

Correlation Risk and Optimal Portfolio Choice

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ABSTRACT

We develop a new framework for multivariate intertemporal portfolio choice that allows us to derive optimal portfolio implications for economies in which the degree of correlation across industries, countries, or asset classes is stochastic. Optimal portfolios include distinct hedging components against both stochastic volatility and correlation risk. We find that the hedging demand is typically larger than in univariate models, and it includes an economically significant covariance hedging component, which tends to increase with the persistence of variance-covariance shocks, the strength of leverage effects, the dimension of the investment opportunity set, and the presence of portfolio constraints.

THIS PAPER DEVELOPS A NEW MULTIVARIATE MODELING FRAMEWORK for intertemporal portfolio choice under a stochastic variance-covariance matrix. We consider an incomplete market economy, in which stochastic volatilities and stochastic correlations follow a multivariate diffusion process. In this setting, volatilities and correlations are conditionally correlated with returns, and optimal portfolio strategies include distinct hedging components against volatility and correlation risk. We solve the optimal portfolio problem and provide simple closed-form solutions that allow us to study the volatility and covariance hedging demands in realistic asset allocation settings. We document the importance of modeling the multivariate nature of second moments, especially in the context of optimal asset allocation, and find that the optimal hedging demand can be significantly different from the one implied by more common models with constant correlations or single-factor stochastic volatility.

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An important thread within the asset pricing literature has documented the characteristics of the time-variation in the covariance matrix of asset returns.¹ The importance of solving portfolio choice models taking into account the time-variation in volatilities and correlations is highlighted by Ball and Torous (2000), who study empirically the comovement of a number of international stock markets. They find that the estimated correlation structure changes over time depending on economic policies, the level of capital market integration, and relative business cycle conditions. They conclude that ignoring the stochastic component of the correlation can easily imply erroneous portfolio choice and risk management decisions.

A revealing example of the importance of modeling time-varying correlations is offered by the comovement of financial markets during the recent 2007 to 2008 financial markets crisis. During the period between April 2005 and April 2008, the sample average correlation of S&P500 and Nikkei (FTSE) weekly stock market returns has been less than 0.20. However, its time-variation has been large: since summer 2007 international equity correlations increased dramatically, with correlations between the S&P500 and the FTSE reaching a value close to 0.80 for the quarter ending in April 2008 (see Panel 1 of Figure 1).

Insert Figure 1 about here.

Another feature highlighted by the data is that correlation processes seem far from being independent: as the correlation with the FTSE has increased, the correlation with the Nikkei has also increased, reaching its highest value of 0.60 in the same month. A last important feature, highlighted in Panel 2 of Figure 1, is the correlation leverage effect: correlations of stock returns tend to be higher in phases of market downturn. While some of these empirical facts have been documented in the literature (see, e.g., Harvey and Siddique (2000), Roll (1988), and Ang and Chen (2002)), little is known about (a) the solution of the optimal portfolio choice problem when correlations are stochastic and (b) the extent to which stochastic correlations affect the characteristics of optimal portfolios in realistic economic settings.

An extensive literature has explored the implications of stochastic volatility for intertemporal portfolio choice. However, the implications of stochastic correlations in a multivariate setting are still not well known. In part, this is due to the difficulty in formulating a flexible and tractable model satisfying the tight nonlinear constraints implied by a well defined corre-

¹Longin and Solnik (1995) reject the null hypothesis of constant international stock market correlations and find that they increase in periods of high volatility. Ledoit, Santa-Clara, and Wolf (2003) show that the level of correlation depends on the phase of the business cycle. Erb, Harvey, and Viskanta (1994) find that international markets tend to be more correlated when countries are simultaneously in a recessionary state. Moskowitz (2003) documents that covariances across portfolio returns are highly correlated with NBER recessions and that average correlations are highly time-varying. Ang and Chen (2002) show that the correlation between U.S. stocks and the aggregate U.S. market is much higher during extreme downside movements than during upside movements. Barndorff-Nielsen and Shephard (2004) find similar results. Bekaert and Harvey (1995, 2000) provide direct evidence that market integration and financial liberalization change the correlation of emerging markets' stock returns with the global stock market index.

lation process. We use our model to address a number of questions on the role of correlation hedging for intertemporal portfolio choice.

First, what is the economic importance of stochastic variancecovariance risk for intertemporal portfolio choice ? We estimate the model using a data set of international stock and U.S. bond returns and find that – even for a moderate number of assets – the hedging demand can be about four to five times larger than in univariate stochastic volatility models. This has two reasons. First, covariance hedging can count for a substantial part of the total hedging demand. Its importance tends to increase with the strength of leverage effects and the dimension of the investment opportunity set. Second, our findings show not only that the joint features of volatility and correlation dynamics are better described by a multivariate model with nonlinear dependence and leverage, but also that they play an important role in the implied optimal portfolios. For instance, in a univariate stochastic volatility model we find that the estimated total hedging demand for S&P500 futures of investors with a relative risk aversion of eight and an investment horizon of 10 years is only about 4.8% of the myopic portfolio. This finding is consistent with the results in Chacko and Viceira (2005). However, in a multivariate (three risky assets) model, the total hedging demand for S&P500 futures is 28% and the covariance hedging demand is 16.9% of the myopic portfolio.

Second, how do both optimal investment in risky assets and covariance hedging demand vary with respect to the investment horizon ? This question is for optimal life-cycle decisions as well as for pension fund managers. We find that the absolute correlation hedging demand increases with the investment horizon. If the correlation hedging demand is positive (negative), this feature implies an optimal investment in risky assets that increases (decreases) in the investment horizon. For instance, in a multivariate model with three risky assets, the estimated total hedging (covariance hedging) demand for S&P500 futures of investors with a relative risk aversion of eight is only about 6.3% (4.5%) of the myopic portfolio at horizons of three months. For horizons of 10 years the total hedging demand increases to 28%.

