} - \nu\| + \left\|\frac{1}{T}\sum_t f_t - E\left[f_t\right]\right\| = o_p\left(1\right)$.

c) Proof of Lemma 1: (i) We use $\hat{\beta}_i - \beta_i = \frac{\tau_{i,T}}{\sqrt{T}}\hat{Q}_{x,i}^{-1}Y_{i,T}$ and $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$. Moreover, $\|\hat{Q}_{x,i}^{-1}\|^2 = Tr\left(\hat{Q}_{x,i}^{-2}\right) = \sum_{k=1}^{K+1}\lambda_{k,i}^{-2} \leq (K+1)CN\left(\hat{Q}_{x,i}\right)^2$, where the $\lambda_{k,i}$ are the eigenvalues of matrix $\hat{Q}_{x,i}$ and we use $eig_{max}\left(\hat{Q}_{x,i}\right) \geq 1$, which implies $\mathbf{1}_i^{\chi}\|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$. Thus, $\sup_i \mathbf{1}_i^{\chi}\|\hat{\beta}_i - \beta_i\| = O_{p,log}\left(T^{-1/2}\sup_i \|Y_{i,T}\|\right)$ from Assumption C.5. Now let $\delta_T := T^{-\eta/2}(\log T)^{(1+\bar{\gamma})/(2C_2)}$, where $\eta, C_2 > 0$ are as in Assumption C.1 and $\bar{\gamma} > 0$ is such that $n = O(T^{\bar{\gamma}})$. We have:

$$\mathbb{P}\left[T^{-1/2}\sup_i \|Y_{i,T}\| \geq \delta_T\right] \leq n\mathbb{P}\left[T^{-1/2}\|Y_{i,T}\| \geq \delta_T\right] = nE\left[\mathbb{P}\left(T^{-1/2}\|Y_{i,T}\| \geq \delta_T|\gamma_i\right)\right] \leq$$

$$n\sup_{\gamma\in[0,1]}\mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_t(\gamma)x_t\varepsilon_t(\gamma)\right\| \geq \delta_T\right] \leq n\left(C_1 T\exp\left\{-C_2\delta_T^2 T^{\eta}\right\} + C_3\delta_T^{-1}\exp\left\{-C_4 T^{\bar{\eta}}\right\}\right) = O(1),$$

from Assumption C.1 c). Part (i) follows. By using $w_i = v_i^{-1}$, $\tau_i \geq 1$ and $eig_{min}(S_{ii}) \geq M^{-1}eig_{min}(Q_x)$ from Assumption A.1 b), part (ii) follows. Part (iii) is proved in the supplementary materials by using Assumptions C.1, C.4 and C.5. Finally, part (iv) follows from $\hat{Q}_b - Q_b = \frac{1}{n}\sum_i(\hat{w}_i\hat{b}_i\hat{b}_i' - w_i b_i b_i')$ $+\frac{1}{n}\sum_i w_i b_i b_i' - Q_b$, by using parts (i)-(iii) and the LLN.

A.2.3 Proof of Proposition 3

a) Asymptotic normality of $\hat{\nu}$. From Equation (a.3) and by using $\hat{\beta}_i - \beta_i = \frac{\tau_{i,T}}{\sqrt{T}} \hat{Q}_{x,i}^{-1} Y_{i,T}$ we get:

$$
\begin{aligned}
\hat{\nu} - \nu &= \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i b_i \left(\hat{\beta}_i - \beta_i\right)' c_\nu + \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \left(\hat{b}_i - b_i\right) \left(\hat{\beta}_i - \beta_i\right)' c_\nu \\
&= \frac{1}{\sqrt{nT}} \hat{Q}_b^{-1} \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}' \hat{Q}_{x,i}^{-1} c_\nu + \frac{1}{T} \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \tau_{i,T}^2 E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} c_\nu. \quad \text{(a.4)}
\end{aligned}
$$

Let $I_1 := \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}' \hat{Q}_{x,i}^{-1} c_\nu$. Then, from Equation (a.4) and the definition of $\hat{B}_\nu$, we get:

$$
\begin{aligned}
\sqrt{nT} \left(\hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu\right) &= \hat{Q}_b^{-1} I_1 + \frac{1}{\sqrt{T}} \hat{Q}_b^{-1} E_2' \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T}^2 \left(\hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} c_\nu - \tau_{i,T}^{-1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}}\right) \\
&=: \hat{Q}_b^{-1} I_1 + \frac{1}{\sqrt{T}} \hat{Q}_b^{-1} E_2' I_2. \quad \text{(a.5)}
\end{aligned}
$$

Let us first show that $\hat{Q}_b^{-1} I_1$ is asymptotically normal. We use the next Lemma, which is proved below in Subsection A.2.3 c).

Lemma 2 *Under Assumptions A.1, A.3, SC.1-SC.2 and C.1, C.3-C.5, we have* $I_1 = \frac{1}{\sqrt{n}} \sum_i w_i \tau_i b_i Y_{i,T}' \hat{Q}_x^{-1} c_\nu + o_p(1)$, *when* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $\bar{\gamma} > 0$.

From Lemmas 1 (iv) and 2, and using $vec\left[ABC\right] = \left[C' \otimes A\right] vec\left[B\right]$ (MN Theorem 2, p. 35), we have:

$$
\hat{Q}_b^{-1} I_1 = \hat{Q}_b^{-1} \left(\frac{1}{\sqrt{n}} \sum_i w_i \tau_i b_i Y_{i,T}'\right) \hat{Q}_x^{-1} c_\nu + o_p(1) = \left(c_\nu' \hat{Q}_x^{-1} \otimes \hat{Q}_b^{-1}\right) \frac{1}{\sqrt{n}} \sum_i w_i \tau_i \left(Y_{i,T} \otimes b_i\right) + o_p(1).
$$

Then, we deduce $\hat{Q}_b^{-1} I_1 \Rightarrow N\left(0, \Sigma_\nu\right)$, by Assumptions A.2a) and C.1a) and Lemma 1 (iv).

Let us now show that $\frac{1}{\sqrt{T}} I_2 = o_p(1)$. We have:

$$
\begin{aligned}
I_2 &= \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T}^2 \hat{Q}_{x,i}^{-1} \left(Y_{i,T} Y_{i,T}' - S_{ii,T}\right) \hat{Q}_{x,i}^{-1} c_\nu - \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T}^2 \hat{Q}_{x,i}^{-1} \left(\tau_{i,T}^{-1} \hat{S}_{ii}^0 - S_{ii,T}\right) \hat{Q}_{x,i}^{-1} c_\nu \\
&\quad - \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} \hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - \hat{S}_{ii}^0\right) \hat{Q}_{x,i}^{-1} c_\nu - \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} \left(c_{\hat{\nu}} - c_\nu\right) \\
&=: (I_{21} - I_{22} - I_{23}) c_\nu - I_{24} \left(c_{\hat{\nu}} - c_\nu\right), \quad \text{(a.6)}
\end{aligned}
$$

where $\hat{S}^0_{ii} := \frac{1}{T_i}\sum_t I_{i,t}\varepsilon^2_{i,t}x_t x_t'$ and $S_{ii,T} = \frac{1}{T}\sum_t I_{i,t}\sigma_{ii,t}x_t x_t'$. The various terms are bounded in the next Lemma, which is proved in Appendix B.

Lemma 3 *Under Assumptions A.1, A.3, SC.1-SC.2, C.1-C.5, (i)* $I_{21} = \frac{1}{\sqrt{n}}\sum_i w_i \tau_i^2 \hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_x^{-1}$ $+O_{p,log}\left(\frac{\sqrt{n}}{T}\right) = O_p(1) + O_{p,log}\left(\frac{\sqrt{n}}{T}\right)$, *(ii)* $I_{22} = O_{p,log}\left(\frac{1}{\sqrt{T}} + \frac{\sqrt{n}}{T}\right)$, *(iii)* $I_{23} = O_{p,log}\left(\frac{\sqrt{n}}{T}\right)$ *(iv)* $I_{24} = O_{p,log}\left(\sqrt{n}\right)$ *and (v)* $c_{\hat{\nu}} - c_\nu = O_{p,log}\left(\frac{1}{\sqrt{nT}} + \frac{1}{T}\right)$*, when* $n, T \to \infty$ *such that* $n = O\left(T^{\bar{\gamma}}\right)$ *for* $\bar{\gamma} > 0$.

From Equation (a.6) and Lemma 3 we get $\frac{1}{\sqrt{T}}I_2 = o_p(1) + O_{p,log}\left(\frac{\sqrt{n}}{T\sqrt{T}}\right)$. From $n = O(T^{\bar{\gamma}})$ with $\bar{\gamma} < 3$, we get $\frac{1}{\sqrt{T}}I_2 = o_p(1)$ and the conclusion follows.

b) Asymptotic normality of $\hat{\lambda}$. We have $\sqrt{T}\left(\hat{\lambda} - \lambda\right) = \frac{1}{\sqrt{T}}\sum_t \left(f_t - E\left[f_t\right]\right) + \sqrt{T}\left(\hat{\nu} - \nu\right)$. By using $\sqrt{T}\left(\hat{\nu} - \nu\right) = O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{T}}\right) = o_p\left(1\right)$, the conclusion follows from Assumption A.2b).

c) Proof of Lemma 2: Write:

$$I_1 = \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}' \hat{Q}_x^{-1} c_\nu + \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}'\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)c_\nu \qquad =: I_{11}\hat{Q}_x^{-1}c_\nu + I_{12}c_\nu.$$

Let us decompose I_{11} as:

$$\begin{aligned} I_{11} &= \frac{1}{\sqrt{n}}\sum_i w_i\tau_i b_i Y_{i,T}' + \frac{1}{\sqrt{n}}\sum_i \left(\mathbf{1}_i^\chi - 1\right) w_i \tau_i b_i Y_{i,T}' + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi w_i\left(\tau_{i,T} - \tau_i\right) b_i Y_{i,T}' \\ &\quad + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi\left(\hat{v}_i^{-1} - v_i^{-1}\right)\tau_{i,T} b_i Y_{i,T}' \qquad =: I_{111} + I_{112} + I_{113} + I_{114}. \end{aligned}$$

Similarly, for I_{12} we have:

$$\begin{aligned} I_{12} &= \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi v_i^{-1}\tau_{i,T} b_i Y_{i,T}'\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi\left(\hat{v}_i^{-1} - v_i^{-1}\right)\tau_{i,T} b_i Y_{i,T}'\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) \\ &=: I_{121} + I_{122}. \end{aligned}$$

The conclusion follows by proving that terms I_{112}, I_{113}, I_{114}, I_{121} and I_{122} are $o_p(1)$.

Proof that $I_{112} = o_p(1)$. We use the next Lemma.

Lemma 4 *Under Assumptions SC.1-SC.2, C.1 b), d) and C.4 a), c):* $\mathbb{P}\left[\mathbf{1}_i^{\chi}=0\right]=O(T^{-\bar{b}})$, *for any* $\bar{b}>0$.

In Lemma 4, the unconditional probability $\mathbb{P}\left[\mathbf{1}_i^{\chi}=0\right]$ is independent of i since the indices (γ_i) are i.i.d. By using the bound $\|I_{112}\| \le \frac{C}{\sqrt{n}}\sum_i (1-\mathbf{1}_i^{\chi})\|Y_{i,T}\|$ from Assumptions C.4 b) and c) and Lemma 1 (ii), the bound $\sup_i E[\|Y_{i,T}\||x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \le C$ from Assumptions A.1 a) and b), and Lemma 4, it follows $I_{112} = O_p(\sqrt{n}T^{-\bar{b}})$, for any $\bar{b}>0$. Since $n=O\left(T^{\bar{\gamma}}\right)$, with $\bar{\gamma}>0$, we get $I_{112}=o_p(1)$.

Proof that $I_{113}=o_p(1)$. We have $E\left[\|I_{113}\|^2|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \le \frac{C}{nT}\sum_{i,j}\sum_t \mathbf{1}_i^{\chi}\mathbf{1}_j^{\chi}|\tau_{i,T}-\tau_i||\tau_{j,T}-\tau_j||\sigma_{ij,t}|$ from Assumption A.1 a). By Cauchy-Schwarz inequality and Assumption A.1 c), we get $E\left[\|I_{113}\|^2|\{\gamma_i\}\right] \le CM\sup_{\gamma\in[0,1]} E\left[\mathbf{1}_i^{\chi}|\tau_{i,T}-\tau_i|^4|\gamma_i=\gamma\right]^{1/2}$. By using $\tau_{i,T}-\tau_i = -\tau_{i,T}\tau_i\frac{1}{T}\sum_t\left(I_{i,t}-E[I_{i,t}|\gamma_i]\right)$ and $\mathbf{1}_i^{\chi}\tau_{i,T}\le\chi_{2,T}$, we get $\sup_{\gamma\in[0,1]} E\left[\mathbf{1}_i^{\chi}|\tau_{i,T}-\tau_i|^4|\gamma_i=\gamma\right] \le C\chi_{2,T}^4 \sup_{\gamma\in[0,1]} E\left[\left|\frac{1}{T}\sum_t\left(I_t(\gamma)-E[I_t(\gamma)]\right)\right|^4\right]=o(1)$ from Assumption C.5 and the next Lemma.

Lemma 5 *Under Assumption C.1 d):* $\sup_{\gamma\in[0,1]} E\left[\left|\frac{1}{T}\sum_t\left(I_t(\gamma)-E[I_t(\gamma)]\right)\right|^4\right]=O(T^{-c})$, *for some* $c>0$.

Then, $I_{113}=o_p(1)$.

Proof that $I_{114}=o_p(1)$. From $\hat{v}_i^{-1}-v_i^{-1} = -v_i^{-2}\left(\hat{v}_i-v_i\right)+\hat{v}_i^{-1}v_i^{-2}\left(\hat{v}_i-v_i\right)^2$, we get:

$$I_{114} = -\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi}v_i^{-2}\left(\hat{v}_i-v_i\right)\tau_{i,T}b_iY_{i,T}' + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi}\hat{v}_i^{-1}v_i^{-2}\left(\hat{v}_i-v_i\right)^2\tau_{i,T}b_iY_{i,T}' =: I_{1141}+I_{1142}.$$

Let us first consider I_{1141}. We have:

$$\begin{aligned}
\hat{v}_i - v_i &= \tau_{i,T}c_{\hat{\nu}_1}'\hat{Q}_{x,i}^{-1}\left(\hat{S}_{ii}-S_{ii}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}_1} + 2\tau_{i,T}(c_{\hat{\nu}_1}-c_\nu)'\hat{Q}_{x,i}^{-1}S_{ii}\hat{Q}_{x,i}^{-1}c_{\hat{\nu}_1} \\
&\quad +\tau_{i,T}(c_{\hat{\nu}_1}-c_\nu)'\hat{Q}_{x,i}^{-1}S_{ii}\hat{Q}_{x,i}^{-1}(c_{\hat{\nu}_1}-c_\nu) + 2\tau_{i,T}c_\nu'\left(\hat{Q}_{x,i}^{-1}-Q_x^{-1}\right)S_{ii}\hat{Q}_{x,i}^{-1}c_\nu \\
&\quad +\tau_{i,T}c_\nu'\left(\hat{Q}_{x,i}^{-1}-Q_x^{-1}\right)S_{ii}\left(\hat{Q}_{x,i}^{-1}-Q_x^{-1}\right)c_\nu + (\tau_{i,T}-\tau_i)c_\nu'Q_x^{-1}S_{ii}Q_x^{-1}c_\nu. \qquad \text{(a.7)}
\end{aligned}$$

The contribution of the first two terms to I_{1141} is:

$$\begin{aligned}
I_{11411} &= -\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi}v_i^{-2}\tau_{i,T}^2c_{\hat{\nu}_1}'\hat{Q}_{x,i}^{-1}\left(\hat{S}_{ii}-S_{ii}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}_1}b_iY_{i,T}', \\
I_{11412} &= -\frac{2}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi}v_i^{-2}\tau_{i,T}^2(c_{\hat{\nu}_1}-c_\nu)'\hat{Q}_{x,i}^{-1}S_{ii}\hat{Q}_{x,i}^{-1}c_{\hat{\nu}_1}b_iY_{i,T}'.
\end{aligned}$$

We first show $I_{11412} = o_p(1)$. For this purpose, it is enough to show that $c_{\hat{\nu}_1} - c_\nu = O_p(T^{-c})$, for some $c > 0$, and $\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^2 \left(\hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} \right)_{kl} b_i Y_{i,T}' = O_p\left(\chi_{1,T}^2 \chi_{2,T}^2\right)$, for any $k, l = 1, ..., K+1$. The first statement follows from the proof of Proposition 2 but with known weights equal to 1. To prove the second statement, we use bounds $\mathbf{1}_i^\chi \tau_{i,T} \le \chi_{2,T}$ and $\mathbf{1}_i^\chi \|\hat{Q}_{x,i}^{-1}\| \le C\chi_{1,T}$ and Assumption A.1 c). Let us now prove that $I_{11411} = o_p(1)$. For this purpose, it is enough to show that

$$J_1 := \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^2 \left(\hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - S_{ii} \right) \hat{Q}_{x,i}^{-1} \right)_{kl} b_i Y_{i,T}' = o_p(1), \tag{a.8}$$

for any k, l. By using $\hat{\varepsilon}_{i,t} = \varepsilon_{i,t} - x_t' \left(\hat{\beta}_i - \beta_i \right) = \varepsilon_{i,t} - \frac{\tau_{i,T}}{\sqrt{T}} x_t' \hat{Q}_{x,i}^{-1} Y_{i,T}$, we get:

$$\begin{aligned} \hat{S}_{ii} - S_{ii} &= \frac{1}{T_i} \sum_t I_{i,t} \left(\varepsilon_{it}^2 x_t x_t' - S_{ii} \right) + \frac{1}{T_i} \sum_t I_{i,t} \left(\hat{\varepsilon}_{i,t}^2 - \varepsilon_{it}^2 \right) x_t x_t' \\ &= \frac{\tau_{i,T}}{\sqrt{T}} W_{1,i,T} + \frac{\tau_{i,T}}{\sqrt{T}} W_{2,i,T} - \frac{2\tau_{i,T}^2}{T} W_{3,i,T} \hat{Q}_{x,i}^{-1} Y_{i,T} + \frac{\tau_{i,T}^3}{T} \hat{Q}_{x,i}^{(4)} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1}, \end{aligned} \tag{a.9}$$

where $W_{1,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} x_t^2 \eta_{i,t}$, $\eta_{i,t} = \varepsilon_{it}^2 - \sigma_{ii,t}$, $W_{2,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \zeta_{i,t}$, $\zeta_{i,t} := \sigma_{ii,t} x_t^2 - S_{ii}$, $W_{3,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \varepsilon_{i,t} x_t^3$, $\hat{Q}_{x,i}^{(4)} := \frac{1}{T} \sum_t I_{i,t} x_t^4$ and x_t is treated as a scalar to ease notation. Then:

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-1} W_{1,i,T} \hat{Q}_{x,i}^{-1} b_i Y_{i,T}' + \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-1} W_{2,i,T} \hat{Q}_{x,i}^{-1} b_i Y_{i,T}' \\ &\quad - \frac{2}{\sqrt{nT}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^4 \hat{Q}_{x,i}^{-1} W_{3,i,T} \hat{Q}_{x,i}^{-1} Y_{i,T} \hat{Q}_{x,i}^{-1} b_i Y_{i,T}' + \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^5 \hat{Q}_{x,i}^{-1} \hat{Q}_{x,i}^{(4)} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-2} b_i Y_{i,T}' \\ &=: J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

Let us consider J_{11}. We have:

$$E\left[J_{11} | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] = \frac{1}{\sqrt{nT^3}} \sum_i \sum_{t,s} \mathbf{1}_i^\chi v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-2} b_i x_t^2 x_s E\left[\varepsilon_{i,t}^2 \varepsilon_{i,s} | x_{\underline{T}}, \gamma_i\right] = 0,$$

from Assumption A.3. Moreover, from Assumption C.4:

$$V\left[J_{11} | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \le \frac{C}{nT^3} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \mathbf{1}_i^\chi \mathbf{1}_j^\chi \tau_{i,T}^3 \tau_{j,T}^3 \|\hat{Q}_{x,i}^{-1}\|^2 \|\hat{Q}_{x,j}^{-1}\|^2 |cov\left(\eta_{i,t_1} \varepsilon_{i,t_2}, \eta_{j,t_3} \varepsilon_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j\right)|.$$

By using $\mathbf{1}_i^{\chi}\|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$, $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$, the Law of Iterated Expectations and Assumptions C.3 c) and C.5, we get $E\left[J_{11}\right] = 0$ and $V[J_{11}] = o(1)$. Thus $J_{11} = o_p(1)$. By similar arguments and using Assumptions A.1 c) and C.3 e), we get $J_{12} = o_p(1)$, $J_{13} = o_p(1)$ and $J_{14} = o_p(1)$. Hence the bound in Equation (a.8) follows, and $I_{11411} = o_p(1)$. Paralleling the detailed arguments provided above, we can show that all other remaining terms making I_{114} are also $o_p(1)$.