Third, what is the link between the persistence of correlation shocks and the demand for correlation hedging ? The persistence of correlation shocks varies across markets. In highly liquid markets like the Treasury and foreign exchange markets, which are less affected by private information issues, correlation shocks are less persistent. In other markets, frictions such as asymmetric information and differences in beliefs about future cash flows make price deviations from the equilibrium more difficult to be arbitrated away. Examples include both developed and emerging equity markets. Consistent with this intuition, we find that the optimal hedging demand against covariance risk increases with the degree of persistence of correlation shocks.

Fourth, what is the impact of discrete trading and portfolio constraints on correlation hedging demands ? In the absence of derivative instruments to complete the market, we

find that the covariance hedging demand in continuous-time and discrete-time settings are comparable. Simple short-selling constraints tend to reduce the covariance hedging demand of risk-tolerant investors, typically by a moderate amount, but Value-at-Risk (VaR) constraints can even reinforce the covariance hedging motive. For instance, in the unconstrained discrete-time model with two risky assets the estimated total hedging (covariance hedging) demand for S&P500 futures of investors with relative risk aversion of two is about 12.5% (4.6%) of the myopic portfolio at horizons of two years. In the VaR-constrained setting, the total hedging (covariance hedging) demand increases to 16.7% (8.1%).

This paper draws upon a large literature on optimal portfolio choice under a stochastic investment opportunity set. One set of papers studies optimal portfolio and consumption problems with a single risky asset and a riskless deposit account.² Portfolio selection problems with multiple risky assets have been considered in a further series of papers, but the majority of these are based on the assumption that volatility and correlation are constant. Examples include Brennan and Xia (2002), who study optimal asset allocation under inflation risk, and Sangvinatsos and Wachter (2005), who investigate the portfolio problem of a long-run investor with both nominal bonds and a stock. A notable exception to the constant volatility assumption is Liu (2007), who shows that the portfolio problem can be characterized by a sequence of differential equations in a model with quadratic returns. However, to solve in closed form a concrete model with a riskless asset, a risky bond, and a stock, he assumes independence between the state variable driving interest rate risk and the additional risk factor influencing the volatility of the stock return. Under these assumptions, correlations are restricted to being functions of stock and bond return volatilities. Therefore, optimal hedging portfolios do not allow volatility and correlation risk to have separate roles. We investigate an economy with multivariate risk factors, where correlations are nonredundant sources of risk and several risky assets can be conveniently considered.

We follow a new approach in modeling stochastic variance-covariance risk and directly specify the covariance matrix process as a Wishart diffusion process.³ This process can

²Kim and Omberg (1996) solve the portfolio problem of an investor optimizing utility of terminal wealth, where the riskless rate is constant and the risky asset has a mean reverting Sharpe ratio and constant volatility. Wachter (2002) extends this setting to allow for intermediate consumption and derives closed-form solutions in a complete markets setting. Chacko and Viceira (2005) relax the assumption on both preferences and volatility. They consider an infinite horizon economy with Epstein-Zin preferences, in which the volatility of the risky asset follows a mean reverting square-root process. Liu, Longstaff, and Pan (2003) model events affecting market prices and volatility, using the double-jump framework in Duffie, Pan, and Singleton (2000). They show that the optimal policy is similar to that of an investor facing short selling and borrowing constraints, even if none are imposed. Although their approach allows for a rather general model with stochastic volatility, they focus on an economy with a single risky asset.

³See Bru (1991). The convenient properties of Wishart processes for modeling multivariate stochastic volatility in finance are exploited first by Gouriéroux and Sufana (2004); Gouriéroux, Jasiak, and Sufana (2009) provide a thorough analysis of the properties of Wishart processes, both in discrete and continuous time.

reproduce several of the empirical features of returns covariance matrices highlighted above. At the same time, it is sufficiently tractable to grant closed-form solutions to the optimal portfolio problem, which we can easily interpret economically. The Wishart process is a single-regime mean-reverting matrix diffusion, in which the strength of the mean reversion can generate different degrees of persistence in volatilities, correlations, and co-volatilities. A completely different approach to modelling co-movement in portfolio choice relies on either a Markov switching regime in correlations or the introduction of a sequence of unpredictable joint Poisson shocks in asset returns. Ang and Bekaert (2002) consider a dynamic portfolio model with two i.i.d. switching regimes, one of which is characterized by higher correlations and volatilities. Das and Uppal (2004) study systemic risk, modeled as an unpredictable common Poisson shock, in a setting with a constant opportunity set and in the context of international equity diversification.

The article proceeds as follows: Section I describes the model, the properties of the implied correlation process, and the solution to the portfolio problem. In Section II we estimate the model in a real data example and quantify the portfolio impact of correlation risk. Section III discusses extensions that study the impact of discrete rebalancing and investment constraints on correlation hedging. Section IV concludes. Proofs are in the Appendix, while an Internet Appendix reports additional results.⁴

I. The Model

An investor with Constant Relative Risk Aversion (CRRA) utility over terminal wealth trades three assets, a riskless asset with instantaneous riskless return r and two risky assets, in a continuous-time frictionless economy on a finite time horizon $[0, T]$.⁵ The dynamics of the price vector $S = (S_1, S_2)'$ are described by the bivariate stochastic differential equation:

$$dS(t) = I_S \left[(r\bar{1}_2 + \Lambda(\Sigma, t))dt + \Sigma^{1/2}(t) dW(t) \right], \quad I_S = \text{Diag}[S_1, S_2], \quad (1)$$

where $r \in \mathbb{R}_+$, $\bar{1}_2 = (1, 1)'$, $\Lambda(\Sigma, t)$ is a vector of possibly state-dependent risk premia, W is a standard two-dimensional Brownian motion, and $\Sigma^{1/2}$ is the positive square root of the conditional covariance matrix Σ . The investment opportunity set is stochastic because of the time-varying market price of risk $\Sigma^{-1/2}(t)\Lambda(\Sigma, t)$. The constant interest rate assumption can

⁴An Internet Appendix for this article is online in the ‘‘Supplements and Datasets’’ section at <http://www.afajof.org/supplements.asp>. It is organized in four sub-appendices, labeled with capital letters from A to D. In the text, we refer to these sub-appendices as ‘Internet Appendix x ’, where x is the letter that identifies the sub-appendix.

⁵Our analysis extends to opportunity sets consisting of any number of risky assets and correlations, without affecting the existence of closed-form solutions and their general structure. We consider a two-dimensional setting to keep our notation simple and focus on the key economic intuition and implications of the solution.

easily be relaxed. Such an extension is investigated in Internet Appendix B. The diffusion process for Σ is detailed below. Let $\pi(t) = (\pi_1(t), \pi_2(t))'$ denote the vector of shares of wealth $X(t)$ invested in the first and the second risky asset, respectively. The agent's wealth evolves according to

$$dX(t) = X(t) [r + \pi(t)' \Lambda(\Sigma, t)] dt + X(t) \pi(t)' \Sigma^{1/2}(t) dW(t). \quad (2)$$

The agent selects the portfolio process π that maximizes CRRA utility of terminal wealth, with RRA coefficient γ . If $X_0 = X(0)$ denotes the initial wealth and $\Sigma_0 = \Sigma(0)$ the initial covariance matrix, the investor's optimization problem is

$$J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E} \left[\frac{X(T)^{1-\gamma} - 1}{1-\gamma} \right], \quad (3)$$

subject to the dynamic budget constraint (2).

A. The Stochastic Variance Covariance Process

To model stochastic covariance matrices, we use the continuous-time Wishart diffusion process introduced by Bru (1991). This process is a matrix-valued extension of the univariate square-root process that gained popularity in the term structure and stochastic volatility literature; see, for instance, Cox, Ingersoll, and Ross (1985) and Heston (1993). Let $B(t)$ be a 2×2 standard Brownian motion. The diffusion process for Σ is defined as

$$d\Sigma(t) = [\Omega\Omega' + M\Sigma(t) + \Sigma(t)M'] dt + \Sigma^{1/2}(t) dB(t)Q + Q' dB(t)' \Sigma^{1/2}(t), \quad (4)$$

where Ω , M , and Q are 2×2 square matrices (with Ω invertible). Matrix M drives the mean reversion of Σ and is assumed to be negative semidefinite to ensure stationarity. Matrix Q determines the co-volatility features of the stochastic variance-covariance matrix of returns.

Process (4) satisfies several properties that make it ideal to model stochastic correlation in finance. First, if $\Omega\Omega' \gg Q'Q$, then Σ is a well-defined covariance matrix process. Second, if $\Omega\Omega' = kQ'Q$ for some $k > n - 1$, then $\Sigma(t)$ follows a Wishart distribution. Third, the process (4) is affine in the sense of Duffie and Kan (1996) and Duffie, Filipovic, and Schachermayer (2003). This feature implies closed-form expressions for all conditional Laplace transforms. Fourth, if $d \ln S_t$ is a vector of returns with a Wishart covariance matrix $\Sigma(t)$, then the variance of the return of a portfolio π is a Wishart process. Fifth, processes (1) and (4) can feature some important empirical regularities of financial asset returns documented in the literature, such as leverage and co-leverage.

To model leverage effects, we assume a nonzero correlation between innovations in stock

returns and innovations in the variance-covariance process. Specifically, we define the standard Brownian motion $W(t)$ in the return dynamics as

$$W(t) = \sqrt{1 - \bar{\rho}'\bar{\rho}}Z(t) + B(t)\bar{\rho}, \quad (5)$$

where Z is a two-dimensional standard Brownian motion independent of B and $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2)'$ is a vector of correlation parameters $\bar{\rho}_i \in [-1, 1]$ such that $\bar{\rho}'\bar{\rho} \leq 1$. Parameters $\bar{\rho}_1$ and $\bar{\rho}_2$ parameterize leverage effects in volatilities and correlations of the multivariate return process (1). Since n risky assets are available for investment and the covariance matrix depends on $n(n+1)/2$ independent Brownian shocks, the market is incomplete when $n \geq 2$.

B. Specification of the Risk Premium

The investment opportunity set can be stochastic due to changes in expected returns or changes in conditional variances and covariances. It is well known that to obtain closed-form solutions one needs to impose restrictions on the functional form of the squared Sharpe ratio. Affine squared Sharpe ratios imply affine solutions if the underlying state process is affine. Thus, we consider risk premium specifications $\Lambda(\Sigma, t)$ that imply an affine dependence of squared Sharpe ratios on the state process.

We consider a setting with a constant market price of variance-covariance risk, $\Lambda(\Sigma, t) = \Sigma(t)\lambda$ for $\lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2$. This assumption implies squared Sharpe ratios that increase with volatility, but that can increase or decrease in the correlation depending on the sign of the prices of risk.⁶ We solve the dynamic portfolio problem implied by this specification in Section I.D.2. In Internet Appendix B, we solve the portfolio problem also under the assumption of a constant risk premium, $\Lambda(\Sigma, t) = \mu^e$, $\mu^e = (\mu_1^e, \mu_2^e)' \in \mathbb{R}^2$, and an affine matrix diffusion (4) for the precision process Σ^{-1} . In this setting, the investment opportunity set is stochastic exclusively due to the stochastic covariance matrix.⁷ However, the disadvantage is that state variables are defined by means of Σ^{-1} , which makes the interpretation of model parameters (e.g., in terms of volatility and correlation leverage effects) less straightforward.