Proof that $I_{121} = o_p(1)$. From:

$$\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1} = -\hat{Q}_{x,i}^{-1}\left(\frac{1}{T_i}\sum_t I_{i,t}x_t x_t' - \hat{Q}_x\right)\hat{Q}_x^{-1} = -\tau_{i,T}\hat{Q}_{x,i}^{-1}W_{i,T}\hat{Q}_x^{-1} + \hat{Q}_{x,i}^{-1}W_T\hat{Q}_x^{-1}, \qquad \text{(a.10)}$$

where $W_{i,T} := \frac{1}{T}\sum_t I_{i,t}(x_t x_t' - Q_x)$ and $W_T := \frac{1}{T}\sum_t (x_t x_t' - Q_x)$, we can write:

$$\begin{aligned} I_{121} &= \left(-\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T}^2 b_i Y_{i,T}'\hat{Q}_{x,i}^{-1}W_{i,T} + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T} b_i Y_{i,T}'\hat{Q}_{x,i}^{-1}W_T\right)\hat{Q}_x^{-1} \\ &=: \left(I_{1211} + I_{1212}\right)\hat{Q}_x^{-1}. \end{aligned}$$

Let us consider term I_{1211}. From Assumption C.4, $\mathbf{1}_i^{\chi}\|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$ and $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$, we have:

$$E\left[\|I_{1211}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \leq \frac{C\chi_{1,T}^2\chi_{2,T}^4}{nT}\sum_{i,j}\sum_t |\sigma_{ij,t}|\|W_{i,T}\|\|W_{j,T}\|.$$

Then, from Cauchy-Schwarz inequality, we get $E\left[\|I_{1211}\|^2|\{\gamma_i\}\right] \leq C\chi_{1,T}^2\chi_{2,T}^4\frac{1}{n}\sum_{i,j} E[\sigma_{ij,t}^2|\gamma_i,\gamma_j]^{1/2}$ $\sup_i E\left[\|W_{i,T}\|^4|\gamma_i\right]^{1/2}$, where $\sup_i E\left[\|W_{i,T}\|^4|\gamma_i\right] \leq \sup_{\gamma\in[0,1]} E\left[\left\|\frac{1}{T}\sum_t I_t(\gamma)\left(x_t x_t' - Q_x\right)\right\|^4\right] = O(T^{-c})$ from Assumptions C.1 b) and C.4 a). Then, from Assumptions A.1 c) and C.5 it follows $E[\|I_{1211}\|^2] = o(1)$ and thus $I_{1211} = o_p(1)$. Similarly we can show $I_{1212} = o_p(1)$, and then $I_{121} = o_p(1)$.

Proof that $I_{122} = o_p(1)$. The statement follows by combining arguments similar as for I_{114} and I_{121}.

A.2.4 Proof of Proposition 4

From Proposition 3, we have to show that $\tilde{\Sigma}_\nu - \Sigma_\nu = o_p(1)$. By $\Sigma_\nu = \left(c'_\nu Q_x^{-1} \otimes Q_b^{-1}\right) S_b \left(Q_x^{-1} c_\nu \otimes Q_b^{-1}\right)$ and $\tilde{\Sigma}_\nu = \left(c'_{\hat{\nu}} \hat{Q}_x^{-1} \otimes \hat{Q}_b^{-1}\right) \tilde{S}_b \left(\hat{Q}_x^{-1} c_{\hat{\nu}} \otimes \hat{Q}_b^{-1}\right)$, where $\tilde{S}_b = \frac{1}{n}\sum_{i,j} \hat{w}_i \hat{w}_j \frac{\tau_{i,T}\tau_{j,T}}{\tau_{ij,T}} \tilde{S}_{ij} \otimes \hat{b}_i \hat{b}'_j$, and the consistency of $\hat{Q}_x$ and $\hat{Q}_b$, the statement follows if $\tilde{S}_b - S_b = o_p(1)$. The leading terms in $\tilde{S}_b - S_b$ are given by $I_3 := \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} \left(\tilde{S}_{ij} - S_{ij}\right) \otimes b_i b'_j$ and $I_4 := \frac{1}{n}\sum_{i} w_i w_j \tau_i \tau_j (\tau_{ij,T}^{-1} - \tau_{ij}^{-1}) S_{ij} \otimes b_i b'_j$, while the other ones can be shown to be $o_p(1)$ by arguments similar to the proofs of Propositions 2 and 3.

Proof of $I_3 = o_p(1)$. By using that $\tau_i \leq M$, $\tau_{ij} \geq 1$, $w_i \leq M$ and $\|b_i\| \leq M$, $I_3 = o_p(1)$ follows if we show: $\frac{1}{n}\sum_{i,j} \left\|\tilde{S}_{ij} - S_{ij}\right\| = o_p(1)$. For this purpose, we introduce the following Lemmas 6 and 7 that extend results in Bickel and Levina (2008) from the i.i.d. case to the time series case including random individual effects.

Lemma 6 *Let* $\psi_{nT} := \max_{i,j} \left\|\hat{S}_{ij} - S_{ij}\right\|$, *and* $\Psi_{nT}(\xi) := \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij} - S_{ij}\right\| \geq \xi\right]$, *for* $\xi > 0$. *Under Assumptions SC.1, SC.2, A.4,* $\frac{1}{n}\sum_{i,j} \left\|\tilde{S}_{ij} - S_{ij}\right\| = O_p\left(\psi_{nT} n^\delta \kappa^{-q} + n^\delta \kappa^{1-q} + \psi_{nT} n^2 \Psi_{nT}((1-v)\kappa)\right)$, *for any* $v \in (0,1)$.

Lemma 7 *Under Assumptions SC.1, SC.2, C.1, C.4 and C.5, if* $\kappa = M\sqrt{\frac{\log n}{T^\eta}}$ *with* M *large, then* $n^2 \Psi_{nT}((1-v)\kappa) = O(1)$, *for any* $v \in (0,1)$, *and* $\psi_{nT} = O_p\left(\sqrt{\frac{\log n}{T^\eta}}\right)$, *when* $n, T \to \infty$ *such that* $n = O(T^{\bar{\gamma}})$ *for* $\bar{\gamma} > 0$.

In Lemma 6, the probability $\mathbb{P}\left[\left\|\hat{S}_{ij} - S_{ij}\right\| \geq \xi\right]$ is the same for all pairs (i,j) with $i = j$, and for all pairs with $i \neq j$, since this probability is marginal w.r.t. the individual random effects. From Lemmas 6 and 7, it follows $\frac{1}{n}\sum_{i,j} \left\|\tilde{S}_{ij} - S_{ij}\right\| = O_p\left(\left(\frac{\log n}{T^\eta}\right)^{(1-q)/2} n^\delta\right) = o_p(1)$, since $n = O(T^{\bar{\gamma}})$ with $\bar{\gamma} < \eta \frac{1-q}{2\delta}$.

Proof of $I_4 = o_p(1)$. From $w_i \leq M$, $\tau_i \leq M$ and $b_i \leq M$, we have $E[\|I_4\| | \{\gamma_i\}] \leq C \sup_{i,j} E[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \gamma_i, \gamma_j] \frac{1}{n}\sum_{i,j} \|S_{ij}\|$. By using the inequalities $\sup_{i,j} E[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \gamma_i, \gamma_j]$ $\leq \sup_{\gamma,\gamma' \in [0,1]} E\left[\left\|\frac{1}{T}\sum_t (I_t(\gamma) I_t(\gamma') - E[I_t(\gamma) I_t(\gamma')]\right\|\right]$ and $\|S_{ij}\| \leq E[|\sigma_{ij,t}| | \gamma_i, \gamma_j]$, from Assumptions A.1 c) and C.1 d) we get $E[\|I_4\|] = o(1)$, which implies $I_4 = o_p(1)$.

A.2.5 Proof of Proposition 5

By definition of $\hat{Q}_e$, we get the following result:

Lemma 8 *Under $\mathcal{H}_0$ and Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.3 and C.1-C.5, we have* $\hat{Q}_e = \frac{1}{n}\sum_i \hat{w}_i \left[c'_{\hat{\nu}}\left(\hat{\beta}_i - \beta_i\right)\right]^2 + O_{p,log}\left(\frac{1}{nT} + \frac{1}{T^2}\right)$*, when $n, T \to \infty$ such that $n = O\left(T^{\bar{\gamma}}\right)$ for $\bar{\gamma} > 0$.*

From Lemma 4 and $n = O(T^{\bar{\gamma}})$ for $0 < \bar{\gamma} < 2$, it follows $\hat{\xi}_{nT} = \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \left\{\left[c'_{\hat{\nu}}\sqrt{T}\left(\hat{\beta}_i - \beta_i\right)\right]^2 + -\tau_{i,T} c'_{\hat{\nu}} \hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1}c_{\hat{\nu}}\right\} + o_p(1)$. By using $\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) = \tau_{i,T}\hat{Q}_{x,i}^{-1}Y_{i,T}$, we get

$$
\begin{aligned}
\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T}^2 c'_{\hat{\nu}}\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - \tau_{i,T}^{-1}\hat{S}_{ii}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T}^2 c'_{\hat{\nu}}\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} - \frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T}^2 c'_{\hat{\nu}}\hat{Q}_{x,i}^{-1}\left(\tau_{i,T}^{-1}\hat{S}_{ii} - S_{ii,T}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} \\
&\quad + o_p(1) \qquad =: c'_{\hat{\nu}}(I_{21} - I_{22} - I_{23})c_{\hat{\nu}} + o_p(1),
\end{aligned}
$$

where I_{21}, I_{22} and I_{23} are defined in (a.6). By Lemma 3 (i)-(iii), and the consistency of $\hat{\nu}$, we have

$$
\hat{\xi}_{nT} = \frac{1}{\sqrt{n}}\sum_i w_i \tau_i^2 c'_{\hat{\nu}}\hat{Q}_x^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)\hat{Q}_x^{-1}c_{\hat{\nu}} + O_{p,log}\left(\frac{\sqrt{n}}{T}\right) + o_p(1).
$$

Moreover, from $n = O(T^{\bar{\gamma}})$ with $\bar{\gamma} < 2$, the remainder term $O_{p,log}(\sqrt{n}/T)$ is $o_p(1)$. Then, by using $tr\left[A'B\right] = vec\left[A\right]' vec\left[B\right]$, and $vec\left[YY'\right] = (Y \otimes Y)$ for a vector Y, we get

$$
\begin{aligned}
\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i w_i \tau_i^2 tr\left[\hat{Q}_x^{-1}c_{\hat{\nu}}c'_{\hat{\nu}}\hat{Q}_x^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)\right] + o_p(1) \\
&= \left(vec\left[\hat{Q}_x^{-1}c_{\hat{\nu}}c'_{\hat{\nu}}\hat{Q}_x^{-1}\right]\right)' \frac{1}{\sqrt{n}}\sum_i w_i \tau_i^2 \left(Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right) + o_p(1).
\end{aligned}
$$

By using Assumption A.5, and by consistency of $\hat{\nu}$ and $\hat{Q}_x$, we get $\hat{\xi}_{nT} \Rightarrow N(0, \Sigma_{\xi})$, where $\Sigma_{\xi} = \left(vec\left[Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}\right]\right)' \Omega \left(vec\left[Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}\right]\right)$. By using MN Theorem 3 Chapter 2, we have

$$
\begin{aligned}
vec\left[Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}\right]'\left(S_{ij} \otimes S_{ij}\right) vec\left[Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}\right] &= tr\left[S_{ij}Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}S_{ij}Q_x^{-1}c_{\nu}c'_{\nu}Q_x^{-1}\right] \\
&= \left(c'_{\nu}Q_x^{-1}S_{ij}Q_x^{-1}c_{\nu}\right)^2, \qquad \text{(a.11)}
\end{aligned}
$$

and

$$vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right]'\left(S_{ij}\otimes S_{ij}\right)W_{K+1}vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right] = \left(c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu\right)^2. \quad \text{(a.12)}$$

Then, from the definition of Ω in Assumption A.5 and Equations (a.11) and (a.12), we deduce $\Sigma_\xi = 2$ a.s.-$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i,j} w_i w_j \dfrac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left(c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu\right)^2$. Finally, $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p(1)$ follows from $\dfrac{1}{n}\sum\limits_{i,j}\|\tilde{S}_{ij}-S_{ij}\| = o_p(1)$ and $\dfrac{1}{n}\sum\limits_{i,j}\|\tilde{S}_{ij}-S_{ij}\|^2 = o_p(1)$.

A.2.6 Proof of Proposition 6

a) Asymptotic normality of $\hat{\nu}$. By definition of $\hat{\nu}$ and under $\mathcal{H}_1$, we have

$$\begin{aligned}
\hat{\nu}-\nu_\infty &= \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\hat{b}_i c_{\nu_\infty}'\hat{\beta}_i = \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\hat{b}_i c_{\nu_\infty}'\left(\hat{\beta}_i-\beta_i\right)+\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\hat{b}_i e_i \quad \text{(a.13)}\\
&= \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i b_i\left(\hat{\beta}_i-\beta_i\right)' c_{\nu_\infty}+\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i E_2'\left(\hat{\beta}_i-\beta_i\right)\left(\hat{\beta}_i-\beta_i\right)' c_{\nu_\infty}\\
&\quad +\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i b_i e_i+\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\left(\hat{b}_i-b_i\right)e_i.
\end{aligned}$$

Equation (a.13) is the analogue of Equation (a.3), and the consistency of $\hat{\nu}$ for ν_∞ follows as in the proof of Proposition 2 and by using $E\left[w_i b_i e_i\right]=0$. Thus, we get:

$$\begin{aligned}
&\sqrt{n}\left(\hat{\nu}-\frac{1}{T}\hat{B}_{\nu_\infty}-\nu_\infty\right)\\
= \; &\hat{Q}_b^{-1}\frac{1}{\sqrt{nT}}\sum_i \hat{w}_i\tau_{i,T}b_i Y_{i,T}'\hat{Q}_{x,i}^{-1}c_{\nu_\infty}+\frac{1}{T}\hat{Q}_b^{-1}E_2'\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1}Y_{i,T}Y_{i,T}'\hat{Q}_{x,i}^{-1}c_{\nu_\infty}-\tau_{i,T}^{-1}\hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1}c_{\hat{\nu}}\right)\\
&+\hat{Q}_b^{-1}\frac{1}{\sqrt{n}}\sum_i w_i b_i e_i+\hat{Q}_b^{-1}\frac{1}{\sqrt{n}}\sum_i(\hat{w}_i-w_i)b_i e_i+\hat{Q}_b^{-1}\frac{1}{\sqrt{nT}}\sum_i \hat{w}_i\tau_{i,T}e_i Y_{i,T}'\hat{Q}_{x,i}^{-1}E_2\\
&=: I_{51}+I_{52}+I_{53}+I_{54}+I_{55}.
\end{aligned}$$

From Assumption SC.2 and $E\left[w_i b_i e_i\right]=0$, we get $\dfrac{1}{\sqrt{n}}\sum\limits_i w_i b_i e_i \Rightarrow N\left(0, E\left[w_i^2 e_i^2 b_i b_i'\right]\right)$ by the CLT. Thus, $I_{53}\Rightarrow N\left(0, Q_b^{-1}E\left[w_i^2 e_i^2 b_i b_i'\right]Q_b^{-1}\right)$. Then, the asymptotic distribution of $\hat{\nu}$ follows if terms I_{51}, I_{52}, I_{54} and I_{55} are $o_p(1)$. From similar arguments as for term I_1 in the proof of Proposition 3, we have

$\frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} b_i Y'_{i,T} \hat{Q}^{-1}_{x,i} c_{\nu_\infty} = O_p(1)$ and $\frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} e_i Y'_{i,T} \hat{Q}^{-1}_{x,i} E_2 = O_p(1)$. Thus $I_{51} = o_p(1)$ and $I_{55} = o_p(1)$. From similar arguments as for term I_2 in the proof of Proposition 3, we have $I_{52} = o_p(1)$. Moreover, term $I_{54} = o_p(1)$ from similar arguments as for I_{112} and I_{114}.

b) Asymptotic normality of $\hat{\lambda}$. We have $\sqrt{T}\left(\hat{\lambda} - \lambda_\infty\right) = \sqrt{T}\left(\hat{\nu} - \nu_\infty\right) + \frac{1}{\sqrt{T}} \sum_t \left(f_t - E\left[f_t\right]\right)$. By using $\bar{\gamma} > 1$ and $\sqrt{T}\left(\hat{\nu} - \nu_\infty\right) = O_p\left(\sqrt{\frac{T}{n}} + \frac{1}{\sqrt{T}}\right) = o_p\left(1\right)$, the conclusion follows.

c) Consistency of the test. By definition of $\hat{Q}_e$, we get the following result:

Lemma 9 *Under $\mathcal{H}_1$ and Assumptions SC.1, SC.2, A.1-A.3, C.1-C.5, we have* $\hat{Q}_e = \frac{1}{n} \sum_i \hat{w}_i \left[c'_{\hat{\nu}} \left(\hat{\beta}_i - \beta_i\right)\right]^2 + \frac{1}{n} \sum_i w_i e_i^2 + O_{p,log}\left(\frac{1}{n} + \frac{1}{\sqrt{nT}} + \frac{1}{\sqrt{T^3}}\right)$, *when $n, T \to \infty$ such that $n = O\left(T^{\bar{\gamma}}\right)$ for $\bar{\gamma} > 0$.*

By similar arguments as in the proof of Proposition 5 and using $\bar{\gamma} < 2$, we get:

$$
\begin{aligned}
\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}} \sum_i w_i \tau_i^2 c'_{\hat{\nu}} \hat{Q}_x^{-1} \left(Y_{i,T} Y'_{i,T} - S_{ii,T}\right) \hat{Q}_x^{-1} c_{\hat{\nu}} + T \frac{1}{\sqrt{n}} \sum_i w_i e_i^2 + O_{p,log}\left(\frac{T}{\sqrt{n}} + \sqrt{T}\right) + o_p\left(1\right) \\
&= T\sqrt{n} E\left[w_i \left(a_i - b'_i \nu_\infty\right)^2\right] + O_p\left(T\right).
\end{aligned}
$$

Under $\mathcal{H}_1$, we have $E\left[w_i \left(a_i - b'_i \nu_\infty\right)^2\right] > 0$, since $w_i > 0$ and $\left(a_i - b_i \nu_\infty\right)^2 > 0$, P-a.s. Moreover, $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p\left(1\right)$. Thus, $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT} = T\sqrt{n}\left(\Sigma_\xi^{-1/2} E\left[w_i \left(a_i - b'_i \nu_\infty\right)^2\right] + o_p\left(1\right)\right)$.

A.3 Conditional factor model

A.3.1 Proof of Proposition 7

Proposition 7 is proved along similar lines as Proposition 1. Hence we only highlight the slight differences. We can work at $t = 1$ because of stationarity, and use that $a_1(\gamma)$, $b_1(\gamma)$, for $\gamma \in [0, 1]$, are $\mathcal{F}_0$-measurable. Then, the proof by contradiction uses the strong LLN applied conditionally on $\mathcal{F}_0$ and Assumption APR.7 as in the proof of Proposition 1. A result similar to Proposition APR also holds true with straightforward modifications to accommodate the conditional case.

A.3.2 Derivation of Equations (3.6) and (3.7)

From Equation (3.5) and by using $vec\,[ABC] = [C' \otimes A]\,vec\,[B]$ (MN Theorem 2, p. 35), we get $Z_{t-1}'B_i'f_t = vec\,[Z_{t-1}'B_i'f_t] = [f_t' \otimes Z_{t-1}']\,vec\,[B_i']$, and $Z_{i,t-1}'C_i'f_t = [f_t' \otimes Z_{i,t-1}']\,vec\,[C_i']$, which gives $Z_{t-1}'B_i'f_t + Z_{i,t-1}'C_i'f_t = x_{2,i,t}'\beta_{2,i}$.

Let us now consider the first two terms in the RHS of Equation (3.5). a) By definition of matrix X_t in Section 3.1, we have

$$\begin{aligned} Z_{t-1}'B_i'(\Lambda - F)\,Z_{t-1} &= \frac{1}{2}Z_{t-1}'\left[B_i'(\Lambda - F) + (\Lambda - F)'\,B_i\right]Z_{t-1} \\ &= \frac{1}{2}vech\,[X_t]'\,vech\left[B_i'(\Lambda - F) + (\Lambda - F)'\,B_i\right]. \end{aligned}$$

By using the Moore-Penrose inverse of the duplication matrix D_p, we get

$$vech\left[B_i'(\Lambda - F) + (\Lambda - F)'\,B_i\right] = D_p^+\left[vec\left[B_i'(\Lambda - F)\right] + vec\left[(\Lambda - F)'\,B_i\right]\right].$$

Finally, by the properties of the vec operator and the commutation matrix $W_{p,K}$, we obtain

$$\frac{1}{2}D_p^+\left[vec\left[B_i'(\Lambda - F)\right] + vec\left[(\Lambda - F)'\,B_i\right]\right] = \frac{1}{2}D_p^+\left[(\Lambda - F)' \otimes I_p + I_p \otimes (\Lambda - F)'\,W_{p,K}\right]vec\left[B_i'\right].$$

b) By the properties of the tr and vec operators, we have

$$\begin{aligned} Z_{i,t-1}'C_i'(\Lambda - F)\,Z_{t-1} &= tr\left[Z_{t-1}Z_{i,t-1}'C_i'(\Lambda - F)\right] = vec\left[Z_{i,t-1}Z_{t-1}'\right]'\,vec\left[C_i'(\Lambda - F)\right] \\ &= (Z_{t-1} \otimes Z_{i,t-1})'\left[(\Lambda - F)' \otimes I_q\right]vec\left[C_i'\right]. \end{aligned}$$

By combining a) and b), we get $Z_{t-1}'B_i'(\Lambda - F)\,Z_{t-1} + Z_{i,t-1}'C_i'(\Lambda - F)\,Z_{t-1} = x_{1,i,t}'\beta_{1,i}$ and $\beta_{1,i} = \Psi\beta_{2,i}$.

A.3.3 Derivation of Equation (3.8)

We use $\beta_{1,i} = \left(\left(\frac{1}{2}D_p^+\left[vec\left[B_i'(\Lambda - F)\right] + vec\left[(\Lambda - F)'\,B_i\right]\right]\right)', \left(vec\left[C_i'(\Lambda - F)\right]\right)'\right)'$ from Section A.3.2. a) From the properties of the vec operator, we get

$$vec\left[B_i'\left(\Lambda-F\right)\right]+vec\left[\left(\Lambda-F\right)'B_i\right]=\left(I_p\otimes B_i'\right)vec\left[\Lambda-F\right]+\left(B_i'\otimes I_p\right)vec\left[\Lambda'-F'\right].$$

Since $vec\left[\Lambda-F\right]=W_{p,K}vec\left[\Lambda'-F'\right]$, we can factorize $\nu=vec\left[\Lambda'-F'\right]$ to obtain

$$\frac{1}{2}D_p^+\left[vec\left[B_i'\left(\Lambda-F\right)\right]+vec\left[\left(\Lambda-F\right)'B_i\right]\right]=\frac{1}{2}D_p^+\left[\left(I_p\otimes B_i'\right)W_{p,K}+B_i'\otimes I_p\right]\nu.$$

By properties of commutation and duplication matrices (MN p. 54-58), we have $\left(I_p\otimes B_i'\right)W_{p,K}=W_p\left(B_i'\otimes I_p\right)$ and $D_p^+W_p=D_p^+$, then $\frac{1}{2}D_p^+\left[\left(I_p\otimes B_i'\right)W_{p,K}+B_i'\otimes I_p\right]=D_p^+\left(B_i'\otimes I_p\right)$.

b) From the properties of the vec operator, we get

$$vec\left[C_i'\left(\Lambda-F\right)\right]=\left(I_p\otimes C_i'\right)vec\left[\Lambda-F\right]=\left(I_p\otimes C_i'\right)W_{p,K}vec\left[\Lambda'-F'\right]=W_{p,q}\left(C_i'\otimes I_p\right)\nu.$$

A.3.4 Derivation of Equation (3.9)

We use $vec\left[\beta_{3,i}'\right]=\left(vec\left[\{D_p^+\left(B_i'\otimes I_p\right)\}'\right]',vec\left[\{W_{p,q}\left(C_i'\otimes I_p\right)\}'\right]'\right)'$.

a) By MN Theorem 2 p. 35 and Exercise 1 p. 56, and by writing $I_{pK}=I_K\otimes I_p$, we obtain

$$\begin{aligned}
vec\left[D_p^+\left(B_i'\otimes I_p\right)\right] &= \left(I_{pK}\otimes D_p^+\right)vec\left[B_i'\otimes I_p\right]\\
&= \left(I_{pK}\otimes D_p^+\right)\{I_K\otimes\left[\left(W_p\otimes I_p\right)\left(I_p\otimes vec\left[I_p\right]\right)\right]\}vec\left[B_i'\right]\\
&= \left\{I_K\otimes\left[\left(I_p\otimes D_p^+\right)\left(W_p\otimes I_p\right)\left(I_p\otimes vec\left[I_p\right]\right)\right]\right\}vec\left[B_i'\right].
\end{aligned}$$

Moreover, $vec\left[\{D_p^+\left(B_i'\otimes I_p\right)\}'\right]=W_{p(p+1)/2,pK}vec\left[D_p^+\left(B_i'\otimes I_p\right)\right]$.

b) Similarly, $vec\left[W_{p,q}\left(C_i'\otimes I_p\right)\right]=\{I_K\otimes\left[\left(I_p\otimes W_{p,q}\right)\left(W_{p,q}\otimes I_p\right)\left(I_q\otimes vec\left[I_p\right]\right)\right]\}vec\left[C_i'\right]$ and $vec\left[\{W_{p,q}\left(C_i'\otimes I_p\right)\}'\right]=W_{pq,pK}vec\left[W_{p,q}\left(C_i'\otimes I_p\right)\right]$.

By combining a) and b) the conclusion follows.

A.3.5 Proof of Proposition 8

a) Consistency of $\hat{\nu}$. By definition of $\hat{\nu}$, we have: $\hat{\nu}-\nu=\hat{Q}_{\beta_3}^{-1}\frac{1}{n}\sum_i\hat{\beta}_{3,i}'\hat{w}_i\left(\hat{\beta}_{1,i}-\hat{\beta}_{3,i}\nu\right)$. From Equation (3.9) and MN Theorem 2 p. 35, we get $\hat{\beta}_{3,i}\nu=vec[\nu'\hat{\beta}_{3,i}']=\left(I_{d_1}\otimes\nu'\right)vec[\hat{\beta}_{3,i}']=\left(I_{d_1}\otimes\nu'\right)J_a\hat{\beta}_{2,i}$.

Moreover, by using matrices E_1 and E_2, we obtain $\left(\hat{\beta}_{1,i} - \hat{\beta}_{3,i}\nu\right) = \left[E_1' - (I_{d_1} \otimes \nu') J_a E_2'\right] \hat{\beta}_i = C_\nu' \hat{\beta}_i = C_\nu' \left(\hat{\beta}_i - \beta_i\right)$, from Equation (3.8). It follows that

$$\hat{\nu} - \nu \quad = \quad \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}_{3,i}' \hat{w}_i C_\nu' \left(\hat{\beta}_i - \beta_i\right). \tag{a.14}$$

By comparing with Equation (a.3) and by using the same arguments as in the proof of Proposition 2 applied to $\beta_{3,i}'$ instead of b_i, the result follows.

b) Consistency of $\hat{\Lambda}$. By definition of $\hat{\Lambda}$, we deduce $\left\|vec\left[\hat{\Lambda}' - \Lambda'\right]\right\| \leq \|\hat{\nu} - \nu\| + \left\|vec\left[\hat{F}' - F'\right]\right\|$. By part a), $\|\hat{\nu} - \nu\| = o_p(1)$. By the LLN and Assumptions C.1a), C.4a) and C.6, we have $\frac{1}{T}\sum_t Z_{t-1}Z_{t-1}' = O_p(1)$ and $\frac{1}{T}\sum_t u_t Z_{t-1}' = o_p(1)$. Then, by Slustky theorem, we get that $\left\|vec\left[\hat{F}' - F'\right]\right\| = o_p(1)$. The result follows.

A.3.6 Proof of Proposition 9

a) Asymptotic normality of $\hat{\nu}$. From Equation (a.14) and by using $\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) = \tau_{i,T} \hat{Q}_{x,i}^{-1} Y_{i,T}$, we get

$$\begin{aligned}
\sqrt{nT}\left(\hat{\nu} - \nu\right) \quad &= \quad \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \hat{\beta}_{3,i}' \hat{w}_i C_\nu' \hat{Q}_{x,i}^{-1} Y_{i,T} = \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \beta_{3,i}' \hat{w}_i C_\nu' \hat{Q}_{x,i}^{-1} Y_{i,T} \\
&\quad + \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \left(\hat{\beta}_{3,i} - \beta_{3,i}\right)' \hat{w}_i C_\nu' \hat{Q}_{x,i}^{-1} Y_{i,T} \quad =: \hat{Q}_{\beta_3}^{-1} I_{61} + I_{62}.
\end{aligned}$$

Term I_{61} is the analogue of term I_1 in the proof of Proposition 3. To analyse I_{62}, we use the following lemma.

Lemma 10 *Let A be a $m \times n$ matrix and b be a $n \times 1$ vector. Then, $Ab = \left(vec\left[I_n\right]' \otimes I_m\right) vec\left[vec\left[A\right] b'\right]$.*

By Lemma 10, Equation (3.9) and $\sqrt{T} vec\left[\left(\hat{\beta}_{3,i} - \beta_{3,i}\right)'\right] = \tau_{i,T} J_a E_2' \hat{Q}_{x,i}^{-1} Y_{i,T}$, we have

$$\begin{aligned}
I_{62} \quad &= \quad \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 \left(vec\left[I_{d_1}\right]' \otimes I_{Kp}\right) vec\left[J_a E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i\right] \\
&= \quad \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 J_b vec\left[E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i\right] = \sqrt{\frac{n}{T}} \hat{B}_\nu + \frac{1}{\sqrt{T}} \hat{Q}_{\beta_3}^{-1} I_{63},
\end{aligned}$$

where $I_{63} := \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 J_b vec \left[E_2' \left(\hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_\nu - \tau_{i,T}^{-1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \right) \hat{w}_i \right]$. We get:

$$\sqrt{nT} \left(\hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu \right) = \hat{Q}_{\beta_3}^{-1} I_{61} + \frac{1}{\sqrt{T}} \hat{Q}_{\beta_3}^{-1} I_{63}, \tag{a.15}$$

which is the analogue of Equation (a.5) in the proof of Proposition 3. Let us now derive the asymptotic behaviour of the terms in the RHS of (a.15). By MN Theorem 2 p. 35, we have $I_{61} = \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \left[(Y_{i,T}' \hat{Q}_{x,i}^{-1}) \otimes (\beta_{3,i}' \hat{w}_i) \right] vec \left[C_\nu' \right]$. Similarly as in Lemma 2, we have $I_{61} = \frac{1}{\sqrt{n}} \sum_i \tau_i \left[(Y_{i,T}' Q_{x,i}^{-1}) \otimes (\beta_{3,i}' w_i) \right] vec \left[C_\nu' \right] + o_p(1)$. Then, by the properties of the vec operator, we get $\hat{Q}_{\beta_3}^{-1} I_{61} = \left(vec \left[C_\nu' \right]' \otimes \hat{Q}_{\beta_3}^{-1} \right) \frac{1}{\sqrt{n}} \sum_i \tau_i vec \left[(Y_{i,T}' Q_{x,i}^{-1}) \otimes (\beta_{3,i}' w_i) \right] + o_p(1)$. Moreover, by using the equality $vec \left[(Y_{i,T}' Q_{x,i}^{-1}) \otimes (\beta_{3,i}' w_i) \right] = (Q_{x,i}^{-1} Y_{i,T}) \otimes vec \left[\beta_{3,i}' w_i \right]$ (see MN Theorem 10 p. 55), we get $\hat{Q}_{\beta_3}^{-1} I_{61} = \left(vec \left[C_\nu' \right]' \otimes \hat{Q}_{\beta_3}^{-1} \right) \frac{1}{\sqrt{n}} \sum_i \tau_i \left[(Q_{x,i}^{-1} Y_{i,T}) \otimes v_{3,i} \right] + o_p(1)$. Then $\hat{Q}_{\beta_3}^{-1} I_{61} \Rightarrow N(0, \Sigma_\nu)$ follows from Assumption B.2 a). Let us now consider I_{63}. By similar arguments as in the proof of Proposition 3 (control of term I_2), $\frac{1}{\sqrt{T}} I_{63} = o_p(1)$. The conclusion follows.

b) Asymptotic normality of $vec \left(\hat{\Lambda}' \right)$**.** We have $\sqrt{T} vec \left[\hat{\Lambda}' - \Lambda' \right] = \sqrt{T} vec \left[\hat{F}' - F' \right] + \sqrt{T} (\hat{\nu} - \nu)$. By using $\sqrt{T} vec \left[\hat{F}' - F' \right] = \left[I_K \otimes \left(\frac{1}{T} \sum_t Z_{t-1} Z_{t-1}' \right)^{-1} \right] \frac{1}{\sqrt{T}} \sum_t u_t \otimes Z_{t-1}$ and $\sqrt{T} (\hat{\nu} - \nu) = O_p \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{T}} \right) = o_p(1)$, the conclusion follows from Assumption B.2b).

A.3.7 Proof of Proposition 10

By similar arguments as in the proof of Proposition 5, we have:

$$\begin{aligned} \hat{Q}_e &= \frac{1}{n} \sum_i \left(\hat{\beta}_i - \beta_i \right)' C_{\hat{\nu}} \hat{w}_i C_{\hat{\nu}}' \left(\hat{\beta}_i - \beta_i \right) + O_{p,log} \left(\frac{1}{nT} + \frac{1}{T^2} \right) \\ &= \frac{1}{nT} \sum_i \tau_{i,T}^2 tr \left[C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] + O_{p,log} \left(\frac{1}{nT} + \frac{1}{T^2} \right). \end{aligned}$$

By using that $\tau_{i,T} tr\left[C'_{\hat{\nu}}\hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1}C_{\hat{\nu}}\hat{w}_i\right] = \mathbf{1}_i^{\chi} d_1$, Lemma 4 in the conditional case and $n = O(T^{\bar{\gamma}})$ with $\bar{\gamma} < 2$, we get:

$$\begin{aligned}\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i \tau_{i,T}^2 tr\left[C'_{\hat{\nu}}\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - \tau_{i,T}^{-1}\hat{S}_{ii}\right)\hat{Q}_{x,i}^{-1}C_{\hat{\nu}}\hat{w}_i\right] + o_p(1) \\ &= \frac{1}{\sqrt{n}}\sum_i \tau_i^2 tr\left[C'_{\hat{\nu}}Q_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)Q_{x,i}^{-1}C_{\hat{\nu}}w_i\right] + o_p(1).\end{aligned}$$

Now, by using $tr(ABCD) = vec(D')'(C' \otimes A)vec(B)$ (MN Theorem 3, p. 35) and $vec(ABC) = (C' \otimes A)vec(B)$ for conformable matrices, we have:

$$\begin{aligned}& tr\left[C'_{\hat{\nu}}Q_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)Q_{x,i}^{-1}C_{\hat{\nu}}w_i\right] = vec[w_i]'\left(C'_{\hat{\nu}} \otimes C'_{\hat{\nu}}\right)vec\left[Q_{x,i}^{-1}\left(Y_{i,T}Y'_{i,T} - S_{ii,T}\right)Q_{x,i}^{-1}\right] \\ =\; & vec[w_i]'\left(C'_{\hat{\nu}} \otimes C'_{\hat{\nu}}\right)\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)vec\left[Y_{i,T}Y'_{i,T} - S_{ii,T}\right] \\ =\; & vec[w_i]'\left(C'_{\hat{\nu}} \otimes C'_{\hat{\nu}}\right)\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec[S_{ii,T}]\right) \\ =\; & vec\left[C'_{\hat{\nu}} \otimes C'_{\hat{\nu}}\right]'\left\{\left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec[S_{ii,T}]\right)\right] \otimes vec[w_i]\right\}.\end{aligned}$$

Thus, we get $\hat{\xi}_{nT} = vec\left[C'_{\hat{\nu}} \otimes C'_{\hat{\nu}}\right]' \frac{1}{\sqrt{n}}\sum_i \tau_i^2\left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right)\right] \otimes vec[w_i]$. From Assumption B.4, we get $\hat{\xi}_{nT} \Rightarrow N(0, \Sigma_\xi)$, where $\Sigma_\xi = vec\left[C'_{\nu} \otimes C'_{\nu}\right]'\Omega vec\left[C'_{\nu} \otimes C'_{\nu}\right]$. Now, by using that $tr(ABCD) = vec(D)'(A \otimes C')vec(B')$ we have:

$$\begin{aligned}& vec\left[C'_{\nu} \otimes C'_{\nu}\right]'\left[(S_{Q,ij} \otimes S_{Q,ij}) \otimes vec[w_i]vec[w_j]'\right]vec\left[C'_{\nu} \otimes C'_{\nu}\right] \\ =\; & tr\left[(S_{Q,ij} \otimes S_{Q,ij})(C_\nu \otimes C_\nu)vec[w_j]vec[w_i]'\left(C'_{\nu} \otimes C'_{\nu}\right)\right] \\ =\; & vec[w_i]'\left[\left(C'_\nu S_{Q,ij}C_\nu\right) \otimes \left(C'_\nu S_{Q,ij}C_\nu\right)\right]vec[w_j] = tr\left[\left(C'_\nu S_{Q,ij}C_\nu\right)w_j\left(C'_\nu S_{Q,ji}C_\nu\right)w_i\right] \\ =\; & tr\left[\left(C'_\nu Q_{x,i}^{-1}S_{ij}Q_{x,j}^{-1}C_\nu\right)w_j\left(C'_\nu Q_{x,j}^{-1}S_{ji}Q_{x,i}^{-1}C_\nu\right)w_i\right],\end{aligned}$$

and similarly we have $vec\left[C'_{\nu} \otimes C'_{\nu}\right]'\left[(S_{Q,ij} \otimes S_{Q,ij})W_d \otimes vec[w_i]vec[w_j]'\right]vec\left[C'_{\nu} \otimes C'_{\nu}\right]$ $= tr\left[\left(C'_\nu Q_{x,i}^{-1}S_{ij}Q_{x,j}^{-1}C_\nu\right)w_j\left(C'_\nu Q_{x,j}^{-1}S_{ji}Q_{x,i}^{-1}C_\nu\right)w_i\right]$. Thus, we get the asymptotic variance matrix $\Sigma_\xi = 2\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j}\frac{\tau_i^2\tau_j^2}{\tau_{ij}^2} tr\left[\left(C'_\nu Q_{x,i}^{-1}S_{ij}Q_{x,j}^{-1}C_\nu\right)w_j\left(C'_\nu Q_{x,j}^{-1}S_{ji}Q_{x,i}^{-1}C_\nu\right)w_i\right]\right]$. From $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p(1)$, the conclusion follows.

A.4 Check of assumptions under block dependence

In this appendix, we verify that the eigenvalue condition in Assumption APR.4 (i), and the cross-sectional/time-series dependence and CLT conditions in Assumptions A.1-A.5, are satisfied under a block-dependence structure in a serially i.i.d. framework. Let us assume that:

BD.1 The errors $\varepsilon_t(\gamma)$ are i.i.d. over time with $E[\varepsilon_t(\gamma)] = 0$ and $E[\varepsilon_t(\gamma)^3] = 0$, for all $\gamma \in [0,1]$. For any n, there exists a partition of the interval $[0,1]$ into $J_n \leq n$ subintervals $I_1, ..., I_{J_n}$, such that $\varepsilon_t(\gamma)$ and $\varepsilon_t(\gamma')$ are independent if γ and γ' belong to different subintervals, and $J_n \to \infty$ as $n \to \infty$.

BD.2 The blocks are such that $n \sum_{m=1}^{J_n} B_m^2 = O(1)$, $n^{3/2} \sum_{m=1}^{J_n} B_m^3 = o(1)$, where $B_m = \int_{I_m} dG(\gamma)$.

BD.3 The factors (f_t) and the indicators $(I_t(\gamma))$, $\gamma \in [0,1]$, are i.i.d. over time, mutually independent, and independent of the errors $(\varepsilon_t(\gamma))$, $\gamma \in [0,1]$.

BD.4 There exists a constant M such that $\|f_t\| \leq M$, P-a.s.. Moreover, $\sup_{\gamma \in [0,1]} E[|\varepsilon_t(\gamma)|^6] < \infty$, $\sup_{\gamma \in [0,1]} \|\beta(\gamma)\| < \infty$ and $\inf_{\gamma \in [0,1]} E[I_t(\gamma)] > 0$.

The block-dependence structure as in Assumption BD.1 is satisfied for instance when there are unobserved industry-specific factors independent among industries and over time, as in Ang, Liu, Schwartz (2010). In empirical applications, blocks can match industrial sectors. Then, the number J_n of blocks amounts to a couple of dozens, and the number of assets n amounts to a couple of thousands. There are approximately nB_m assets in block m, when n is large. In the asymptotic analysis, Assumption BD.2 on block sizes and block number requires that the largest block size shrinks with n and that there are not too many large blocks, i.e., the partition in independent blocks is sufficiently fine grained asymptotically. Within blocks, covariances do not need to vanish asymptotically.