C. Correlation Process and Leverage

An application of Itô's Lemma gives us the correlation dynamics implied by the Wishart diffusion (4).

PROPOSITION 1: *Let ρ be the correlation diffusion process implied by the covariance matrix*

⁶The assumption of a constant market price of variance-covariance risk implies a positive risk-return tradeoff and embeds naturally the univariate model studied, among others, in Heston (1993) and Liu (2001).

⁷Buraschi, Trojani, and Vedolin (2009) investigate a general equilibrium orchard economy in which both volatility and correlations are priced in equilibrium and are affected by time-varying economic uncertainty.

dynamics (4). The instantaneous drift and conditional variance of $d\rho(t)$ are given by

$$\mathbb{E}_t[d\rho(t)] = [E_1(t)\rho(t)^2 + E_2(t)\rho(t) + E_3(t)] dt, \quad (6)$$

$$\mathbb{E}_t[d\rho(t)^2] = [(1 - \rho^2(t))(E_4(t) + E_5(t)\rho(t))] dt, \quad (7)$$

where the coefficients E_1, E_2, E_3, E_4 , and E_5 depend exclusively on Σ_{11}, Σ_{22} , and the model parameters Ω, M , and Q . The explicit expression for the coefficients E_1, \dots, E_n in the correlation dynamics is derived in the Appendix.

The correlation dynamics are not affine, because the correlation is a nonlinear function of variances and covariances. The nonlinear drift and volatility functions imply nonlinear mean reversion and persistence properties, depending on the model parameters. Moreover, the drift and volatility coefficients are functions of the volatility of asset returns, showing the intrinsic multivariate nature of the correlation in our model. This property is a clear distinction from approaches that model correlations with a scalar diffusion (see, for instance, Driessen, Maenhout, and Vilkov (2007)).

Black's volatility "leverage" effect, that is, the negative dependence between returns and volatility, is an empirical feature of stock returns, which has important implications for optimal portfolio choice. Roll's (1988) correlation "leverage" effect, that is, the negative dependence between returns and average correlation shocks, is also a well-established stylized fact; see, for example, Ang and Chen (2002). In our model, these effects depend on parameter $\bar{\rho}$ and the matrix Q . To see this explicitly, one can use the properties of the Wishart process to obtain

$$\text{corr}_t \left(\frac{dS_1}{S_1}, d\Sigma_{11} \right) = \frac{q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2}{\sqrt{q_{11}^2 + q_{21}^2}}, \quad \text{corr}_t \left(\frac{dS_1}{S_1}, d\rho \right) = \frac{(q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2)(1 - \rho^2(t))}{\sqrt{(\mathbb{E}_t[d\rho^2])/dt}\Sigma_{22}(t)}, \quad (8)$$

where for any $i, j = 1, 2$, parameters q_{ij} denote the ij^{th} element of matrix Q and the expression for $\mathbb{E}_t[d\rho^2]$ is given in equation (7). The expressions for the second asset are symmetric, with q_{12} replacing q_{11} and q_{22} replacing q_{21} , both in the first and the second equality. The element Σ_{11} replaces Σ_{22} in the second equality. From these formulas, the parameter vector $Q'\bar{\rho}$ controls the dependence between returns, volatility, and correlation shocks: volatility and correlation leverage effects arise for all assets if both components of $Q'\bar{\rho}$ are negative.

D. The Solution of the Investment Problem

The first challenge in solving investment problem (3) subject to the covariance matrix dynamics (4) is that markets are incomplete. If we consider a market with only primary risky securities, then there is no (nondegenerate) specification of the model that allows the

number of available risky assets to match the dimensionality of the Brownian motions.⁸

D.1. Incomplete Market Solution Approach

To solve the portfolio problem, we follow He and Pearson (1991) and solve the following static problem:

$$J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E} \left[\frac{X(T)^{1-\gamma} - 1}{1-\gamma} \right], \quad (9)$$

$$\text{s.t. } \mathbb{E} [\xi_{\nu}(T)X(T)] \leq x, \quad (10)$$

where ν indexes the set of all equivalent martingale measures and ξ_{ν} is in the set of associated state price densities. The optimality condition for the optimization over π in problem (9) is $X(T) = (\psi \xi_{\nu}(T))^{-1/\gamma}$, where ψ is the multiplier of the constraint (10), so that we can focus without loss of generality on the solution of the problem:⁹

$$\widehat{J}(0, \Sigma_0) = \inf_{\nu} \mathbb{E} [\xi_{\nu}(T)^{(\gamma-1)/\gamma}]. \quad (11)$$

To obtain the value function of this problem in closed form, we take advantage of the fact that the Wishart process (4) is an affine stochastic process.

D.2. Exponentially Affine Value Function and Optimal Portfolios

The solution of the dynamic portfolio problem is exponentially affine in Σ , with coefficients obtained as solutions of a system of matrix Riccati differential equations.

PROPOSITION 2: *Given the covariance matrix dynamics in (4), the value function of problem (3) takes the form*

$$J(X_0, \Sigma_0) = \frac{X_0^{1-\gamma} \widehat{J}(0, \Sigma_0)^{\gamma} - 1}{1-\gamma},$$

⁸In order to hedge volatility and correlation risk, one may consider derivatives with a pay-off that depends on the variances of a portfolio of the primary assets, for instance variance swaps or options on a “market” index; see for example Leippold, Egloff, and Wu (2007) for a univariate dynamic portfolio choice problem with variance swaps. If these derivatives completely span the state space generated by variances and covariances, then they can be used to complete the market and solve in closed form the optimal portfolio choice problem. The extent to which volatility and correlation hedging demands in the basic securities will arise depends on the ability of these additional derivatives to span the variance-covariance state space. Since variance swaps are available only in some specific markets, in many cases variance-covariance risk is not likely to be completely hedgeable, which makes the incomplete market case of primary interest.