Lemma 11 *Let Assumptions BD.1-4 on block dependence and Assumptions SC.1-SC.2 on random sampling hold. Then, Assumptions APR.4 (i), A.1, A.2, A.3, A.4 (with any $q \in (0,1)$ and $\delta \in (1/2,1)$) and A.5 are satisfied.*

The proof of Lemma 11 uses a result on almost sure convergence in Stout (1974), a large deviation theorem based on the Hoeffding's inequality in Bosq (1998), and CLTs for martingale difference arrays in Davidson (1994) and White (2001).

Instead of a block structure, we can also assume that the covariance matrix is full, but with off-diagonal elements vanishing asymptotically. In that setting, we can carry out similar checks.

Appendix B: Supplementary materials

These supplementary materials provide the proofs of the technical lemmas used in Appendix A. Finally, we derive inference for the cost of equity and include some empirical results for Ford Motor, Disney Walt, Motorola and Sony (Appendix B.2).

B.1 Proofs of the technical lemmas

B.1.1 Proof of Lemma 1 (iii)

We have $\hat{w}_i - w_i = \mathbf{1}_i^{\chi}(\hat{v}_i^{-1} - v_i^{-1}) + (\mathbf{1}_i^{\chi} - 1)v_i^{-1}$ and $\hat{v}_i^{-1} - v_i^{-1} = -\hat{v}_i^{-1}v_i^{-1}(\hat{v}_i - v_i)$. Since v_i is uniformly lower bounded from part (ii), we have $\frac{1}{n}\sum_i |\hat{w}_i - w_i| \leq C\frac{1}{n}\sum_i \mathbf{1}_i^{\chi}\frac{|\hat{v}_i - v_i|}{C - |\hat{v}_i - v_i|} + C\frac{1}{n}\sum_i(1 - \mathbf{1}_i^{\chi})$. The second term in the RHS is $o_p(1)$ from Lemma 4. To prove that the first term is $o_p(1)$ it is sufficient to show:

$$\sup_i \mathbf{1}_i^{\chi}|\hat{v}_i - v_i| = o_p(1). \tag{b.1}$$

We use Equation (a.7). Since $\hat{\nu}_1 - \nu = O_p(T^{-c})$, for some $c > 0$ (by repeating the proof of Proposition 2 with known weights equal to 1), $\mathbf{1}_i^{\chi}\|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$, $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$, $\|S_{ii}\| \leq M$, and by using Assumption C.5, the uniform bound in (b.1) follows if we prove:

$$\sup_i \mathbf{1}_i^{\chi}\|\hat{S}_{ii} - S_{ii}\| = O_p(T^{-c}), \tag{b.2}$$
$$\sup_i \mathbf{1}_i^{\chi}\|\hat{Q}_{x,i}^{-1} - Q_x^{-1}\| = O_p(T^{-c}), \tag{b.3}$$
$$\sup_i \mathbf{1}_i^{\chi}|\tau_{i,T} - \tau_i| = O_p(T^{-c}), \tag{b.4}$$

for some $c > 0$. To prove the uniform bound (b.2), we use Equation (a.9). As in the proof of Lemma 1 (i), we have $\sup_i T^{-1/2}\|Y_{i,T}\| = O_{p,log}(T^{-\eta/2})$ from Assumption C.1 c), and similarly $\sup_i T^{-1/2}\|W_{1,i,T} + W_{2,i,T}\|$ $= O_{p,log}(T^{-\eta/2})$ and $\sup_i T^{-1/2}\|W_{3,i,T}\| = O_p(T^{-\eta/2})$, from Assumptions C.1 e) and f), respectively. Moreover, $\|\hat{Q}_{x,i}^{(4)}\| \leq M$ and $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$. Thus, from Assumption C.5, bound (b.2) follows. To prove (b.3) we use $\hat{Q}_{x,i}^{-1} - Q_x^{-1} = -\tau_{i,T}\hat{Q}_{x,i}^{-1}W_{i,T}Q_x^{-1}$, where $W_{i,T}$ is defined as in Equation (a.10) and is such that $\sup_i \|W_{i,T}\| = O_{p,log}(T^{-\eta/2})$ from Assumption C.1 b). Finally, (b.4) follows from $|\tau_{i,T} - \tau_i| \leq \tau_{i,T}\tau_i \left|\frac{1}{T}\sum_t (I_{i,t} - E[I_{i,t}|\gamma_i])\right|$, $\mathbf{1}_i^{\chi}\tau_{i,T} \leq \chi_{2,T}$, $\tau_i \leq M$ and by using $\sup_i \left|\frac{1}{T}\sum_t (I_{i,t} - E[I_{i,t}|\gamma_i])\right| = O_{p,log}(T^{-\eta/2})$ from Assumption C.1 d).

B.1.2 Proof of Lemma 3

B.1.2.1 Part i)

Let us write I_{21} as:

$$
\begin{aligned}
I_{21} &= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_{x,i}^{-1} \\
&= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_x^{-1} + \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_x^{-1} \\
&\quad + \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) \\
&\quad + \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) \\
&=: \hat{Q}_x^{-1}I_{211}\hat{Q}_x^{-1} + I_{212}\hat{Q}_x^{-1} + \hat{Q}_x^{-1}I_{212}' + I_{213}.
\end{aligned}
$$

We control the terms separately.

Proof that $I_{211} = \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) + O_{p,log}(\sqrt{n}/T) = O_p(1) + O_{p,log}(\sqrt{n}/T)$. We use a decomposition similar to term I_{11} in the proof of Lemma 2:

$$\begin{aligned} I_{211} &= \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right)+\frac{1}{\sqrt{n}}\sum_i\left(\mathbf{1}_i^\chi-1\right)w_i\tau_i^2\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right) \\ &+\frac{1}{\sqrt{n}}\sum_i\mathbf{1}_i^\chi w_i\left(\tau_{i,T}^2-\tau_i^2\right)\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right) \\ &+\frac{1}{\sqrt{n}}\sum_i\mathbf{1}_i^\chi\left(\hat{v}_i^{-1}-v_i^{-1}\right)\tau_{i,T}^2\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right) \qquad =: I_{2111}+I_{2112}+I_{2113}+I_{2114}. \end{aligned}$$

To prove $I_{2111}=O_p(1)$, take $k,l=1,...,K$, and consider $\zeta_{nT}:=\frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2\left(Y_{i,k,T}Y_{i,l,T}-S_{ii,kl,T}\right)$. Then:

$$\begin{aligned} &E[\zeta_{nT}^2|x_{\underline{T}},I_{\underline{T}},\{\gamma_i\}]=\frac{1}{n}\sum_{i,j}w_iw_j\tau_i^2\tau_j^2 cov\left(Y_{i,k,T}Y_{i,l,T},Y_{j,k,T}Y_{j,l,T}|x_{\underline{T}},I_{\underline{T}},\gamma_i,\gamma_j\right) \\ &= \frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4}w_iw_j\tau_i^2\tau_j^2 cov\left(\varepsilon_{i,t_1}\varepsilon_{i,t_2},\varepsilon_{j,t_3}\varepsilon_{j,t_4}|x_{\underline{T}},\gamma_i,\gamma_j\right)I_{i,t_1}I_{i,t_2}I_{j,t_3}I_{j,t_4}x_{t_1,k}x_{t_2,l}x_{t_3,k}x_{t_4,l}. \end{aligned}$$

From Assumptions A.1 c), C.3 b) and C.4, it follows $E[\zeta_{nT}^2]=O(1)$. Hence, $\zeta_{nT}=O_p(1)$ and $I_{2111}=O_p(1)$. We can prove that $I_{2112}=o_p(1)$ and $I_{2113}=o_p(1)$ by using arguments similar to terms I_{112} and I_{113} in the proof of Lemma 2. Finally, let us prove that $I_{2114}=O_{p,log}(\sqrt{n}/T)$. Similarly to I_{114} in the proof of Lemma 2, we use

$$\hat{v}_i^{-1}-v_i^{-1}=-v_i^{-2}\left(\hat{v}_i-v_i\right)+\hat{v}_i^{-1}v_i^{-2}\left(\hat{v}_i-v_i\right)^2, \tag{b.5}$$

and Equation (a.7). We focus on term:

$$I_{21141}=-\frac{1}{\sqrt{n}}\sum_i\mathbf{1}_i^\chi v_i^{-2}\tau_{i,T}^3 c_{\hat{\nu}_1}'\hat{Q}_{x,i}^{-1}\left(\hat{S}_{ii}-S_{ii}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}_1}\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right),$$

the other contributions to I_{2114} can be controlled similarly. Now, we use Equation (a.9) and treat x_t as a scalar to ease notation. We have:

$$\begin{aligned} I_{21141} &= -\frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^4 c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} W_{1,i,T}\hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &\quad -\frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^4 c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} W_{2,i,T}\hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &\quad +2\frac{1}{\sqrt{n}T}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^5 c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} W_{3,i,T}\hat{Q}_{x,i}^{-1} Y_{i,T}\hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &\quad -\frac{1}{\sqrt{n}T}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^6 c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \hat{Q}_{x,i}^{(4)}\hat{Q}_{x,i}^{-1} Y_{i,T}Y_{i,T}'\hat{Q}_{x,i}^{-2} c_{\hat{\nu}_1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &=: -c_{\hat{\nu}_1}'(I_{211411} + I_{211412} + I_{211413} + I_{211414})c_{\hat{\nu}_1}. \end{aligned}$$

Let us focus on term I_{211411} and prove that it is $O_{p,log}(\sqrt{n}/T)$. We have:

$$I_{211411} = \frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^4 \hat{Q}_{x,i}^{-2} W_{1,i,T} Y_{i,T}^2 - \frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi} v_i^{-2}\tau_{i,T}^4 \hat{Q}_{x,i}^{-2} W_{1,i,T} S_{ii,T} =: I_{2114111} + I_{2114112}.$$

Term $I_{2114111}$ is such that:

$$|E[I_{2114111}|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}]| \leq \frac{C\chi_{1,T}^2\chi_{2,T}^4}{\sqrt{n}T^2}\sum_i \sum_{t_1,t_2,t_3} |E[\eta_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{i,t_3}|x_{\underline{T}}, \gamma_i]|,$$

and

$$V[I_{2114111}|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq \frac{C\chi_{1,T}^4\chi_{2,T}^8}{nT^4}\sum_{i,j}\sum_{t_1,\ldots,t_6} |cov(\eta_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{i,t_3}, \eta_{j,t_4}\varepsilon_{j,t_5}\varepsilon_{j,t_6}|x_{\underline{T}}, \gamma_i, \gamma_j)|.$$

From Assumptions C.2, C.3 f) and C.5, we get $E[I_{2114111}] = O_{log}(\sqrt{n}/T)$ and $V[I_{2114111}] = o(1)$, which implies $I_{2114111} = O_{p,log}(\sqrt{n}/T)$. The other terms making I_{2114} can be controlled similarly, and we get $I_{2114} = O_{p,log}(\sqrt{n}/T)$.

Proof that $I_{212} = o_p(1)$. We have:

$$\begin{aligned} I_{212} &= \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &\quad +\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi}(\hat{v}_i^{-1} - v_i^{-1})\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) =: I_{2121} + I_{2122}. \end{aligned}$$

We focus on term I_{2121}, use Equation (a.10) and treat x_t as a scalar to ease notation. We have:

$$\begin{aligned} I_{2121} &= -\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T}^3 \hat{Q}_{x,i}^{-1} W_{i,T}\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) \\ &\quad + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T}^2 \hat{Q}_{x,i}^{-1} W_{T}\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right) =: (I_{21211} + I_{21212})\hat{Q}_x^{-1}. \end{aligned}$$

Let us focus on I_{21211}. We have:

$$E[\|I_{21211}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq \frac{C\chi_{1,T}^2\chi_{2,T}^6}{nT^2}\sum_{i,j}\sum_{t_1,\ldots,t_4}\|W_{i,T}\|\|W_{j,T}\||cov(\varepsilon_{i,t_1}\varepsilon_{i,t_2},\varepsilon_{j,t_3}\varepsilon_{j,t_4}|x_{\underline{T}},\gamma_i,\gamma_j)|.$$

By the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} E[\|I_{21211}\|^2|\{\gamma_i\}] &\leq C\chi_{1,T}^2\chi_{2,T}^6 \sup_i E[\|W_{i,T}\|^4|\gamma_i]^{1/2} \\ &\quad \frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4} E\left[|cov(\varepsilon_{i,t_1}\varepsilon_{i,t_2},\varepsilon_{j,t_3}\varepsilon_{j,t_4}|x_{\underline{T}},\gamma_i,\gamma_j)|^2|\gamma_i,\gamma_j\right]^{1/2}. \end{aligned}$$

From Assumptions C.1 b), C.3 b), C.4 a), and C.5, we deduce $E[\|I_{21211}\|^2|] = o(1)$, which implies $I_{21211} = o_p(1)$. Similar argument can be used to prove that the other terms making I_{212} are $o_p(1)$.

Proof that $I_{213} = o_p(1)$. This step uses arguments similar as for I_{212}.

B.1.2.2 Part (ii)

We have $I_{22} = \frac{1}{\sqrt{nT}}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_{x,i}^{-1}W_{1,i,T}\hat{Q}_{x,i}^{-1}$, where $W_{1,i,T}$ is as in Equation (a.9). Write:

$$I_{22} = \frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi} v_i^{-1}\tau_{i,T}^2\hat{Q}_{x,i}^{-1}W_{1,i,T}\hat{Q}_{x,i}^{-1} + \frac{1}{\sqrt{nT}}\sum_i \mathbf{1}_i^{\chi}(\hat{v}_i^{-1} - v_i^{-1})\tau_{i,T}^2\hat{Q}_{x,i}^{-1}W_{1,i,T}\hat{Q}_{x,i}^{-1} =: I_{221} + I_{222}.$$

Let us first consider I_{221}. We have:

$$E[\|I_{221}\|^2|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq C\chi_{1,T}^4\chi_{2,T}^4\frac{1}{nT^2}\sum_{i,j}\sum_{t_1,t_2}|cov(\eta_{i,t_1},\eta_{j,t_2}|x_{\underline{T}},\gamma_i,\gamma_j)|.$$

From Assumptions C.3 a) and C.5, it follows $E[\|I_{222}\|^2] = O_{log}(1/T)$, and thus $I_{222} = O_{p,log}(1/\sqrt{T})$.

Let us now consider term I_{222}. We use Equation (b.5), and plug in the decompositions (a.7) and (a.9).

We focus on term $c_{\hat{\nu}_1}^2 I_{2221}$ of the resulting expansion, where:

$$I_{2221} = -\frac{1}{\sqrt{n}T}\sum_i \mathbf{1}_i^\chi v_i^{-2}\tau_{i,T}^4 \hat{Q}_{x,i}^{-4} W_{1,i,T}^2.$$

The other terms can be treated similarly. We have:

$$E[I_{2221}|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq C\chi_{1,T}^4\chi_{2,T}^4 \frac{1}{\sqrt{n}T^2}\sum_i \sum_{t_1,t_2} |cov(\varepsilon_{i,t_1}^2, \varepsilon_{i,t_2}^2 | x_{\underline{T}}, \gamma_i)|,$$

and

$$V[I_{2221}|x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq C\chi_{1,T}^8\chi_{2,T}^8 \frac{1}{nT^4}\sum_{i,j} \sum_{t_1,t_2,t_3,t_4} |cov(\eta_{i,t_1}\eta_{i,t_2}, \eta_{j,t_3}\eta_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j)|.$$

From Assumptions C.3 a) and C.5, it follows $E[I_{2221}] = O_{log}(\sqrt{n}/T)$. By Assumptions C.3 d) and C.5 we can prove that $V[I_{2221}] = o(1)$, and it follows $I_{2221} = O_p(\sqrt{n}/T)$.

B.1.2.3 Part (iii)

We have $I_{23} = -\frac{2}{\sqrt{n}T}\sum_i \hat{w}_i \tau_{i,T}^3 \hat{Q}_{x,i}^{-3} W_{3,i,T} Y_{i,T} + \frac{1}{\sqrt{n}T}\sum_i \hat{w}_i \tau_{i,T}^4 \hat{Q}_{x,i}^{-4} \hat{Q}_{x,i}^{(4)} Y_{i,T}^2$, where $W_{3,i,T}$ and $\hat{Q}_{x,i}^{(4)}$ are as in Equation (a.9) and we treat x_t as a scalar to ease notation. By similar arguments as in part (ii) we can prove that $I_{23} = O_{p,log}(\sqrt{n}/T)$.

B.1.2.4 Part (iv)

The statement follows from Lemma 1 (ii)-(iii), $\mathbf{1}_i^\chi \tau_{i,T} \leq \chi_{2,T}$, $\mathbf{1}_i^\chi \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$, bound (b.2), $\|S_{ii}\| \leq M$ and Assumption C.5.

B.1.2.5 Part (v)

The statement follows from Equation (a.4), Lemma 1 (iv), $I_1 = O_p(1)$ and $\frac{1}{n}\sum_i \hat{w}_i \tau_{i,T}^2 E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1}$ $= O_{p,log}(1)$.

B.1.3 Proof of Lemma 4

We have $\mathbb{P}\left[\mathbf{1}_i^{\chi}=0\right] \leq \mathbb{P}\left[\tau_{i,T} \geq \chi_{2,T}\right] + \mathbb{P}\left[CN\left(\hat{Q}_{x,i}\right) \geq \chi_{1,T}\right] =: P_{1,nT} + P_{2,nT}$. Let us first control $P_{1,nT}$. We have $P_{1,nT} \leq \mathbb{P}\left[\frac{1}{T}\sum_t I_{i,t} \leq \chi_{2,T}^{-1}\right] \leq \mathbb{P}\left[\frac{1}{T}\sum_t \left(I_{i,t} - \tau_i^{-1}\right) \leq \chi_{2,T}^{-1} - M^{-1}\right]$, where we use $\tau_i \leq M$ for all i (Assumption C.4 c)). Then, for $0 < \delta < M^{-1}/2$ and T large such that $M^{-1} - \chi_{2,T}^{-1} > \delta$, we get the upper bound $P_{1,nT} \leq \mathbb{P}\left[\left|\frac{1}{T}\sum_t \left(I_{i,t} - \tau_i^{-1}\right)\right| \geq \delta\right]$. By using that $\tau_i^{-1} = E[I_{i,t}|\gamma_i]$ and $\mathbb{P}\left[\left|\frac{1}{T}\sum_t \left(I_{i,t} - \tau_i^{-1}\right)\right| \geq \delta\right] = E\left[\mathbb{P}\left[\left|\frac{1}{T}\sum_t \left(I_{i,t} - E[I_{i,t}|\gamma_i]\right)\right| \geq \delta | \gamma_i\right]\right] \leq \sup_{\gamma\in[0,1]} \mathbb{P}\left[\left|\frac{1}{T}\sum_t \left(I_t(\gamma) - E[I_t(\gamma)]\right)\right| \geq \delta\right]$, from Assumption C.1 d) it follows $P_{1,nT} = O(T^{-\bar{b}})$, for any $\bar{b} > 0$.

Let us now consider $P_{2,nT}$. By using $\|\hat{Q}_{x,i}\| \leq M$ (Assumption C.4 a)), we get $eig_{max}(\hat{Q}_{x,i}) \leq M$, and thus $CN\left(\hat{Q}_{x,i}\right) \leq M^{1/2}\left[eig_{min}(\hat{Q}_{x,i})\right]^{-1/2}$. Hence $P_{2,nT} \leq \mathbb{P}\left[eig_{min}(\hat{Q}_{x,i}) \leq M/\chi_{1,T}^2\right]$. By using that $eig_{min}(\hat{Q}_{x,i}) \geq eig_{min}(Q_x) - \|\hat{Q}_{x,i} - Q_x\|$, we get $P_{2,nT} \leq \mathbb{P}\left[\|\hat{Q}_{x,i} - Q_x\| \geq eig_{min}(Q_x) - M/\chi_{1,T}^2\right]$. Now, let $0 < \delta \leq eig_{min}(Q_x)/2$ and T large such that $eig_{min}(Q_x) - M/\chi_{1,T}^2 > \delta$. Then, by using $\mathbb{P}\left[\|\hat{Q}_{x,i} - Q_x\| \geq \delta\right] \leq \mathbb{P}\left[\left|\frac{1}{T}\sum_t I_{i,t}(x_t x_t - Q_x)\right| \geq \sqrt{\delta}\right] + \mathbb{P}\left[\tau_{i,T} \geq \sqrt{\delta}\right]$ we get $P_{2,nT} \leq P\left[\left|\frac{1}{T}\sum_t I_{i,t}(x_t x_t - Q_x)\right| \geq \sqrt{\delta}\right] + O(T^{-\bar{b}})$. The first term in the RHS is $O(T^{-\bar{b}})$ by using $\mathbb{P}\left[\left|\frac{1}{T}\sum_t I_{i,t}(x_t x_t - Q_x)\right| \geq \sqrt{\delta}\right] \leq \sup_{\gamma\in[0,1]} \mathbb{P}\left[\left|\frac{1}{T}\sum_t I_t(\gamma)(x_t x_t - Q_x)\right| \geq \sqrt{\delta}\right]$ and Assumption C.1 b). Then, $P_{2,nT} = O(T^{-\bar{b}})$, for any $\bar{b} > 0$.

B.1.4 Proof of Lemma 5

Let $W_T(\gamma) := \frac{1}{T}\sum_t (I_t(\gamma) - E[I_t(\gamma)])$ and $r_T := T^{-a}$ for $0 < a < \eta/2$. Since $|W_T(\gamma)| \leq 1$ for all $\gamma \in [0,1]$, and from Assumption C.1 d), we have:

$$\begin{aligned}
\sup_{\gamma\in[0,1]} E[|W_T(\gamma)|^4] &\leq \sup_{\gamma\in[0,1]} E[|W_T(\gamma)|] = \sup_{\gamma\in[0,1]} \int_0^1 \mathbb{P}[|W_T(\gamma)| \geq \delta]d\delta \leq r_T + \sup_{\gamma\in[0,1]} \int_{r_T}^1 \mathbb{P}[|W_T(\gamma)| \geq \delta]d\delta \\
&\leq r_T + C_1 T \int_{r_T}^1 \exp\left\{-C_2\delta^2 T^{\eta}\right\} d\delta + C_3 \exp\left\{-C_4 T^{\bar{\eta}}\right\} \int_{r_T}^1 \frac{1}{\delta} d\delta \\
&\leq r_T + C_1 T \exp\left\{-C_2 r_T^2 T^{\eta}\right\} + C_3 \exp\left\{-C_4 T^{\bar{\eta}}\right\} \log(1/r_T) = o(1).
\end{aligned}$$

B.1.5 Proof of Lemma 6

By definition of $\tilde{S}_{ij}$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i,j}\left\|\tilde{S}_{ij}-S_{ij}\right\| &= \frac{1}{n}\sum_{i,j}\left\|\hat{S}_{ij}\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa\right\}}-S_{ij}\right\| \\
&\leq \frac{1}{n}\sum_{i,j}\left\|S_{ij}\mathbf{1}_{\{\|S_{ij}\|\geq\kappa\}}-S_{ij}\right\|+\frac{1}{n}\sum_{i,j}\left\|\hat{S}_{ij}\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa\right\}}-S_{ij}\mathbf{1}_{\{\|S_{ij}\|\geq\kappa\}}\right\| \\
&=: I_{31}+I_{32}.
\end{aligned}$$

By Assumption A.4,

$$I_{31}=\frac{1}{n}\sum_{i,j}\|S_{ij}\|\,\mathbf{1}_{\{\|S_{ij}\|<\kappa\}}\leq\max_i\sum_j\|S_{ij}\|^q\,\kappa^{1-q}\leq\kappa^{1-q}c_0\,(n)=O_p\left(\kappa^{1-q}n^{\delta}\right), \tag{b.6}$$

where $c_0(n):=\max_i\sum_j\|S_{ij}\|^q=O_p(n^{\delta})$.

Let us now consider I_{32}:

$$\begin{aligned}
I_{32} &= \frac{1}{n}\sum_{i,j}\left\|\hat{S}_{ij}\right\|\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa,\|S_{ij}\|<\kappa\right\}}+\frac{1}{n}\sum_{i,j}\|S_{ij}\|\,\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|<\kappa,\|S_{ij}\|\geq\kappa\right\}} \\
&\quad+\frac{1}{n}\sum_{i,j}\left\|\hat{S}_{ij}-S_{ij}\right\|\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa,\|S_{ij}\|\geq\kappa\right\}} \\
&\leq \max_i\sum_j\left\|\hat{S}_{ij}\right\|\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa,\|S_{ij}\|<\kappa\right\}}+\max_i\sum_j\|S_{ij}\|\,\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|<\kappa,\|S_{ij}\|\geq\kappa\right\}} \\
&\quad+\max_i\sum_j\left\|\hat{S}_{ij}-S_{ij}\right\|\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa,\|S_{ij}\|\geq\kappa\right\}}=: I_{33}+I_{34}+I_{35}.
\end{aligned}$$

From Assumption A.4, we have:

$$I_{35}\leq\max_{i,j}\left\|\hat{S}_{ij}-S_{ij}\right\|\max_i\sum_j\|S_{ij}\|^q\,\kappa^{-q}=O_p\left(\psi_{nT}c_0\,(n)\,\kappa^{-q}\right). \tag{b.7}$$

Let us study I_{33}:

$$I_{33} \leq \max_i\sum_j\left\|\hat{S}_{ij}-S_{ij}\right\|\mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa,\|S_{ij}\|<\kappa\right\}}+\max_i\sum_j\|S_{ij}\|\,\mathbf{1}_{\{\|S_{ij}\|<\kappa\}}=: I_{36}+I_{37}.$$

By Assumption A.4,

$$I_{37} \leq \kappa^{1-q} c_0\left(n\right). \tag{b.8}$$

Now take $v \in (0,1)$. Let $N_i\left(\epsilon\right) := \sum_j \mathbf{1}_{\left\{\left\|\hat{S}_{ij}-S_{ij}\right\|>\epsilon\right\}}$, for $\epsilon > 0$, then

$$\begin{aligned} I_{36} &= \max_i \sum_j \left\|\hat{S}_{ij}-S_{ij}\right\| \mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa, \left\|S_{ij}\right\|\leq v\kappa\right\}} + \max_i \sum_j \left\|\hat{S}_{ij}-S_{ij}\right\| \mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|\geq\kappa, v\kappa<\left\|S_{ij}\right\|<\kappa\right\}} \\ &\leq \max_{i,j}\left\|\hat{S}_{ij}-S_{ij}\right\| \max_i N_i\left(\left(1-v\right)\kappa\right) + \max_{i,j}\left\|\hat{S}_{ij}-S_{ij}\right\| c_0\left(n\right)\left(v\kappa\right)^{-q}. \end{aligned}$$

Moreover, by the Chebyschev inequality, for any positive sequence R_{nT} we have:

$$\mathbb{P}\left[\max_i N_i(\epsilon) \geq R_{nT}\right] \leq n\mathbb{P}\left[N_i(\epsilon) \geq R_{nT}\right] \leq \frac{n}{R_{nT}} E[N_i(\epsilon)] \leq \frac{n^2}{R_{nT}} \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij}-S_{ij}\right\| \geq \epsilon\right],$$

which implies $\max_i N_i\left(\epsilon\right) = O_p\left(n^2 \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij}-S_{ij}\right\| \geq \epsilon\right]\right)$. Thus,

$$I_{36} = O_p\left(\psi_{nT} n^2 \Psi_{nT}\left(\left(1-v\right)\kappa\right) + \psi_{nT} c_0\left(n\right)\left(v\kappa\right)^{-q}\right). \tag{b.9}$$

Finally, we consider I_{34}. We have

$$\begin{aligned} I_{34} &\leq \max_i \sum_j \left(\left\|\hat{S}_{ij}-S_{ij}\right\| + \left\|\hat{S}_{ij}\right\|\right) \mathbf{1}_{\left\{\left\|\hat{S}_{ij}\right\|<\kappa, \left\|S_{ij}\right\|\geq\kappa\right\}} \\ &\leq \max_{i,j}\left\|\hat{S}_{ij}-S_{ij}\right\| \max_i \sum_j \mathbf{1}_{\left\{\left\|S_{ij}\right\|\geq\kappa\right\}} + \kappa \max_i \sum_j \mathbf{1}_{\left\{\left\|S_{ij}\right\|\geq\kappa\right\}} \\ &= O_p\left(\psi_{nT} c_0\left(n\right)\kappa^{-q} + c_0\left(n\right)\kappa^{1-q}\right). \end{aligned} \tag{b.10}$$

Combining (b.6)-(b.10) the result follows.

B.1.6 Proof of Lemma 7

By using $\hat{\varepsilon}_{i,t} = \varepsilon_{i,t} - x_t' \left(\hat{\beta}_i - \beta_i\right)$ and $\hat{S}_{ij}^0 = \frac{1}{T_{ij}} \sum_t I_{ij,t} \varepsilon_{i,t} \varepsilon_{j,t} x_t x_t'$, we have:

$$
\begin{aligned}
\hat{S}_{ij} &= \hat{S}_{ij}^0 - \frac{1}{T_{ij}} \sum_t I_{ij,t} \varepsilon_{i,t} x_t' \left(\hat{\beta}_j - \beta_j\right) x_t x_t' - \frac{1}{T_{ij}} \sum_t I_{ij,t} \varepsilon_{j,t} x_t' \left(\hat{\beta}_i - \beta_i\right) x_t x_t' \\
&\quad + \frac{1}{T_{ij}} \sum_t I_{ij,t} \left(\hat{\beta}_i - \beta_i\right)' x_t x_t' \left(\hat{\beta}_j - \beta_j\right) x_t x_t' \\
&=: \hat{S}_{ij}^0 - A_{ij} - B_{ij} + C_{ij},
\end{aligned}
$$

where $A_{ij} = B_{ji}$. Then, for any i, j, we have $\left\|\hat{S}_{ij} - S_{ij}\right\| \leq \left\|\hat{S}_{ij}^0 - S_{ij}\right\| + \|A_{ij}\| + \|B_{ij}\| + \|C_{ij}\|$. We get for any $\xi \geq 0$:

$$
\begin{aligned}
\Psi_{nT}(\xi) &\leq \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij}^0 - S_{ij}\right\| \geq \frac{\xi}{4}\right] + \max_{i,j} \mathbb{P}\left[\|A_{ij}\| \geq \frac{\xi}{4}\right] + \max_{i,j} \mathbb{P}\left[\|B_{ij}\| \geq \frac{\xi}{4}\right] \\
&\quad + \max_{i,j} \mathbb{P}\left[\|C_{ij}\| \geq \frac{\xi}{4}\right] = \Psi_{nT}^0(\xi/4) + 2P_{1,nT}(\xi/4) + P_{2,nT}(\xi/4), \qquad \text{(b.11)}
\end{aligned}
$$

where $\Psi_{nT}^0(\xi/4) := \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij}^0 - S_{ij}\right\| \geq \frac{\xi}{4}\right]$, $P_{1,nT}(\xi/4) := \max_{i,j} \mathbb{P}\left[\|A_{ij}\| \geq \frac{\xi}{4}\right]$, and $P_{2,nT}(\xi/4) := \max_{i,j} \mathbb{P}\left[\|C_{ij}\| \geq \frac{\xi}{4}\right]$. Let us bound the three terms in the RHS of Inequality (b.11).

a) Bound of $\Psi_{nT}^0(\xi/4)$. We use that $\hat{S}_{ij}^0 - S_{ij} = \frac{1}{T_{ij}} \sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' - S_{ij}\right)$ $= \tau_{ij,T} \frac{1}{T} \sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' | \gamma_i \gamma_j\right]\right)$ and $\tau_{ij} \leq M$. Then:

$$
\begin{aligned}
\|\hat{S}_{ij}^0 - S_{ij}\| &\leq M \left\| \frac{1}{T} \sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' | \gamma_i \gamma_j\right]\right) \right\| \\
&\quad + |\tau_{ij,T} - \tau_{ij}| \left\| \frac{1}{T} \sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' | \gamma_i \gamma_j\right]\right) \right\|.
\end{aligned}
$$

We deduce:

$$
\begin{aligned}
& \Psi_{nT}^{0}(\xi/4) \\
\leq \quad & \max_{i,j} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t'|\gamma_i\gamma_j\right]\right)\right\| \geq \frac{\xi}{8M}\right] + \max_{i,j}\mathbb{P}\left[|\tau_{ij,T}-\tau_{ij}| \geq \sqrt{\frac{\xi}{8}}\right] \\
& + \max_{i,j} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t'|\gamma_i\gamma_j\right]\right)\right\| \geq \sqrt{\frac{\xi}{8}}\right] \\
\leq \quad & 2\max_{i,j} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{ij,t}\left(\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t' - E\left[\varepsilon_{i,t}\varepsilon_{j,t}x_t x_t'|\gamma_i\gamma_j\right]\right)\right\| \geq \frac{\xi}{8M}\right] + \max_{i,j}\mathbb{P}\left[|\tau_{ij,T}-\tau_{ij}| \geq \sqrt{\frac{\xi}{8}}\right] \\
=: \quad & 2P_{3,nT} + P_{4,nT},
\end{aligned}
$$

for small ξ. We use $P_{3,nT} \leq \sup_{\gamma,\gamma' \in [0,1]} \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_t(\gamma)I_t(\gamma')\left(\varepsilon_t(\gamma)\varepsilon_t(\gamma')x_t x_t' - E\left[\varepsilon_t(\gamma)\varepsilon_t(\gamma')x_t x_t'\right]\right)\right\| \geq \frac{\xi}{8M}\right]$ and Assumption C.1 e) to get $P_{3,nT} \leq C_1 T \exp\left\{-C_2^* \xi^2 T^{\eta}\right\} + C_3^* \xi^{-1} \exp\left\{-C_4 T^{\bar{\eta}}\right\}$, for some constants $C_1, C_2^*, C_3^*, C_4 > 0$. To bound $P_{4,nT}$, we use $\tau_{ij} \leq M$ and $|\tau_{ij,T} - \tau_{ij}| \leq \tau_{ij}\tau_{ij,T}|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \leq \tau_{ij}\frac{|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}|}{\tau_{ij}^{-1} - |\tau_{ij,T}^{-1} - \tau_{ij}^{-1}|} \leq 2M^2|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}|$, if $|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \leq M^{-1}/2$. Thus, we have $P_{4,nT} \leq 2\max_{i,j}\mathbb{P}\left[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \geq \frac{1}{2M^2}\sqrt{\frac{\xi}{8}}\right]$, for small ξ. By using $\tau_{ij,T}^{-1} = \frac{1}{T}\sum_t I_{ij,t}$ and $\tau_{ij}^{-1} = E[I_{ij,t}|\gamma_i, \gamma_j]$, from Assumption C.1 d) we get:

$$
\begin{aligned}
\max_{i,j}\mathbb{P}\left[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \geq \frac{1}{2M^2}\sqrt{\frac{\xi}{8}}\right] & \leq \sup_{\gamma,\gamma' \in [0,1]} \mathbb{P}\left[\left|\frac{1}{T}\sum_t (I_t(\gamma)I_t(\gamma') - E[I_t(\gamma)I_t(\gamma')])\right| \geq \frac{1}{2M^2}\sqrt{\frac{\xi}{8}}\right] \\
& \leq C_1 T \exp\left\{-C_2^* \xi T^{\eta}\right\} + C_3^* \xi^{-1/2} \exp\left\{-C_4 T^{\bar{\eta}}\right\}.
\end{aligned}
$$

We deduce:

$$
\Psi_{nT}^{0}(\xi/4) \leq C_1^* T \exp\left\{-C_2^* \xi^2 T^{\eta}\right\} + C_3^* \xi^{-1} \exp\left\{-C_4 T^{\bar{\eta}}\right\}. \tag{b.12}
$$

b) Bound of $P_{1,nT}(\xi/4)$. For some constant C, we have

$$
\|A_{ij}\| \leq C\tau_{ij,T} \max_{k,l,m}\left|\frac{1}{T}\sum_t I_{ij,t}\varepsilon_{i,t}x_{t,k}x_{t,l}x_{t,m}\right| \left\|\hat{\beta}_j - \beta_j\right\|.
$$

Let $\chi_{3,T} = (\log T)^a$, for $a > 0$. From a similar argument as in the proof of Lemma 4, and Assumption C.1

d), we have $\max_{i,j} \mathbb{P}\left[\tau_{ij,T} \geq \chi_{3,T}\right] = O(T^{-\bar{b}})$, for any $\bar{b} > 0$. Thus,

$$\begin{aligned}
& P_{1,nT}\left(\xi/4\right) \\
\leq\ & \max_{i,j} \mathbb{P}\left[\tau_{ij,T} \max_{k,l,m} \left| \frac{1}{T} \sum_{t} I_{ij,t} \varepsilon_{i,t} x_{t,k} x_{t,l} x_{t,m} \right| \left\| \hat{\beta}_j - \beta_j \right\| \geq \frac{\xi}{4C} \right] \\
\leq\ & \max_{i,j} \mathbb{P}\left[\tau_{ij,T} \geq \chi_{3,T}\right] + \max_{i,j} \mathbb{P}\left[\max_{k,l,m} \left| \frac{1}{T} \sum_{t} I_{ij,t} \varepsilon_{i,t} x_{t,k} x_{t,l} x_{t,m} \right| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \text{ and } \tau_{ij,T} \leq \chi_{3,T} \right] \\
& + \max_{i,j} \mathbb{P}\left[\left\| \hat{\beta}_j - \beta_j \right\| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \text{ and } \tau_{ij,T} \leq \chi_{3,T} \right] \\
\leq\ & (K+1)^3 \max_{i,j} \max_{k,l,m} \mathbb{P}\left[\left| \frac{1}{T} \sum_{t} I_{ij,t} \varepsilon_{i,t} x_{t,k} x_{t,l} x_{t,m} \right| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \right] \\
& + \mathbb{P}\left[\left\| \hat{\beta}_j - \beta_j \right\| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \text{ and } \tau_{j,T} \leq \chi_{3,T} \right] + O(T^{-\bar{b}}). \qquad \text{(b.13)}
\end{aligned}$$

By Assumption C.1 f),

$$\begin{aligned}
\max_{i,j} \max_{k,l,m} \mathbb{P}\left[\left| \frac{1}{T} \sum_{t} I_{ij,t} \varepsilon_{i,t} x_{t,k} x_{t,l} x_{t,m} \right| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \right] \quad \leq\ & C_1 T \exp\left\{ -\frac{C_2^* \xi}{\chi_{3,T}} T^{\eta} \right\} \\
& + C_3^* \sqrt{\frac{\chi_{3,T}}{\xi}} \exp\left\{ -C_4 T^{\bar{\eta}} \right\}. \qquad \text{(b.14)}
\end{aligned}$$

Let us now focus on $\mathbb{P}\left[\left\| \hat{\beta}_j - \beta_j \right\| \geq \sqrt{\frac{\xi}{4\chi_{3,T} C}} \text{ and } \tau_{j,T} \leq \chi_{3,T} \right]$. By using

$$\left\| \hat{\beta}_j - \beta_j \right\| \leq \chi_{3,T} \left\| Q_x^{-1} \right\| \left\| \frac{1}{T} \sum_{t} I_{j,t} x_t \varepsilon_{j,t} \right\| + \chi_{3,T} \left\| \hat{Q}_{x,j}^{-1} - Q_x^{-1} \right\| \left\| \frac{1}{T} \sum_{t} I_{j,t} x_t \varepsilon_{j,t} \right\|,$$

when $\tau_{j,T} \leq \chi_{3,T}$, we get

$$\begin{aligned}
&\mathbb{P}\left[\left\|\hat{\beta}_j - \beta_j\right\| \geq \sqrt{\frac{\xi}{4\chi_{3,T}C}} \text{ and } \tau_{j,T} \leq \chi_{3,T}\right] \\
\leq\ & \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \frac{1}{2}\sqrt{\frac{\xi}{4\chi_{3,T}C}}\chi_{3,T}^{-1}\left\|Q_x^{-1}\right\|^{-1}\right] \\
&+\mathbb{P}\left[\left\|\hat{Q}_{x,j}^{-1} - Q_x^{-1}\right\|\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \frac{1}{2}\sqrt{\frac{\xi}{4\chi_{3,T}C}}\chi_{3,T}^{-1}\right] \\
\leq\ & \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \sqrt{\frac{\xi}{16\chi_{3,T}^3C}}\left\|Q_x^{-1}\right\|^{-1}\right] \\
&+\mathbb{P}\left[\left\|\hat{Q}_{x,j}^{-1} - Q_x^{-1}\right\| \geq \left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\right] + \mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\right] \\
\leq\ & 2\mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \sqrt{\frac{\xi}{16\chi_{3,T}^3C}}\left\|Q_x^{-1}\right\|^{-1}\right] + \mathbb{P}\left[\left\|\hat{Q}_{x,j}^{-1} - Q_x^{-1}\right\| \geq \left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\right], \qquad \text{(b.15)}
\end{aligned}$$

for small ξ. From Assumption C.1c), the first probability in the RHS of Inequality (b.15) is such that:

$$\mathbb{P}\left[\left\|\frac{1}{T}\sum_t I_{j,t}x_t\varepsilon_{j,t}\right\| \geq \sqrt{\frac{\xi}{16\chi_{3,T}^3C}}\left\|Q_x^{-1}\right\|^{-1}\right] \leq C_1 T\exp\left\{-\frac{C_2^*\xi}{\chi_{3,T}^3}T^\eta\right\} + C_3^*\sqrt{\frac{\chi_{3,T}^3}{\xi}}\exp\left\{-C_4T^{\bar{\eta}}\right\}. \qquad \text{(b.16)}$$

To bound the second probability in the RHS of Inequality (b.15) we use the next Lemma.