⁹Results in Schroder and Skiadas (2003) imply that if the original optimization problem has a solution, the value function of the static problem coincides with the value function of the original problem. Cvitanic and Karatzas (1992) show that the solution to the original problem exists under additional restrictions on the utility function, most importantly that the relative risk aversion does not exceed one. Cuoco (1997) proves a more general existence result, imposing minimal restrictions on the utility function.

where the function $\widehat{J}(t, \Sigma)$ is given by

$$\widehat{J}(t, \Sigma) = \exp(B(t, T) + \text{tr}(A(t, T) \Sigma)), \quad (12)$$

with $B(t, T)$ and the symmetric matrix-valued function $A(t, T)$ solving the system of matrix Riccati differential equations

$$0 = \frac{dB}{dt} + \text{tr}[A\Omega\Omega'] - \frac{\gamma - 1}{\gamma} r, \quad (13)$$

$$0 = \frac{dA}{dt} + \Gamma' A + A\Gamma + 2A\Lambda A + C, \quad (14)$$

under the terminal conditions $B(T, T) = 0$ and $A(T, T) = 0$. Constant matrices Γ , Λ , and C , as well as the closed-form solution of the system of matrix Riccati differential equations (13) to (14), are reported in the Appendix.

REMARK. In the term structure literature, it is well known that modeling correlated stochastic factors is not straightforward. Duffie and Kan (1996) show that for a well-defined affine process to exist, parametric restrictions on the drift matrix of the factor dynamics have to be satisfied. These features restrict the correlation structures that many affine models can fit (see, for example, Duffie, 2002). In the Dai and Singleton (2000) classification for affine $A_m(n)$ models, restrictions need to be imposed for the model to be solved in closed form. This issue is well acknowledged also in the portfolio choice literature. For instance, Liu (2007) addresses it by assuming a triangular factor structure in an affine portfolio problem with two risky assets. Using the Wishart specification (4), we obtain a simple affine solution for problem (3), which does not imply excessive restrictions on the dependence of variance-covariance factors.¹⁰ \square

An advantage of the exponentially affine solution \widehat{J} in Proposition 2 is that it allows for a simple description of the partial derivatives of the marginal indirect utility of wealth with respect to the variance-covariance factors. This property implies a simple and easily interpretable solution to the multivariate portfolio choice problem.

PROPOSITION 3: *Let π be the optimal portfolio obtained under the assumptions of Proposition 2. It then follows that*

$$\pi = \frac{\lambda}{\gamma} + 2 \begin{bmatrix} (q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2)A_{11} + (q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2)A_{12} \\ (q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2)A_{22} + (q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2)A_{12} \end{bmatrix}, \quad (15)$$

¹⁰The Wishart state space is useful also more generally, for example, for term structure modeling. Buraschi, Cieslak, and Trojani (2007) develop a completely affine model with a Wishart state space to explain several empirical regularities of the term structure at the same time.

where A_{ij} denotes the ij^{th} component of the matrix A , which characterizes the function $\hat{J}(t, \Sigma)$ in Proposition 2, and the coefficients q_{ij} are the entries of the matrix Q appearing in the Wishart dynamics (4).

The portfolio policy $\pi = (\pi_1, \pi_2)'$ is the sum of a myopic demand and a hedging demand. The interpretation is simple and can easily be linked to Merton's (1969) solution. The myopic portfolio is the optimal portfolio that would prevail in an economy with a constant opportunity set, that is, a constant covariance matrix. When the opportunity set is stochastic, the optimal portfolio also consists of an intertemporal hedging demand. This portfolio component reduces the impact of shocks to the indirect utility of wealth. The size of intertemporal hedging depends on two components: (a) the extent to which investors' marginal utility of wealth is indeed affected by shocks in the state variables, and (b) the extent to which these state variables are correlated with returns. Using Merton's notation, the optimal hedging demand, denoted π_h , can be written as

$$\pi_h = - \sum_{i,j} \frac{J_{X\Sigma_{ij}}}{X J_{XX}} \Sigma^{-1} \frac{Cov_t(I_S^{-1} dS, d\Sigma_{ij})}{dt} . \quad (16)$$

The term $-\frac{J_{X\Sigma_{ij}}}{X J_{XX}} = -\frac{J_X}{X J_{XX}} \frac{J_{X\Sigma_{ij}}}{J_X} = A_{ij}$ is a risk tolerance weighted sensitivity of the log of the indirect marginal utility of wealth with respect to the state variable Σ_{ij} . The regression coefficient $\Sigma^{-1} Cov_t(I_S^{-1} dS, d\Sigma_{ij})$ captures the ability of asset returns to hedge unexpected changes in this state variable. The hedging portfolio is zero if and only if either $J_{X\Sigma_{ij}} = 0$ for all i and j (e.g., log utility investors) or $\Sigma^{-1} Cov_t(I_S^{-1} dS, d\Sigma_{ij}) = 0$. Using the properties of the Wishart process, the hedging portfolio then follows, in explicit form:

$$\pi_h = \Sigma^{-1} \frac{Cov_t(I_S^{-1} dS, \sum_{i,j} A_{ij} d\Sigma_{ij})}{dt} = 2 \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \\ q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \end{pmatrix} . \quad (17)$$

This is the second term in the sum on the right-hand side of formula (15). Hedging demands are generated by the willingness to hedge unexpected changes in the portfolio total utility due to shocks in the state variables Σ_{ij} . Hedging demands proportional to A_{ij} are demands against unexpected changes in Σ_{ij} . It follows that hedging portfolios proportional to A_{11} and A_{22} are volatility hedging portfolios, and hedging portfolios proportional to A_{12} are covariance hedging portfolios. The role of parameters Q and $\bar{\rho}$ in the hedging portfolio is clarified by writing equation (17) in the equivalent form:

$$\pi_h = 2A_{11} \begin{pmatrix} q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \\ 0 \end{pmatrix} + 2A_{22} \begin{pmatrix} 0 \\ q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \end{pmatrix} + 2A_{12} \begin{pmatrix} q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \\ q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \end{pmatrix} . \quad (18)$$

The parameters Q and $\bar{\rho}$ determine the ability of asset returns to span shocks in risk factors, because they determine the regression coefficients $\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij})$ in equation (16):

$$\Sigma^{-1} \frac{Cov_t(I_S^{-1}dS, d\Sigma_{11})}{dt} = 2 \begin{pmatrix} q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \\ 0 \end{pmatrix}, \quad (19)$$

$$\Sigma^{-1} \frac{Cov_t(I_S^{-1}dS, d\Sigma_{22})}{dt} = 2 \begin{pmatrix} 0 \\ q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \end{pmatrix}, \quad (20)$$

$$\Sigma^{-1} \frac{Cov_t(I_S^{-1}dS, d\Sigma_{12})}{dt} = 2 \begin{pmatrix} q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \\ q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \end{pmatrix}. \quad (21)$$

By comparing (19) to (21) with (8), the sign of each component of $\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij})$ equals the sign of the co-movement between returns, variances, and correlations. It follows that the first and second columns of Q impact the volatility and covariance hedging demand for the first and second assets, respectively, via the coefficient vectors $(q_{11}, q_{21})'$ and $(q_{12}, q_{22})'$. In contrast, parameter $\bar{\rho}$ directly impacts all hedging portfolios. Risky assets are better at spanning the risk of variance-covariance shocks when $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ are large in absolute value. Moreover, asset i is a better hedging instrument against its stochastic volatility Σ_{ii} , and less so against shocks in the covariance Σ_{ij} , when the coefficient $q_{1i}\bar{\rho}_1 + q_{2i}\bar{\rho}_2$ is the largest one. Despite the simple hedging portfolio (15), a variety of other hedging implications can arise. For instance, when $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ are both negative, volatility and correlation leverage effects arise for all returns. However, if parameters $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ have mixed signs some returns will feature leverage effects, but others will not.

D.3. Sensitivity of the Marginal Utility of Wealth to the State Variables

The second determinant of the hedging demand is the sensitivity of the marginal utility of wealth to the state variables Σ_{ij} . This effect is summarized by the components A_{ij} . Therefore, it is useful to gain intuition on the dependence of A_{ij} on the structural model parameters. For brevity, we focus on investors with risk aversion above one and a vector λ such that $\lambda_1\lambda_2 \geq 0$. This setting includes the choice of parameters implied by the model estimation results in Section II.

PROPOSITION 4: *Consider an investor with risk aversion parameter $\gamma > 1$. (i) The following inequalities, describing the properties of the sensitivity of the indirect marginal utility of wealth with respect to changes in the state variables Σ_{ij} , hold true: $A_{11}, A_{22} \leq 0$ and $|A_{12}| \leq |A_{11} + A_{22}|/2$. (ii) If it is additionally assumed that either $\lambda_1 \geq \lambda_2 \geq 0$ or $\lambda_1 \leq \lambda_2 \leq 0$, then $A_{12} \leq 0$ and $|A_{22}| \leq |A_{12}| \leq |A_{11}|$.*

This result describes the link between the indirect marginal utility of wealth and the state variables Σ_{ij} : $A_{ij} = -\frac{J_{X\Sigma_{ij}}}{XJ_{XX}}$. This sensitivity is increasing with the sensitivity of the stochastic opportunity set, that is, the squared Sharpe ratio, to unexpected changes in Σ_{ij} . The squared Sharpe ratio is given by $\lambda_1^2\Sigma_{11} + \lambda_2^2\Sigma_{22} + 2\lambda_1\lambda_2\Sigma_{12}$. Its sensitivity to the variance risk factor Σ_{11} is highest when $|\lambda_1| \geq |\lambda_2|$, and vice versa. The sensitivity to the covariance risk factor is bounded by the absolute average sensitivity to the variance factors, because squared Sharpe ratios depend on Σ_{12} via a loading that is twice the product of λ_1 and λ_2 . To understand the sign of A_{ij} , recall that investors with risk aversion above one have a negative utility function bounded from above. Wealth homogeneity of the solution implies $J_X(t) = X(t)^{\gamma-1}\hat{J}(t, \Sigma(t))^{1-\gamma}$, so that $J_{\Sigma_{ij}}$ and $J_{X\Sigma_{ij}}$ have the same sign. An increase in the variance Σ_{ii} of one risky asset increases the squared Sharpe ratio of the optimal portfolio, but at the same time it increases the squared Sharpe ratio variance.¹¹ Investors with risk aversion above one dislike this effect, because ex-ante they profit less from higher future Sharpe ratios than they suffer from higher future Sharpe ratio variances. These features imply the negative sign of A_{ii} . The sign of $J_{\Sigma_{12}}$ depends on how squared Sharpe ratios depend on Σ_{12} . If $\lambda_1\lambda_2 \geq 0$, Σ_{12} affects the squared Sharpe ratios positively, which implies $A_{12} \leq 0$.

II. Hedging Stochastic Variance-Covariance Risk

We quantify volatility and covariance hedging for a realistic stock-bond portfolio problem in which a portfolio manager allocates wealth between the S&P500 Index futures contract, traded at the Chicago Mercantile Exchange, the Treasury bond futures contract, traded at the Chicago Board of Trade, and a riskless asset.