Lemma 12 *For any two non-singular matrices A and B such that $\|A - B\| < \frac{1}{2}\|A^{-1}\|^{-1}$ we have:*

$$\|B^{-1} - A^{-1}\| \leq 2\|A^{-1}\|^2\|A - B\|.$$

From Lemma 12, we get:

$$
\begin{aligned}
\mathbb{P}\left[\left\|\hat{Q}_{x,j}^{-1}-Q_x^{-1}\right\|\geq\left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\right] &\leq \mathbb{P}\left[\left\|\hat{Q}_{x,j}-Q_x\right\|\geq\frac{1}{2}\left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\|Q_x^{-1}\|^{-2}\right]\\
&\quad+\mathbb{P}\left[\left\|\hat{Q}_{x,j}-Q_x\right\|\geq\frac{1}{2}\|Q_x^{-1}\|^{-1}\right]\\
&\leq 2\mathbb{P}\left[\left\|\hat{Q}_{x,j}-Q_x\right\|\geq\frac{1}{2}\left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\|Q_x^{-1}\|^{-2}\right],
\end{aligned}
$$

for small $\xi>0$. From Assumption C.1b),

$$
\begin{aligned}
\mathbb{P}\left[\left\|\hat{Q}_{x,j}-Q_x\right\|\geq\frac{1}{2}\left(\frac{\xi}{16\chi_{3,T}^3C}\right)^{1/4}\|Q_x^{-1}\|^{-2}\right] &\leq C_1T\exp\left\{-C_2^*\sqrt{\frac{\xi}{\chi_{3,T}^3}}T^{\eta}\right\}\\
&\quad+2C_3^*\left(\frac{\chi_{3,T}^3}{\xi}\right)^{1/4}\exp\left\{-C_4T^{\bar{\eta}}\right\}. \qquad \text{(b.17)}
\end{aligned}
$$

Then, from (b.13)-(b.17) we get:

$$
P_{1,nT}\left(\xi/4\right)\leq C_1^*T\exp\left\{-C_2^*\xi T^{\eta}/\chi_{3,T}^3\right\}+\frac{C_3^*\chi_{3,T}^{3/2}}{\sqrt{\xi}}\exp\left\{-C_4T^{\bar{\eta}}\right\}+O(T^{-\bar{b}}), \qquad \text{(b.18)}
$$

for small $\xi>0$ and some constants $C_1^*,C_2^*,C_3^*,C_4>0$.

c) Bound of $P_{2,nT}\left(\xi/4\right)$. We have from Assumption C.4

$$
\begin{aligned}
\|C_{ij}\| &\leq \left\|\hat{\beta}_i-\beta_i\right\|\left\|\hat{\beta}_j-\beta_j\right\|\sup_{k,l,m,p}\left|\frac{1}{T_{ij}}\sum_t I_{ij,t}x_{t,k}x_{t,l}x_{t,m}x_{t,p}\right|\\
&\leq C\left\|\hat{\beta}_i-\beta_i\right\|\left\|\hat{\beta}_j-\beta_j\right\|.
\end{aligned}
$$

Thus, we have:

$$
P_{2,nT}\left(\xi/4\right) \leq \max_{i,j}\mathbb{P}\left[C\left\|\hat{\beta}_i-\beta_i\right\|\left\|\hat{\beta}_j-\beta_j\right\|\geq\frac{\xi}{4}\right]\leq 2\mathbb{P}\left[\left\|\hat{\beta}_i-\beta_i\right\|\geq\left(\frac{\xi}{4C}\right)^{1/2}\right].
$$

By the same arguments as above, we get:

$$P_{2,nT}\left(\xi/4\right) \leq C_1^* T \exp\left\{-C_2^* \xi T^{\eta}/\chi_{3,T}^3\right\} + \frac{C_3^* \chi_{3,T}^{3/2}}{\sqrt{\xi}} \exp\left\{-C_4 T^{\bar{\eta}}\right\}, \tag{b.19}$$

for small $\xi > 0$ and some constants $C_1^*, C_2^*, C_3^*, C_4 > 0$.

d) Conclusion. From inequalities (b.11), (b.12), (b.18) and (b.19) we deduce:

$$\Psi_{nT}\left(\xi\right) \leq C_1^* T \exp\left\{-C_2^* \xi_T^2 T^{\eta}\right\} + \frac{C_3^*}{\xi_T} \exp\left\{-C_4 T^{\bar{\eta}}\right\} + O(T^{-\bar{b}}),$$

where $\xi_T := \min\{\xi, \sqrt{\xi/\chi_{3,T}^3}\}$, for small $\xi > 0$ and constants $C_1^*, C_2^*, C_3^*, C_4 > 0$. For $\xi = (1-v)\,\kappa$ and $\kappa = M\sqrt{\frac{\log n}{T^{\eta}}}$, we get $\xi_T = (1-\nu)\,\kappa$ for large T and

$$\begin{aligned} n^2 \Psi_{nT}\left((1-v)\,\kappa\right) \quad &\leq \quad C_1^* n^2 T \exp\left\{-C_2^* M^2 \left(1-v\right)^2 \log n\right\} + \frac{n^2 C_3^*}{\left(1-v\right) M} \sqrt{\frac{T^{\eta}}{\log n}} \exp\left\{-C_4^* T^{\bar{\eta}}\right\} \\ &\quad +O(n^2 T^{-\bar{b}}) = O\left(1\right), \end{aligned}$$

for $\bar{b}$ and M sufficiently large, when $n, T \to \infty$ such that $n = O\left(T^{\bar{\gamma}}\right)$ for $\bar{\gamma} > 0$.

Finally, let us prove that $\psi_{nT} = O_p\left(\sqrt{\frac{\log n}{T^{\eta}}}\right)$. Let $\epsilon > 0$. Then,

$$\begin{aligned} \mathbb{P}\left[\psi_{nT} \geq \sqrt{\frac{\log n}{T^{\eta}}}\epsilon\right] \quad &\leq \quad n^2 \max_{i,j} \mathbb{P}\left[\left\|\hat{S}_{ij} - S_{ij}\right\| \geq \sqrt{\frac{\log n}{T^{\eta}}}\epsilon\right] \\ &= \quad n^2 \Psi_{nT}\left(\sqrt{\frac{\log n}{T^{\eta}}}\epsilon\right) \leq n^2 \Psi_{nT}\left((1-v)\kappa\right) = O\left(1\right), \end{aligned}$$

for large ϵ. The conclusion follows.

B.1.7 Proof of Lemma 8

Under the null hypothesis $\mathcal{H}_0$, and by definition of the fitted residual $\hat{e}_i$, we have

$$\begin{aligned} \hat{e}_i &= a_i - b_i'\hat{\nu} + \hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right) \\ &= a_i - b_i'\nu + \hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right) - b_i' \left(\hat{\nu} - \nu\right) \\ &= \hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right) - b_i' \left(\hat{\nu} - \nu\right). \end{aligned} \tag{b.20}$$

By definition of $\hat{Q}_e$, it follows

$$\begin{aligned} \hat{Q}_e &= \frac{1}{n}\sum_i \hat{w}_i \left[\hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right)\right]^2 - 2\left(\hat{\nu} - \nu\right)' \frac{1}{n}\sum_i \hat{w}_i b_i \left(\hat{\beta}_i - \beta_i\right)' \hat{c}_\nu + \left(\hat{\nu} - \nu\right)' \frac{1}{n}\sum_i \hat{w}_i b_i b_i' \left(\hat{\nu} - \nu\right) \\ &=: \frac{1}{n}\sum_i \hat{w}_i \left[\hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right)\right]^2 - 2I_{71} + I_{72}. \end{aligned}$$

Let us study the second term in the RHS:

$$I_{71} = \frac{1}{\sqrt{nT}} \left(\hat{\nu} - \nu\right)' \frac{1}{\sqrt{n}} \sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}' \hat{Q}_{x,i}^{-1} \hat{c}_\nu =: \frac{1}{\sqrt{nT}} (\hat{\nu} - \nu)' I_{711} \hat{c}_\nu,$$

where $I_{711} = O_p(1)$ by the same arguments used to control term I_1 in the proof of Proposition 3. We have $\hat{\nu} - \nu = O_{p,log}\left(\frac{1}{\sqrt{nT}} + \frac{1}{T}\right)$ and $\hat{c}_\nu = O_p(1)$ by Lemma 3 (v). Thus, $I_{71} = O_{p,log}\left(\frac{1}{nT} + \frac{1}{T\sqrt{nT}}\right)$.

Let us now consider I_{72}. From Lemma 1 (ii)-(iii) and Lemma 3 (v), we have $I_{72} = O_{p,log}\left(\frac{1}{nT} + \frac{1}{T^2}\right)$. The conclusion follows.

B.1.8 Proof of Lemma 9

Under $\mathcal{H}_1$, and using Equation (b.20), we have $\hat{e}_i = e_i + \hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right) - b_i' \left(\hat{\nu} - \nu\right)$. By definition of $\hat{Q}_e$, it follows:

$$\begin{aligned} \hat{Q}_e &= \frac{1}{n}\sum_i \hat{w}_i e_i^2 + 2\frac{1}{n}\sum_i \hat{w}_i \hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right) e_i - 2\left(\hat{\nu} - \nu\right)' \frac{1}{n}\sum_i \hat{w}_i b_i e_i \\ &\quad + \frac{1}{n}\sum_i \hat{w}_i \left[\hat{c}_\nu' \left(\hat{\beta}_i - \beta_i\right)\right]^2 - 2\left(\hat{\nu} - \nu\right)' \frac{1}{n}\sum_i \hat{w}_i b_i \left(\hat{\beta}_i - \beta_i\right)' \hat{c}_\nu + \left(\hat{\nu} - \nu\right)' \frac{1}{n}\sum_i \hat{w}_i b_i b_i' \left(\hat{\nu} - \nu\right) \\ &=: I_{81} + I_{82} + I_{83} + I_{84} + I_{85} + I_{86}. \end{aligned} \tag{b.21}$$

From Equations (a.7) and (a.9) and similar arguments as in Section A.2.3 c), we have $I_{81} = \frac{1}{n}\sum_i w_i e_i^2 + O_{p,log}\left(\frac{1}{\sqrt{T}}\right)$. By similar arguments as for term I_1 in the proof of Proposition 3, we have $I_{82} = \frac{1}{\sqrt{nT}}\left(\frac{1}{\sqrt{n}}\sum_i \hat{w}_i \tau_{i,T} e_i Y_{i,T}' \hat{Q}_{x,i}^{-1}\right)\hat{c}_\nu = O_p\left(\frac{1}{\sqrt{nT}}\right)$. By using $\frac{1}{n}\sum_i \hat{w}_i b_i e_i = \frac{1}{n}\sum_i w_i b_i e_i + O_{p,log}\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\frac{1}{\sqrt{n}}\right) + O_{p,log}\left(\frac{1}{\sqrt{T}}\right)$ and $\hat{\nu} - \nu_\infty = O_{p,log}\left(\frac{1}{\sqrt{n}} + \frac{1}{T}\right)$, we get $I_{83} = O_{p,log}\left(\frac{1}{n} + \frac{1}{\sqrt{nT}} + \frac{1}{\sqrt{T^3}}\right)$. Similar as for I_{82} we have $I_{85} = O_{p,log}\left(\frac{1}{n\sqrt{T}} + \frac{1}{\sqrt{nT^3}}\right)$. From $\hat{\nu} - \nu_\infty = O_{p,log}\left(\frac{1}{\sqrt{n}} + \frac{1}{T}\right)$, we have $I_{86} = O_{p,log}\left(\frac{1}{n} + \frac{1}{T^2}\right)$. The conclusion follows.

B.1.9 Proof of Lemma 10

By applying MN Theorem 2 p.35, Theorem 10 p. 55 and using $W_{n,1} = I_n$, we have

$$\begin{aligned}
Ab = vec\,(Ab) &= \left(b' \otimes A\right) vec\,(I_n) \\
&= vec\left[\left(b' \otimes A\right) vec\,(I_n)\right] \\
&= \left(vec\,(I_n)' \otimes I_m\right) vec\left(b' \otimes A\right) \\
&= \left(vec\,(I_n)' \otimes I_m\right)(I_n \otimes W_{n,1} \otimes I_m)\left(vec\left(b'\right) \otimes vec\,(A)\right) \\
&= \left(vec\,(I_n)' \otimes I_m\right)(I_n \otimes I_m)\, vec\left(vec\,(A)\, b'\right) \\
&= \left(vec\,(I_n)' \otimes I_m\right) vec\left(vec\,(A)\, b'\right).
\end{aligned}$$

B.1.10 Proof of Lemma 11

B.1.10.1 Assumption APR.4 (i)

We use that $eig_{\max}(A) \leq \max_{i=1,\ldots,n} \sum_{j=1} |a_{i,j}|$ for any matrix $A = [a_{ij}]_{i,j=1,\ldots,n}$. Then, for any sequence (γ_i) in $[0,1]$ we have:

$$eig_{\max}(\Sigma_{\varepsilon,1,n}) \leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} |Cov[\varepsilon_t(\gamma_i), \varepsilon_t(\gamma_j)]| \leq C \max_{m=1,\ldots,J_n} \sum_{j=1}^{n} 1\{\gamma_j \in I_m\} \qquad \text{(b.22)}$$

where $C := \sup_{\gamma\in[0,1]} E[\varepsilon_t(\gamma)^2]$. Define:

$$\mathcal{J} = \left\{ (\gamma_i) : \max_{m=1,\ldots,J_n} \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} = o(1) \right\}.$$

Then Assumption APR.4 (i) holds if $\mu_\Gamma(\mathcal{J}) = 1$. From Theorem 2.1.1 in Stout (1974), it is enough to show that $\sum_{n=1}^{\infty} \mu_\Gamma \left(\max_{m=1,\ldots,J_n} \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} > \varepsilon \right) < \infty$, for any $\varepsilon > 0$. Now, since $\max_{m=1,\ldots,J_n} B_m = o(1)$, we have $\mu_\Gamma \left(\max_{m=1,\ldots,J_n} \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} > \varepsilon \right) \leq \mu_\Gamma \left(\max_{m=1,\ldots,J_n} \left| \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} - B_m \right| > \varepsilon/2 \right)$, for large n. Thus, we get:

$$\mu_\Gamma \left(\max_{m=1,\ldots,J_n} \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} > \varepsilon \right) \leq J_n \max_{m=1,\ldots,J_n} \mu_\Gamma \left(\left| \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} - B_m \right| > \varepsilon/2 \right),$$

for large n. To bound the probability in the RHS, we use $|1\{\gamma_i \in I_m\} - B_m| \leq 1$ and the Hoeffding's inequality (see Bosq (1998), Theorem 1.2) to get:

$$\mu_\Gamma \left(\left| \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} - B_m \right| > \varepsilon/2 \right) \leq 2 \exp\left(-n\varepsilon^2/8\right).$$

Then, since $J_n \leq n$, we get:

$$\sum_{n=1}^{\infty} \mu_\Gamma \left(\max_{m=1,\ldots,J_n} \frac{1}{n} \sum_{i=1}^{n} 1\{\gamma_i \in I_m\} > \varepsilon \right) \leq 2 \sum_{n=1}^{\infty} n \exp\left(-n\varepsilon^2/8\right) < \infty,$$

and the conclusion follows.

B.1.10.2 Assumption A.1

Conditions a) and b) are clearly satisfied under BD.1, BD.3 and BD.4. Let us now consider condition c). We have $\sigma_{ij,t} = E[\varepsilon_t(\gamma_i)\varepsilon_t(\gamma_j)|\gamma_i,\gamma_j] =: \sigma_{ij}$ independent of t. Thus, $E[\sigma_{ij,t}^2|\gamma_i,\gamma_j]^{1/2} = \sigma_{ij}$. By BD.1, BD.4 and the Cauchy-Schwarz inequality $\sigma_{ij} = \sum_{m=1}^{J_n} 1\{\gamma_i,\gamma_j \in I_m\} E[\varepsilon_t(\gamma_i)\varepsilon_t(\gamma_j)|\gamma_i,\gamma_j] \leq C \sum_{m=1}^{J_n} 1\{\gamma_i,\gamma_j \in I_m\}$, where $C = \sup_{\gamma \in [0,1]} E[\varepsilon_t(\gamma)^2]$. Hence, we get:

$$
\begin{aligned}
E\left[\frac{1}{n}\sum_{i,j} E[\sigma_{ij,t}^2|\gamma_i,\gamma_j]^{1/2}\right] \quad &\leq \quad C\frac{1}{n}\sum_i \sum_{m=1}^{J_n} E[1\{\gamma_i \in I_m\}] + C\frac{1}{n}\sum_{i \neq j}\sum_{m=1}^{J_n} E[1\{\gamma_i, \gamma_j \in I_m\}] \\
&= \quad C\sum_{m=1}^{J_n} B_m + C(n-1)\sum_{m=1}^{J_n} B_m^2 = O\left(1 + n\sum_{m=1}^{J_n} B_m^2\right).
\end{aligned}
$$

From BD.2, the RHS is $O(1)$, and condition c) in Assumption A.1 follows.

B.1.10.3 Assumption A.2

Let us consider condition a). Under BD.1 and BD.3, we have $S_{ij} = \sigma_{ij}Q_x$ and $S_b = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}}\sigma_{ij}(Q_x \otimes b_i b_j')\right]$. This limit is finite (if it exists), since from BD.4 we have $\left\|\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}}\sigma_{ij}(Q_x \otimes b_i b_j')\right\| \leq C\frac{1}{n}\sum_{i,j}|\sigma_{i,j}|$, and $E\left[\frac{1}{n}\sum_{i,j}|\sigma_{i,j}|\right] = O(1)$ from Assumption A.1. Moreover:

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i\tau_i Y_{i,T} \otimes b_i = \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\sum_{i=1}^{n} w_i\tau_i I_{i,t}\left(x_t \otimes b_i\right)\varepsilon_{i,t} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{n,t},
$$

where $\xi_{n,t} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i\tau_i I_{i,t}\left(x_t \otimes b_i\right)\varepsilon_{i,t}$. The triangular array $(\xi_{n,t})$ is a martingale difference sequence w.r.t. the sigma-field $\mathcal{F}_{n,t} = \{f_{\underline{t}}, \varepsilon_{i,\underline{t}}, \gamma_i, i = 1, ..., n\}$. From a multivariate version of Corollary 5.26 in White (2001), the CLT in condition a) follows if we show:

(i) $\frac{1}{T}\sum_{t=1}^{T} E[\xi_{n,t}\xi_{n,t}^{'}] \to S_b$,

(ii) $\frac{1}{T}\sum_{t=1}^{T}\left(\xi_{n,t}\xi_{n,t}^{'} - E[\xi_{n,t}\xi_{n,t}^{'}]\right) = o_p(1)$,

(iii) $\sup_{t=1,\ldots,T} E[\|\xi_{n,t}\|^{2+\delta}] = O(1)$, for some $\delta > 0$.

Moreover, we prove the alternative characterization of the asymptotic variance-covariance matrix:

(iv) $S_b = \text{a.s.-}\lim_{n\to\infty}\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}}\sigma_{ij}(Q_x \otimes b_i b_j')$.

Let us check these conditions. (i) Let $\mathcal{G}_n = \{\gamma_i, i = 1, ..., n\}$. We have:

$$\begin{aligned}
\frac{1}{T}\sum_t E[\xi_{n,t}\xi_{n,t}^{'}|\mathcal{G}_n] &= \frac{1}{Tn}\sum_t\sum_{i,j} w_i w_j \tau_i \tau_j E\left[I_{i,t}I_{j,t}\left(x_t x_t^{'} \otimes b_i b_j^{'}\right)\varepsilon_{i,t}\varepsilon_{j,t}|\gamma_i,\gamma_j\right] \\
&= \frac{1}{Tn}\sum_t\sum_{i,j} w_i w_j \tau_i \tau_j E[I_{i,t}I_{j,t}|\gamma_i,\gamma_j]\left(E[x_t x_t^{'}]\otimes b_i b_j^{'}\right)E[\varepsilon_{i,t}\varepsilon_{j,t}|\gamma_i,\gamma_j] \\
&= \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{i,j}}\sigma_{ij}\left(Q_x \otimes b_i b_j^{'}\right).
\end{aligned}$$

By taking expectation on both sides, condition (i) follows.