A. Data and Estimation Results

The model is estimated by GMM (Generalized Method of Moments) using the conditional moment conditions of the process, derived in closed form.¹² The methodology is easily implemented and provides asymptotic tests for overidentifying restrictions. As a first step, we use the methodology proposed by Andersen et al. (2003) to obtain model-free realized volatilities and covariances from daily quadratic variations and covariations of log prices.¹³ The high-frequency data set we use is from “Price-Data” and “Tick-Data” and it includes

¹¹This variance equals $4\lambda'\Sigma\lambda\lambda'Q'Q\lambda$, using the properties of Wishart processes.

¹²See Internet Appendix A. A closed form expression for the moments of the Wishart process can be found, for instance, in the Appendix of Buraschi, Cieslak, and Trojani (2007).

¹³Bandorff-Nielsen and Shephard (2002) apply quasi-likelihood methods to time series of realized volatilities, and they estimate the parameters of continuous-time stochastic volatility models. Bollerslev and Zhou (2002) apply GMM to high frequency foreign exchange and equity index returns to estimate stochastic volatility models. Monte Carlo simulations indicate that the estimation procedure is accurate and the statistical inference reliable.

tick-by-tick futures returns for the S&P500 index and the 30-year Treasury bond, from January 1990 to October 2003.¹⁴

We use both weekly and monthly returns, realized volatilities, and covariances, to investigate the impact of different exact discretizations of the model on the estimated parameters. Let $\theta := (\text{vec}(M)', \text{vec}(Q)', \lambda', \bar{\rho}')'$. A GMM estimator of θ is given by

$$\hat{\theta} = \arg \min_{\theta} (\mu(\theta) - \mu_T)' V(\theta) (\mu(\theta) - \mu_T) , \quad (22)$$

where μ_T is the vector of empirical moments implied by the historical returns and their realized variance-covariance matrices, and $\mu(\theta)$ is the theoretical vector of moments in the model. The term $V(\theta)$ is the GMM optimal weighting matrix in the sense of Hansen (1982), estimated using a Newey-West estimator with 12 lags. We estimate θ using moment conditions that provide information about returns, their realized volatilities and correlations, and the leverage effects. The term μ_T consists of the following moment restrictions: unconditional risk premia of log-returns, unconditional first and second moments of variances and covariances of log-returns, and unconditional covariances between returns and each element of the variance-covariance matrix of returns. This leaves us with 17 moment restrictions for a 13-dimensional parameter vector, so that we have four overidentifying restrictions.

A.1. Basic Estimation Results

Table I presents results of our GMM model estimation.

Insert Table I about here

Hansen's test of overidentifying restrictions does not reject the model specification at the weekly and monthly frequency.¹⁵ The parameter estimates support the multivariate specification of the correlation process. The null hypothesis that the volatility of volatility matrix Q is identically zero is rejected at the 5% significance level, which supports the hypothesis of a stochastic correlation process. Parameter estimates for the components of M are also almost all significant, supporting a multivariate mean reversion and persistence in variances and covariances. The estimated eigenvalues of matrix M imply clear-cut evidence for two

¹⁴See the web pages www.grainmarketresearch.com and www.tickdata.com for details. When we estimate the model with three risky assets, in Internet Appendix D, we also use returns for the Nikkei225 Index futures contract.

¹⁵We also estimate the model using daily returns, realized volatilities, and covariances. Results are reported in Internet Appendix D. In this case, the Hansen's statistic rejected the model. We find that jumps in returns and realized conditional second moments are mainly responsible for this rejection, suggesting the misspecification of a pure multivariate diffusion in this context. The extension of our setting to a matrix-valued affine jump diffusion would be an interesting topic for future research.

very different mean reversion frequencies, a high one and a low one, underlying the returns covariance matrix.¹⁶ All estimated eigenvalues are negative, which supports the stationarity of the variance-covariance process. The larger eigenvalues estimated with monthly returns imply a larger persistence of variance-covariance shocks at monthly frequencies. The estimated components of vector $\bar{\rho}$ are all negative and significant. Together with the positive point estimates for the coefficients of Q , this feature implies volatility and correlation leverage features for all risky assets. The point estimates for Q highlight a large estimated parameter q_{11} . Thus, the first leverage parameter $\bar{q}_{11}\rho_1 + q_{21}\bar{\rho}_2$ is about twice the size of the leverage parameter $\bar{q}_{12}\rho_1 + q_{22}\bar{\rho}_2$. This implies that S&P500 returns are better vehicles to hedge their volatility risk than 30-year Treasury returns. At the same time, 30-year Treasury returns are better hedging instruments for hedging the covariance risk.

A.2. Estimated Correlation Process

Using the model parameter estimates, we can study the nonlinear dynamic properties of the implied correlation process. A convenient measure of the mean reversion properties of a nonlinear diffusion is given by its pull function - see Conley et al. (1997). The pull function $\wp(x)$ of a process X is the conditional probability that X_t reaches the value $x + \epsilon$ before $x - \epsilon$, if initialized at $X_0 = x$. To first order in ϵ , this probability is given by

$$\wp(x) = \frac{1}{2} + \frac{\mu_X(x)}{2\sigma_X^2(x)}\epsilon + o(\epsilon), \quad (23)$$

where μ_X and σ_X are the drift and the volatility function of X . Figure 2 presents non-parametric estimates of the pull function for the correlation and volatility processes of the S&P500 futures and 30-year Treasury futures returns, shifted by the factor 1/2 in equation (23).