Let us now consider condition (ii). Define $\zeta_{n,T} = \frac{1}{T}\sum_t (\xi_{n,t,k}\xi_{n,t,l} - E[\xi_{n,t,k}\xi_{n,t,l}])$, where $\xi_{n,t,k}$ is the k-th element of $\xi_{n,t}$. Since $E[\zeta_{n,T}] = 0$, it is enough to show $V[\zeta_{n,T}] = o(1)$, for any k, l. We show this for $k = l$, the proof for $k \neq l$ is similar. For expository purpose we omit the index k, and we write $x_{t,k}^2 \equiv x_t^2$. We have:

$$V[\zeta_{n,T}] = \frac{1}{T^2}\sum_t V[\xi_{n,t}^2] + \frac{1}{T^2}\sum_{t\neq s} Cov\left(\xi_{n,t}^2, \xi_{n,s}^2\right), \tag{b.23}$$

where:

$$\xi_{n,t}^2 = \frac{1}{n}\sum_{i,j} w_i w_j \tau_i \tau_j I_{i,t} I_{j,t} x_t^2 b_i b_j \varepsilon_{i,t}\varepsilon_{j,t}.$$

- Consider first the terms $Cov(\xi_{n,t}^2, \xi_{n,s}^2)$ for $t \neq s$. By the variance decomposition formula:

$$Cov(\xi_{n,t}^2, \xi_{n,s}^2) = E\left[Cov(\xi_{n,t}^2, \xi_{n,s}^2|\mathcal{G}_n)\right] + Cov\left[E(\xi_{n,t}^2|\mathcal{G}_n), E(\xi_{n,s}^2|\mathcal{G}_n)\right].$$

We have $Cov(\xi_{n,t}^2, \xi_{n,s}^2|\mathcal{G}_n) = 0$ from the i.i.d. assumption over time. Moreover:

$$E[\xi_{n,t}^2|\mathcal{G}_n] = \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}} Q_x \sigma_{ij} b_i b_j = \frac{1}{n}\sum_{m=1}^{J_n}\sum_{i,j}\alpha_{ij}\sigma_{ij}1\{\gamma_i,\gamma_j \in I_m\},$$

where $\alpha_{ij} = w_i w_j \frac{\tau_i\tau_j}{\tau_{ij}} b_i b_j Q_x$ and $Q_x = E[x_t^2]$. Thus:

$$Cov\left[E(\xi_{n,t}^2|\mathcal{G}_n), E(\xi_{n,s}^2|\mathcal{G}_n)\right] = \frac{1}{n^2}\sum_{m,p=1}^{J_n}\sum_{i,j,k,l} Cov\left(\alpha_{ij}\sigma_{ij}1\{\gamma_i,\gamma_j \in I_m\}, \alpha_{kl}\sigma_{kl}1\{\gamma_k,\gamma_l \in I_p\}\right).$$

In the above sum, the terms such that sets $\{i,j\}$ and $\{k,l\}$ do not have a common element, vanish. Consider now the sum of the terms such that $i = k$ (terms such that $i = l$, or $j = k$, or $j = l$ are symmetric). Therefore, let us focus on the sum

$$\begin{aligned} S_n &:= \frac{1}{n^2}\sum_{m,p=1}^{J_n}\sum_{i,j,l} Cov\left(\alpha_{ij}\sigma_{ij}1\{\gamma_i,\gamma_j \in I_m\}, \alpha_{il}\sigma_{il}1\{\gamma_i,\gamma_l \in I_p\}\right) \\ &= \frac{1}{n^2}\sum_{m=1}^{J_n}\sum_{i,j,l} Cov\left(\alpha_{ij}\sigma_{ij}1\{\gamma_i,\gamma_j \in I_m\}, \alpha_{il}\sigma_{il}1\{\gamma_i,\gamma_l \in I_m\}\right) \\ &\quad -\frac{1}{n^2}\sum_{m,p=1,m\neq p}^{J_n}\sum_{i,j,l} E\left[\alpha_{ij}\sigma_{ij}1\{\gamma_i,\gamma_j \in I_m\}\right] E\left[\alpha_{il}\sigma_{il}1\{\gamma_i,\gamma_l \in I_p\}\right]. \end{aligned}$$

From BD.4, we have $\alpha_{ij} \leq C$ and $\sigma_{ij} \leq C$. Thus, we get $S_n = O\left(\frac{1}{n^2}\sum_{m=1}^{J_n}\sum_{i,j,l} E[1\{\gamma_i,\gamma_j,\gamma_l \in I_m\}]\right) + O\left(\frac{1}{n^2}\sum_{m,p=1,m\neq p}^{J_n}\sum_{i,j,l} E\left[1\{\gamma_i,\gamma_j \in I_m\}\right] E\left[1\{\gamma_i,\gamma_l \in I_p\}\right]\right)$. By using that $\sum_{i,j,l} E[1\{\gamma_i,\gamma_j,\gamma_l \in I_m\}] = O\left(nB_m + n^2B_m^2 + n^3B_m^3\right)$ and $\sum_{i,j,l} E\left[1\{\gamma_i,\gamma_j \in I_m\}\right] E\left[1\{\gamma_i,\gamma_l \in I_p\}\right] = O\left(nB_mB_p + n^2(B_m^2B_p + B_mB_p^2) + n^3B_m^2B_p^2\right)\right)$, we get $S_n = O\left(1/n + \sum_{m=1}^{J_n} B_m^2 + n\sum_{m=1}^{J_n} B_m^3 + n\left(\sum_{m=1}^{J_n} B_m^2\right)^2\right)$. The RHS is $o(1)$ from BD.2. Thus, we have shown that:

$$Cov(\xi_{n,t}^2, \xi_{n,s}^2) = o(1), \tag{b.24}$$

uniformly in $t \neq s$.

- Consider now $V[\xi_{n,t}^2]$. By the variance decomposition formula:

$$V[\xi_{n,t}^2] = E\left[V(\xi_{n,t}^2|\mathcal{G}_n)\right] + V\left[E(\xi_{n,t}^2|\mathcal{G}_n)\right].$$

By similar arguments as above, we have $V\left[E(\xi_{n,t}^2|\mathcal{G}_n)\right] = o(1)$ uniformly in t. Consider now term

$E\left[V(\xi_{n,t}^2|\mathcal{G}_n)\right]$. We have:

$$
\begin{aligned}
V(\xi_{n,t}^2|\mathcal{G}_n) &= \frac{1}{n^2}\sum_{i,j,k,l} w_i w_j w_k w_l \tau_i\tau_j\tau_k\tau_l b_i b_j b_k b_l \\
&\qquad\qquad \cdot Cov\left(I_{i,t}I_{j,t}x_t^2\varepsilon_{i,t}\varepsilon_{j,t}, I_{k,t}I_{l,t}x_t^2\varepsilon_{k,t}\varepsilon_{l,t}|\gamma_i,\gamma_j,\gamma_k,\gamma_l\right).
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
&Cov\left(I_{i,t}I_{j,t}x_t^2\varepsilon_{i,t}\varepsilon_{j,t}, I_{k,t}I_{l,t}x_t^2\varepsilon_{k,t}\varepsilon_{l,t}|\gamma_i,\gamma_j,\gamma_k,\gamma_l\right) \\
&= E\left[I_{i,t}I_{j,t}I_{k,t}I_{l,t}|\gamma_i,\gamma_j,\gamma_k,\gamma_l\right]E\left[\varepsilon_{i,t}\varepsilon_{j,t}\varepsilon_{k,t}\varepsilon_{l,t}|\gamma_i,\gamma_j,\gamma_k,\gamma_l\right]E[x_t^4] - \sigma_{ij}\sigma_{kl}\tau_{ij}^{-1}\tau_{kl}^{-1}E[x_t^2]^2.
\end{aligned}
$$

From the block dependence structure in BD.1, the expectation $E\left[\varepsilon_{i,t}\varepsilon_{j,t}\varepsilon_{k,t}\varepsilon_{l,t}|\gamma_i,\gamma_j,\gamma_k,\gamma_l\right]$ is different from zero only if a pair of indices are in a same block I_m, and the other pair is also in a same block I_p, say, possibly with $m = p$. Similarly, $\sigma_{ij}\sigma_{kl}$ is different from zero only if γ_i and γ_j are in the same block and γ_k and γ_l are in the same block. From BD.4, we deduce that $V(\xi_{n,t}^2|\mathcal{G}_n) \leq C\frac{1}{n^2}\sum_{i,j,k,l}\sum_{m,p=1}^{J_n} 1\{\gamma_i,\gamma_j\in I_m\}1\{\gamma_k,\gamma_l\in I_p\}$, where in the double sum the elements with $m\neq p$ are not zero only if the pairs (γ_i,γ_j) and (γ_k,γ_l) have no element in common. Thus:

$$
\begin{aligned}
E\left[V(\xi_{n,t}^2|\mathcal{G}_n)\right] &= O\left(\frac{1}{n^2}\sum_{i,j,k,l}\sum_{m=1}^{J_n} E[1\{\gamma_i,\gamma_j,\gamma_k,\gamma_l\in I_m\}]\right) \\
&\quad + O\left(\frac{1}{n^2}\sum_{i,j,k,l:i\neq k,l;j\neq k,l}\;\sum_{m,p=1:m\neq p}^{J_n} E[1\{\gamma_i,\gamma_j\in I_m\}]E[1\{\gamma_k,\gamma_l\in I_p\}]\right).
\end{aligned}
$$

By using $\sum_{i,j,k,l}\sum_{m=1}^{J_n} E[1\{\gamma_i,\gamma_j,\gamma_k,\gamma_l\in I_m\}] = O\left(\sum_{m=1}^{J_n}(nB_m + n^2B_m^2 + n^3B_m^3 + n^4B_m^4)\right)$ and $\sum_{i,j,k,l}\sum_{m,p=1}^{J_n} E[1\{\gamma_i,\gamma_j\in I_m\}]E[1\{\gamma_k,\gamma_l\in I_p\}] = O\left(\sum_{m,p=1}^{J_n}(n^2B_mB_p + n^3B_m^2B_p + n^4B_m^2B_p^2)\right)$, we get:

$$
E\left[V(\xi_{n,t}^2|\mathcal{G}_n)\right] = O\left(1 + n\sum_{m=1}^{J_n}B_m^2 + (n\sum_{m=1}^{J_n}B_m^2)^2 + n^2\sum_{m=1}^{J_n}B_m^4\right).
$$

By BD.2, $n\max_{m=1,\ldots,n}B_m^2 = O(1)$, and we get $E\left[V(\xi_{n,t}^2|\mathcal{G}_n)\right] = O(1)$.

Thus, we have shown:

$$V(\xi_{n,t}^2) = O(1), \tag{b.25}$$

uniformly in t.

From (b.23), (b.24) and (b.25), we get $V[\zeta_{nT}] = o(1)$, and condition (ii) follows. From (b.25) and by using $E[\xi_{n,t}^2] = O(1)$, condition (iii) follows for $\delta = 2$. Finally, condition (iv) follows from $\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} \sigma_{ij} b_i b_j' = (1+\lambda' V[f_t]\lambda)^{-2} \frac{1}{n}\sum_{i,j} \frac{1}{\tau_{ij}} \frac{\sigma_{i,j}}{\sigma_{ii}\sigma_{jj}} b_i b_j'$ and the next Lemma 13.

Lemma 13 *Under Assumptions BD.1-BD.4:* $\frac{1}{n}\sum_{i,j} \frac{1}{\tau_{ij}} \frac{\sigma_{i,j}}{\sigma_{ii}\sigma_{jj}} b_i b_j' \to L$, *P-a.s., where:*

$$L = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} \frac{1}{\tau_{ij}} \frac{\sigma_{i,j}}{\sigma_{ii}\sigma_{jj}} b_i b_j'\right] = \int_0^1 \omega(\gamma) d\gamma + \lim_{n\to\infty} n \sum_{m=1}^{J_n} \int_{I_m}\int_{I_m} \omega(\gamma,\gamma') d\gamma d\gamma',$$

with $\omega(\gamma,\gamma') := E[I_t(\gamma) I_t(\gamma')] \frac{E[\varepsilon_t(\gamma)\varepsilon_t(\gamma')]}{E[\varepsilon_t(\gamma)^2] E[\varepsilon_t(\gamma')^2]} b(\gamma) b(\gamma')'$ *and* $\omega(\gamma) := \omega(\gamma,\gamma)$.

Then, we have proved part a). Part b) follows by a standard CLT.

B.1.10.4 Assumption A.3

Assumption A.3 is satisfied since the errors are i.i.d. and have zero third moment (Assumption BD.1).

B.1.10.5 Assumption A.4

We have to show that $\max_i \sum_j \|S_{ij}\|^q = O_p(n^\delta)$, for any $q \in (0,1)$ and $\delta > 1/2$. From $S_{ij} = \sigma_{ij} Q_x$, and an argument similar to (b.22):

$$\max_i \sum_j \|S_{ij}\|^q \le C \max_{m=1,\dots,J_n} \sum_{j=1}^n 1\{\gamma_j \in I_m\} \le Cn \max_{m=1,\dots,J_n} B_m + C \max_{m=1,\dots,J_n} \left|\sum_{j=1}^n [1\{\gamma_j \in I_m\} - B_m]\right|,$$

for any $q > 0$. Let us derive (probability) bounds for the two terms in the RHS. From BD.2:

$$n \max_m |B_m| \le \sqrt{n} \left(n \sum_m |B_m|^2\right)^{1/2} = O\left(\sqrt{n}\right).$$

Let $\varepsilon_n := n^{\delta}$, with $\delta > 1/2$. Then:

$$\begin{aligned} P\left[\max_{m=1,\dots,J_n}\left|\sum_{j=1}^{n}[1\{\gamma_j \in I_m\} - B_m]\right| \geq \varepsilon_n\right] &\leq J_n \max_{m=1,\dots,J_n} P\left[\left|\sum_{j=1}^{n}[1\{\gamma_j \in I_m\} - B_m]\right| \geq \varepsilon_n\right] \\ &\leq 2J_n \exp(-\varepsilon_n^2/(2n)) = o(1), \end{aligned}$$

from the Hoeffding's inequality (see Bosq (1998), Theorem 1.2), and $J_n \leq n$. Thus, we have shown that $\max_{m=1,\dots,J_n}\left|\sum_{j=1}^{n}[1\{\gamma_j \in I_m\} - B_m]\right| = o_p(n^{\delta})$, and the conclusion follows.

B.1.10.6 Assumption A.5

We have $S_{ii,T} = \sigma_{ii}\hat{Q}_{x,i}$ and $S_{ij} = \sigma_{ij}Q_x$. Let us denote by $\mathcal{H} = \sigma\left((f_t), (I_t(\gamma)), \gamma \in [0,1], \gamma_i, i = 1, 2, ...\right)$ the information in the factor path, the indicators paths and the individual random effects. The proof is in two steps.

STEP 1: We first show that conditional on $\mathcal{H}$ we have

$$\Upsilon_{nT} := \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2\left[Y_{i,T} \otimes Y_{i,T} - \widetilde{S}_{ii,T}\right] \Rightarrow N(0,\Omega), \quad n, T \to \infty, \tag{b.26}$$

P-a.s., where $\widetilde{S}_{ii,T} = \sigma_{ii}vec(\hat{Q}_{x,i})$ and $\Omega = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\sigma_{ij}^2\right]\left[Q_x \otimes Q_x + (Q_x \otimes Q_x)W_{K+1}\right]$.

For this purpose, we apply the Lyapunov CLT for heterogenous independent arrays (see Davidson (1994), Theorem 23.11). Write

$$\Upsilon_{nT} = \frac{1}{\sqrt{n}}\sum_i \sum_{m=1}^{J_n} 1\{\gamma_i \in I_m\} w_i\tau_i^2\left[Y_{i,T} \otimes Y_{i,T} - \widetilde{S}_{ii,T}\right] = \frac{1}{\sqrt{J_n}}\sum_{m=1}^{J_n} W_{m,nT},$$

where

$$W_{m,nT} := \sqrt{\frac{J_n}{n}}\sum_i 1\{\gamma_i \in I_m\} w_i\tau_i^2\left[Y_{i,T} \otimes Y_{i,T} - \widetilde{S}_{ii,T}\right].$$

Conditional on $\mathcal{H}$, the variables $W_{m,nT}$, for $m = 1, ..., J_n$ are independent, with zero mean. The conclusion follows if we prove:

(i) $\lim_{n,T} \frac{1}{J_n}\sum_m V\left[W_{m,nT}|\mathcal{H}\right] = \Omega$, P-a.s, and

(ii) $\lim_{n,T} \frac{1}{J_n^{3/2}} \sum_m E\left[\|W_{m,nT}\|^3 |\mathcal{H}\right] = 0$, P-a.s..

To show (i), we use:

$$
\begin{aligned}
V\left[W_{m,nT}|\mathcal{H}\right] &= \frac{J_n}{n} \sum_{i,j \in I_m} w_i w_j \tau_i^2 \tau_j^2 Cov\left[Y_{i,T} \otimes Y_{i,T}, Y_{j,T} \otimes Y_{j,T}|\mathcal{H}\right] \\
&= \frac{J_n}{n} \sum_{i,j \in I_m} w_i w_j \tau_i^2 \tau_j^2 \left\{ E\left[\left(Y_{i,T} \otimes Y_{i,T}\right)\left(Y_{j,T} \otimes Y_{j,T}\right)' |\mathcal{H}\right] - \widetilde{S}_{ii,T} \widetilde{S}_{jj,T}' \right\},
\end{aligned}
$$

where $\sum_{i,j \in I_m}$ denotes double sum over all $i, j = 1, ..., n$ such that $\gamma_i, \gamma_j \in I_m$. Now, we have by the independence property over time:

$$
\begin{aligned}
& E\left[\left(Y_{i,T} \otimes Y_{i,T}\right)\left(Y_{j,T} \otimes Y_{j,T}\right)' |\mathcal{H}\right] \\
=\; & \frac{1}{T^2} \sum_t \sum_s \sum_p \sum_q E\left[\varepsilon_{i,t}\varepsilon_{i,p}\varepsilon_{j,s}\varepsilon_{j,q} | \left(f_t\right), \gamma_i, \gamma_j\right] I_{i,t} I_{i,p} I_{j,s} I_{j,q} \left(x_t x_s' \otimes x_p x_q'\right) \\
=\; & E\left[\varepsilon_{it}^2 \varepsilon_{jt}^2 | \gamma_i, \gamma_j\right] \frac{1}{T^2} \sum_t I_{i,t} I_{j,t} \left(x_t x_t' \otimes x_t x_t'\right) + \sigma_{ij}^2 \frac{1}{T^2} \sum_t \sum_{p \neq t} I_{ij,t} I_{ij,p} \left(x_t x_t' \otimes x_p x_p'\right) \\
& + \sigma_{ii}^2 \sigma_{jj}^2 \frac{1}{T^2} \sum_t \sum_{s \neq t} I_{i,t} I_{j,s} \left(x_t x_s' \otimes x_t x_s'\right) + \sigma_{ij}^2 \frac{1}{T^2} \sum_t \sum_{s \neq t} I_{ij,t} I_{ij,s} \left(x_t x_s' \otimes x_s x_t'\right) \\
=:\; & E\left[\varepsilon_{it}^2 \varepsilon_{jt}^2 | \gamma_i, \gamma_j\right] A_{1,T} + \sigma_{ij}^2 A_{2,T} + \sigma_{ii}^2 \sigma_{jj}^2 A_{3,T} + \sigma_{ij}^2 A_{4,T}.
\end{aligned}
$$

Moreover, $A_{1,T} = \frac{T_{ij}}{T^2} \sum_t \frac{I_{ij,t}}{T_{ij}} \left(x_t x_t' \otimes x_t x_t'\right) = O\left(T_{ij}/T^2\right) = O(1/T)$, uniformly in $\mathcal{H}$. Let us define $\hat{Q}_{x,ij} = \frac{1}{T_{ij}} \sum_t I_{ij,t} x_t x_t'$, then

$$
\begin{aligned}
A_{2,T} &= \frac{1}{T^2} \sum_t \sum_p I_{ij,t} I_{ij,p} \left(x_t x_t' \otimes x_p x_p'\right) - A_{1,T} = \frac{1}{\tau_{ij,T}^2} \left(\hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij}\right) + O\left(1/T\right), \\
A_{3,T} &= \frac{1}{T^2} \sum_t \sum_s I_{i,t} I_{j,s} \left(x_t x_s' \otimes x_t x_s'\right) - A_{1,T} = vec\left(\hat{Q}_{x,i}\right) vec\left(\hat{Q}_{x,j}\right)' + O\left(1/T\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{4,T} &= \frac{1}{T^2}\sum_t \sum_s I_{ij,t} I_{ij,s}\left(x_t x_s^{'} \otimes x_s x_t^{'}\right) - A_{1,T} \\
&= \frac{1}{T^2}\sum_t \sum_s I_{ij,t} I_{ij,s}\left(x_t \otimes x_s\right)\left(x_s \otimes x_t\right)^{'} - A_{1,T} \\
&= \frac{1}{T^2}\sum_t \sum_s I_{ij,t} I_{ij,s}\left(x_t \otimes x_s\right)\left(x_t \otimes x_s\right)^{'} W_{K+1} - A_{1,T} \\
&= \frac{1}{\tau_{ij,T}^2}\left(\hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij}\right) W_{K+1} + O\left(1/T\right).
\end{aligned}
$$

Then, it follows that:

$$
\begin{aligned}
V\left[W_{m,nT}|\mathcal{H}\right] &= \frac{J_n}{n}\left[\sum_{i,j\in I_m} w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij,T}^2}\sigma_{ij}^2\left(\hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij} + \hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij} W_{K+1}\right)\right] \\
&\quad + O\left(\frac{J_n}{n}\frac{1}{T}\sum_{i,j\in I_m} w_i w_j \tau_i^2\tau_j^2\right),
\end{aligned}
$$

where the O term is uniform w.r.t. $\mathcal{H}$. Thus, we get:

$$
\begin{aligned}
\frac{1}{J_n}\sum_m V\left[W_{m,nT}|\mathcal{H}\right] &= \left(\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\sigma_{ij}^2\right)\left(Q_x \otimes Q_x + Q_x \otimes Q_x W_{K+1}\right) \\
&\quad + \frac{1}{n}\sum_m \sum_{i,j\in I_m} w_i w_j \tau_i^2\tau_j^2\sigma_{ij}^2\alpha_{ij} + O\left(\frac{1}{T}\frac{1}{n}\sum_m \sum_{i,j\in I_m} w_i w_j \tau_i^2\tau_j^2\right),
\end{aligned}
$$

where the $\alpha_{ij} = \frac{1}{\tau_{ij,T}^2}\left(\hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij} + \hat{Q}_{x,ij} \otimes \hat{Q}_{x,ij} W_{K+1}\right) - \frac{1}{\tau_{ij}^2}\left(Q_x \otimes Q_x + Q_x \otimes Q_x W_{K+1}\right)$ are $o(1)$ uniformly in i,j, and $w_i w_j \frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\sigma_{ij}^2 = (1+\lambda'\Sigma_f^{-1}\lambda)^{-2}\frac{\tau_i\tau_j}{\tau_{ij}^2}\frac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}}$. Then, point i) follows from $\frac{1}{n}\sum_{i,j}\frac{\tau_i\tau_j}{\tau_{ij}^2}\frac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}} \to L$, P-a.s., where $L = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j}\frac{\tau_i\tau_j}{\tau_{ij}^2}\frac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}}\right]$, which is proved by similar arguments as Lemma 13.

Let us now prove point ii). We have:

$$\begin{aligned}\frac{1}{J_n^{3/2}}\sum_m E\left[\|W_{m,nT}\|^3|\mathcal{H}\right] &\leq \frac{1}{n^{3/2}}\sum_m\left[\sum_{i\in I_m} w_i\tau_i^2\left(E\left[\|(Y_{i,T}\otimes Y_{i,T})\|^3|\mathcal{H}\right]^{1/3}+\left\|\widetilde{S}_{ii,T}\right\|\right)\right]^3 \\ &\leq \frac{1}{n^{3/2}}\left(\sum_m\left(\sum_{i\in I_m} w_i\tau_i^2\right)^3\right)\left(\sup_i E\left[\|Y_{i,T}\otimes Y_{i,T}\|^3|\mathcal{H}\right]^{1/3}+\sup_i\left\|\widetilde{S}_{ii,T}\right\|\right)^3.\end{aligned}$$

Now,

$$\begin{aligned}E\left[\|Y_{i,T}\otimes Y_{i,T}\|^3|\mathcal{H}\right] &\leq E\left[\|Y_{i,T}\|^6|\mathcal{H}\right]=E\left[\left(Y_{i,T}^{'}Y_{i,T}\right)^3|\mathcal{H}\right] \\ &= \frac{1}{T^3}\sum_{t_1,\ldots,t_6} I_{i,t_1}\ldots I_{i,t_6}E\left[\varepsilon_{i,t_1}\ldots\varepsilon_{i,t_6}|\gamma_i\right]\left(x_{t_1}'x_{t_2}\right)\left(x_{t_3}'x_{t_4}\right)\left(x_{t_5}'x_{t_6}\right).\end{aligned}$$

By the independence property, the non-zero terms $E\left[\varepsilon_{i,t_1}\ldots\varepsilon_{i,t_6}|\gamma_i\right]$ involve at most 3 different time indices, which implies together with BD.4 that $\sup_i E\left[\|Y_{i,T}\otimes Y_{i,T}\|^3|\mathcal{H}\right]=O(1)$, P-a.s. Similarly $\sup_i\left\|\widetilde{S}_{ii,T}\right\|=O(1)$, P-a.s. Thus, we get:

$$\frac{1}{J_n^{3/2}}\sum_{m=1}^{J_n} E\left[\|W_{m,nT}\|^3|\mathcal{H}\right]\leq C\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i\in I_m\}\right)^3.$$

Then, point ii) follows from the next Lemma 14.

Lemma 14 *Under Assumptions BD.1-BD.4:* $\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i\in I_m\}\right)^3\to 0$*, P-a.s.*

STEP 2: We show that (b.26) implies the asymptotic normality condition in Assumption A.4. Indeed, from (b.26) we have:

$$\lim_{n,T\to\infty} P\left[\alpha'\Upsilon_{nT}\leq z|\mathcal{H}\right]=\Phi\left(\frac{z}{\sqrt{\alpha'\Omega\alpha}}\right),$$

for any $\alpha\in\mathbb{R}^{2(K+1)}$ and for any $z\in\mathbb{R}$, and P-a.s. We now apply the Lebesgue dominated convergence theorem, by using that the sequence of random variables $P\left[\alpha'\Upsilon_{nT}\leq z|\mathcal{H}\right]$ are such that $P\left[\alpha'\Upsilon_{nT}\leq z|\mathcal{H}\right]\leq 1$, uniformly in n and T. We conclude that, for any $\alpha\in\mathbb{R}^{2(K+1)}$, $z\in\mathbb{R}$:

$$\lim_{n,T\to\infty} P\left[\alpha'\Upsilon_{nT}\leq z\right]=\lim_{n,T\to\infty} E\left(P\left[\alpha'\Upsilon_{nT}\leq z|\mathcal{H}\right]\right)=\Phi\left(\frac{z}{\sqrt{\alpha'\Omega\alpha}}\right),$$

since $\Phi\left(\frac{z}{\sqrt{\alpha'\Omega\alpha}}\right)$ is independent of the information set $\mathcal{H}$. The conclusion follows.

B.1.11 Proof Lemma 12

Write:

$$B^{-1}-A^{-1}=\left[A\left(I-A^{-1}\left(A-B\right)\right)\right]^{-1}-A^{-1}=\left\{\left[I-A^{-1}\left(A-B\right)\right]^{-1}-I\right\}A^{-1},$$

and use that, for a square matrix C such that $\|C\|<1$, we have

$$(I-C)^{-1}=I+C+C^2+C^3+...$$

and

$$\left\|(I-C)^{-1}-I\right\| \quad \leq \|C\|+\|C\|^2+... \quad \leq \frac{\|C\|}{1-\|C\|}.$$

Thus, we get:

$$\begin{aligned}\left\|B^{-1}-A^{-1}\right\| \quad &\leq \quad \frac{\left\|A^{-1}\left(A-B\right)\right\|}{1-\left\|A^{-1}\left(A-B\right)\right\|}\left\|A^{-1}\right\| \\ &\leq \quad \frac{\left\|A^{-1}\right\|^2\|A-B\|}{1-\|A^{-1}\|\,\|A-B\|} \\ &\leq \quad 2\left\|A^{-1}\right\|^2\|A-B\|,\end{aligned}$$

if $\|A-B\|<\dfrac{1}{2}\|A^{-1}\|^{-1}$.

B.1.12 Proof of Lemma 13

Let us denote $\xi_{i,j}=\dfrac{1}{\tau_{ij}}\dfrac{\sigma_{ij}}{\sigma_{ii}\sigma_{jj}}b_i b_j'=w\left(\gamma_i,\gamma_j\right)$. We have $\dfrac{1}{n}\sum\limits_{i,j}\xi_{i,j}=\dfrac{1}{n}\sum\limits_{i}\xi_{ii}+\dfrac{1}{n}\sum\limits_{i\neq j}\xi_{i,j}$. By the LLN we get $\dfrac{1}{n}\sum\limits_{i}\xi_{ii}=\dfrac{1}{n}\sum\limits_{i}\omega(\gamma_i)\to\displaystyle\int_0^1\omega(\gamma)d\gamma$, P-a.s.. Let us now consider the double sum $\dfrac{1}{n}\sum\limits_{i\neq j}\xi_{i,j}$. The proof proceeds in three steps.

STEP 1: We first prove that $\dfrac{1}{n}\sum\limits_{i\neq j}\xi_{i,j}=L'+o_p(1)$, where $L':=\lim\limits_{n\to\infty}n\sum\limits_{m=1}^{J_n}\displaystyle\int_{I_m}\int_{I_m}\omega(\gamma,\gamma')d\gamma d\gamma'$.

For this purpose, write $\frac{1}{n}\sum_{i\neq j}\xi_{i,j}=\sum_{m=1}^{J_n}X_m$, where $X_m:=\frac{1}{n}\sum_{i\neq j}\omega(\gamma_i,\gamma_j)1\{\gamma_i,\gamma_j\in I_m\}$, by using block-dependence. Then, we have:

$$E[X_m]=\frac{1}{n}\sum_{i\neq j}E[\omega(\gamma_i,\gamma_j)1\{\gamma_i,\gamma_j\in I_m\}]=(n-1)\int_{I_m}\int_{I_m}\omega(\gamma,\gamma')d\gamma d\gamma'=:(n-1)\bar{\omega}_m,$$

which implies:

$$E\left[\frac{1}{n}\sum_{i\neq j}\xi_{i,j}\right]=(n-1)\sum_{m=1}^{J_n}\bar{\omega}_m\to L'.$$

Moreover:

$$\begin{aligned}V\left[X_m\right] &= \frac{1}{n^2}\sum_{i\neq j}\sum_{k\neq l}E\left[\omega(\gamma_i,\gamma_j)\omega(\gamma_k,\gamma_l)1\{\gamma_i,\gamma_j,\gamma_k,\gamma_l\in I_m\}\right]-E[X_m]^2\\ &= \frac{1}{n^2}\left[n(n-1)(n-2)(n-3)\bar{\omega}_m^2+O(n^3B_m^3)+O(n^2B_m^2)\right]-(n-1)^2\bar{\omega}_m^2\\ &= O(nB_m^4)+O(nB_m^3)+O(B_m^2),\end{aligned}$$

and:

$$\begin{aligned}Cov(X_m,X_p) &= \frac{1}{n^2}\sum_{i\neq j}\sum_{k\neq l}E\left[\omega(\gamma_i,\gamma_j)\omega(\gamma_k,\gamma_l)1\{\gamma_i,\gamma_j\in I_m\}1\{\gamma_k,\gamma_l\in I_p\}\right]-E[X_m]E[X_p]\\ &= \frac{1}{n^2}\left[n(n-1)(n-2)(n-3)\bar{\omega}_m\bar{\omega}_p\right]-(n-1)^2\bar{\omega}_m\bar{\omega}_p=O(nB_m^2B_p^2),\end{aligned}$$

for $m\neq p$, which implies:

$$V\left[\frac{1}{n}\sum_{i\neq j}\xi_{i,j}\right] = \sum_{m=1}^{J_n}V[X_m]+\sum_{m,p=1,m\neq p}^{J_n}Cov(X_m,X_p)=o(1),$$

from BD.2. Then, Step 1 follows.

STEP 2: There exists a random variable $\tilde{L}$ such that $\frac{1}{n}\sum_{i\neq j}\xi_{i,j} \to \tilde{L}$, P-a.s.. To show this statement, we use that the event in which series $\frac{1}{n}\sum_{i\neq j}\xi_{i,j}$ converges is a tail event for the i.i.d. sequence (γ_i). Indeed, we have that $\frac{1}{n}\sum_{i\neq j}\xi_{i,j}$ converges if, and only if, $\frac{1}{n}\sum_{i,j\geq N, i\neq j}\xi_{i,j}$ converges, for any integer N. Then, by the Kolmogorov zero-one law, the event in which series $\frac{1}{n}\sum_{i\neq j}\xi_{i,j}$ converges has probability either 1 or 0. The latter case however is excluded by Step 1. Therefore, the sequence $\frac{1}{n}\sum_{i\neq j}\xi_{i,j}$ converges with probability 1, and Step 2 follows.

STEP 3: We have $\tilde{L} = L'$, with probability 1. Indeed, by Steps 1 and 2 it follows $\frac{1}{n}\sum_{i\neq j}\xi_{i,j} - L' = o_p(1)$ and $\frac{1}{n}\sum_{i\neq j}\xi_{i,j} - \tilde{L} = o_p(1)$. These equations imply that $\tilde{L} - L' = o_p(1)$, which holds if and only if $\tilde{L} = L$ with probability 1 (since $\tilde{L}$ and L' are independent of n).

B.1.13 Proof of Lemma 14

The proof is similar to the one of Lemma 13 and we give only the main steps. First, we prove that $\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i \in I_m\}\right)^3 = o_p(1)$. Indeed, we have:

$$E\left[\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i \in I_m\}\right)^3\right] = \frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\sum_{i,j,k} E\left[1\gamma_i,\gamma_j,\gamma_k \in I_m\right] = O\left(n^{3/2}\sum_{m=1}^{J_n} B_m^3\right) = o(1),$$

from Assumption BD.2, and we can show $V\left[\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i \in I_m\}\right)^3\right] = o(1)$. Second, by using the monotone convergence theorem and the Kolmogorov zero-one law, we can show that sequence $\frac{1}{n^{3/2}}\sum_{m=1}^{J_n}\left(\sum_i 1\{\gamma_i \in I_m\}\right)^3$ converges with probability 1. Third, we conclude that the limit is 0 with probability 1.

B.2 Cost of equity

We can use the results in Chapter 3 for estimation and inference on the cost of equity in conditional factor models. We can estimate the time varying cost of equity $CE_{i,t} = r_{f,t} + b'_{i,t}\lambda_t$ of firm i with $\widehat{CE}_{i,t} = r_{f,t} + \hat{b}'_{i,t}\hat{\lambda}_t$, where $r_{f,t}$ is the risk-free rate. We have (see Appendix B.2.1)

$$\begin{aligned}\sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) &= \psi'_{i,t}E'_2\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) \\ &\quad + \left(Z'_{t-1} \otimes b'_{i,t}\right) W_{p,K}\sqrt{T} vec\left[\hat{\Lambda}' - \Lambda'\right] + o_p(1), \qquad \text{(b.27)}\end{aligned}$$

where $\psi_{i,t} = \left(\lambda'_t \otimes Z'_{t-1}, \lambda'_t \otimes Z'_{i,t-1}\right)'$. Standard results on OLS imply that estimator $\hat{\beta}_i$ is asymptotically normal, $\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) \Rightarrow N\left(0, \tau_i Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1}\right)$, and independent of estimator $\hat{\Lambda}$. Then, from Proposition 9, we deduce that $\sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) \Rightarrow N\left(0, \Sigma_{CE_{i,t}}\right)$, conditionally on Z_{t-1}, where

$$\Sigma_{CE_{i,t}} = \tau_i \psi'_{i,t} E'_2 Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} E_2 \psi_{i,t} + \left(Z'_{t-1} \otimes b'_{i,t}\right) W_{p,K} \Sigma_\Lambda W_{K,p} \left(Z_{t-1} \otimes b_{i,t}\right).$$

Figure 1 plots the path of the estimated annualized costs of equity for Ford Motor, Disney, Motorola and Sony. The cost of equity has risen tremendously during the recent subprime crisis.

B.2.1 Proof of Equation (b.27)

We have:

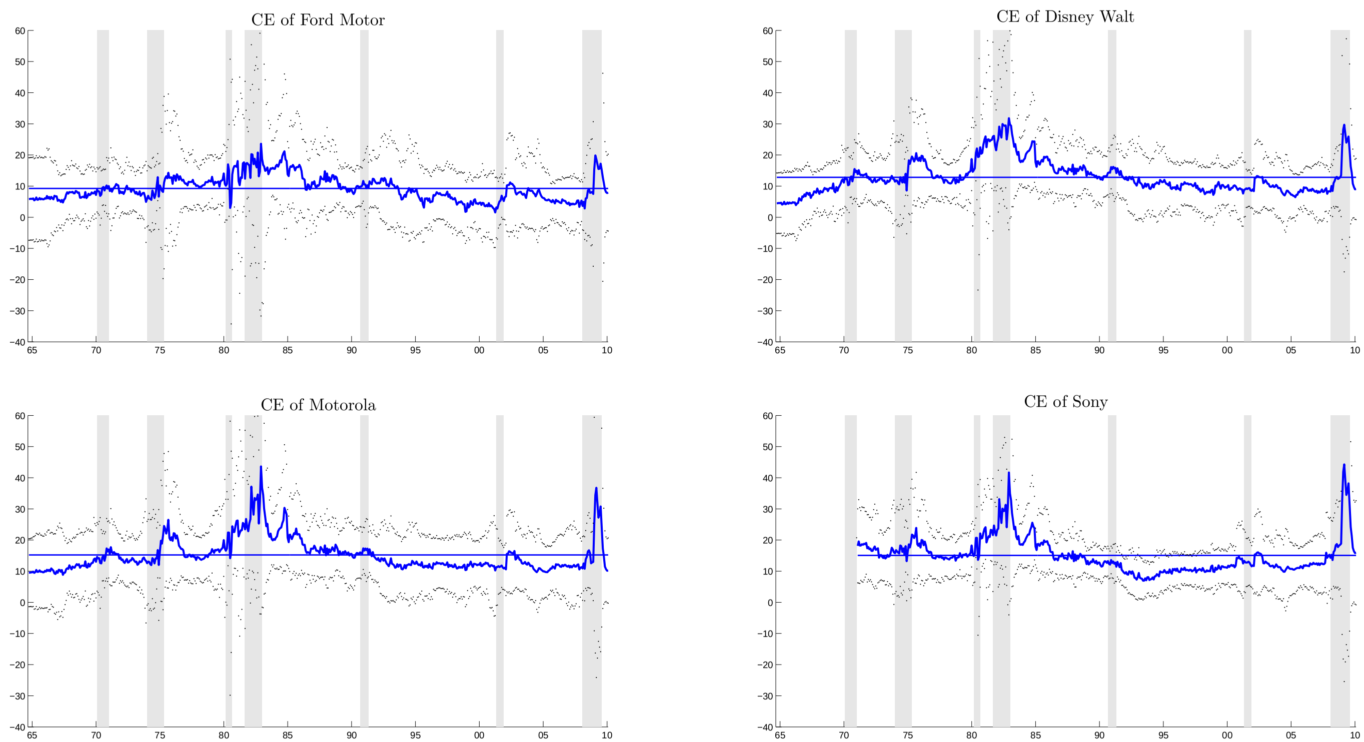
$$\hat{b}'_{i,t}\hat{\lambda}_t = tr\left[Z_{t-1}Z'_{t-1}\hat{B}'_i\hat{\Lambda}\right] + tr\left[Z_{t-1}Z'_{i,t-1}\hat{C}'_i\hat{\Lambda}\right] = \left(Z'_{t-1} \otimes Z'_{t-1}\right) vec\left[\hat{B}'_i\hat{\Lambda}\right] + \left(Z'_{t-1} \otimes Z'_{i,t-1}\right) vec\left[\hat{C}'_i\hat{\Lambda}\right].$$

Thus, we get:

$$\begin{aligned}&\sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) \\ = \;& \left(Z'_{t-1} \otimes Z'_{t-1}\right)\sqrt{T}\left(vec\left[\hat{B}'_i\hat{\Lambda}\right] - vec\left[B'_i\Lambda\right]\right) + \left(Z'_{t-1} \otimes Z'_{i,t-1}\right)\sqrt{T}\left(vec\left[\hat{C}'_i\hat{\Lambda}\right] - vec\left[C'_i\Lambda\right]\right) \\ = \;& \left(Z'_{t-1} \otimes Z'_{t-1}\right)\left[\left(\hat{\Lambda}' \otimes I_p\right)\sqrt{T} vec\left[\hat{B}'_i - B'_i\right] + \left(I_p \otimes B'_i\right)\sqrt{T} vec\left[\hat{\Lambda} - \Lambda\right]\right] \\ & + \left(Z'_{t-1} \otimes Z'_{i,t-1}\right)\left[\left(\hat{\Lambda}' \otimes I_q\right)\sqrt{T} vec\left[\hat{C}'_i - C'_i\right] + \left(I_p \otimes C'_i\right)\sqrt{T} vec\left[\hat{\Lambda} - \Lambda\right]\right].\end{aligned}$$

By using that $\hat{\Lambda} = \Lambda + o_p(1)$ and $vec\left[\hat{\Lambda} - \Lambda\right] = W_{p,K} vec\left[\hat{\Lambda}' - \Lambda'\right]$, Equation (b.27) follows.

Figure 1: Path of estimated annualized costs of equity



The figure plots the path of estimated annualized costs of equity for Ford Motor, Disney Walt, Motorola and Sony and their pointwise confidence intervals at 95% probability level. We also report the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determinated by the National Bureau of Economic Research (NBER).

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