Insert Figure 2 about here

Each panel in the left column plots the estimated pull functions for volatilities and correlations for weekly and monthly data. The panels in the middle (right) column plot pull functions estimated from a long time series of observations simulated from our model. These pull functions are all inside a two-sided 95% confidence interval around the empirical pull functions, which indicates that our model can capture adequately the nonlinear mean rever-

¹⁶Evidence for a multifactor structure in the variance-covariance of asset returns is provided by Da and Schaumburg (2006), who apply Asymptotic Principal Component analysis to a panel of realized volatilities for U.S. stock returns. They find that three to four factors explain no more than 60% of the variation in realized volatilities and that the forecasting power of multifactor volatility models is superior to that of univariate ones. Similar findings are obtained by Andersen and Benzoni (2007). Calvet and Fisher (2007) develop an equilibrium model in which innovations in dividend volatility depend on shocks that decay with different frequencies. They show that this feature of volatility is crucial for the model's forecasting performance.

sion properties of volatilities and correlations in our data. Estimated pull functions for the correlation are highly asymmetric and are typically smaller in absolute value for positive correlations above 0.3 than for negative correlations below -0.4. This feature indicates a higher persistence of correlation shocks when correlations are positive and large. The pull function for the volatility of S&P500 futures returns in the first row of Figure 2 is almost flat and moderately positive for volatilities larger than 10%. On average, the pull function estimated for the volatility of 30-year Treasury futures returns tends to be larger in absolute value, which indicates a lower persistence of shocks in the volatility of Treasury futures returns.

Given the evidence of a nonlinear mean reversion of volatilities and correlations in the data, it is natural to ask whether univariate Markov continuous-time models can reproduce these features accurately. We estimate the Heston (1993) square-root processes for the volatility and an autonomous specification for the correlation process, as in Driessen, Maenhout, and Vilkov (2007). When we compute the model-implied pull functions, we find that (i) they are often outside the 95% confidence band around the empirical pull function and (ii) they are almost linear in shape, which is difficult to reconcile with our data. This suggests the importance of using an explicitly multivariate modeling approach.

B. The Size of Variance Covariance Hedging

We study the structure of the hedging demands based on our parameter estimates, and compute the optimal hedging components in Proposition 3 as a function of the relative risk aversion and the investment horizon.

B.1. Basic Results

Table II summarizes the estimated volatility and covariance hedging demands, as a percentage of the myopic portfolio allocations.

Insert Table II about here

Overall, monthly estimates of the hedging demands are greater than weekly estimates.¹⁷ A more persistent variance-covariance process implies that variance-covariance shocks have more persistent effects on future squared Sharpe ratios and their volatility. This feature yields a higher absolute sensitivity of the marginal utility of wealth to variance-covariance shocks and greater absolute hedging demands on average.

Consider, for illustration purposes, the hedging demands estimated for monthly returns under an investment horizon of $T = 5$ years and a relative risk aversion parameter of six. The estimated risk premium for the S&P500 futures returns implies a higher loading of

¹⁷Weekly estimates are reported in Internet Appendix D.

volatility than the one for the risk premium of 30-year Treasury futures returns: $\lambda_1 \geq \lambda_2$. Thus, the estimated sensitivity of the marginal utility of wealth to the returns volatility is highest for S&P500 futures returns: $|A_{11}| > |A_{22}|$. Moreover, according to the GMM parameter estimates, stocks are better instruments to hedge their volatility than bonds: $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2 \geq q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$. These features imply a higher volatility hedging demand for stocks (about 13% of the myopic portfolio) relative to the volatility hedging component for bonds (about 8% of the myopic portfolio). The total average volatility hedging demand is approximately 10.5% of the myopic portfolio, while the total average covariance hedging demand on the two risky assets is slightly higher (about 11%). According to the GMM point estimates, bonds are better hedging vehicles than stocks to hedge covariance risk: $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2 \geq q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$. This effect determines the higher covariance hedging demand for bonds (about 17%) than for stocks (about 7%).

Given the evidence of a misspecification of univariate stochastic volatility models in our data, we compare the portfolio implications of our setting with those of univariate portfolio choice models with stochastic volatility; see Heston (1993) and Liu (2001), among others. This is an easy task since these models are nested in our setting for the special case in which the dimension of the investment opportunity set is equal to one. For each risky asset in our data set, we estimate these univariate stochastic volatility models by GMM.

Internet Appendix D presents the estimated volatility hedging demands as a percentage of the myopic portfolio. For illustration purposes, consider a relative risk aversion coefficient of $\gamma = 8$ and an investment horizon of $T = 5$ years. The volatility hedging demands estimated for the univariate models are 4.8% and 4%, respectively, for stocks and bonds. In the multivariate model, the corresponding pure volatility hedging demands are 13.6% and 8.8%, respectively, and the average total hedging demand is as large as 21.1%. One explanation for this finding is the very different mean reversion and persistence properties of second moments in the data, relative to those implied by univariate stochastic volatility models. A second reason is the fact that univariate models cannot capture the correlation and co-volatility dynamics, which generate a good portion of the total hedging demand. This is important since the optimal portfolio, as the results show, presents a substantial bias.

B.2. Comparative Statics

To get more insight into the determinants of hedging motives, it is useful to study comparative statics with respect to model parameters. To this end, we modify the value of these parameters in an interval of one sample standard deviation around the true parameter estimate, and compute the implied hedging demands.

It is natural to focus on parameters that have an impact on the indirect marginal utility sensitivities A_{ij} , and the volatility and correlation leverage effects. Matrix M drives the

persistence on the variance-covariance process, but leaves unaffected the leverage properties of asset returns. For brevity, we consider comparative statics with respect to the parameters m_{12} . The matrix Q and vector $\bar{\rho}$ affect primarily the ability of each asset to span unexpected variance-covariance shocks. To study the effects of these parameters, we consider for brevity comparative statics with respect to q_{11} and both components of $\bar{\rho}$. In doing so, we decompose the total effect on hedging demands into one part due to a modification of the leverage properties of returns and one part due to the change in the marginal utility sensitivity coefficients A_{ij} . The investment horizon we consider is $T = 5$ years and the relative risk aversion is $\gamma = 6$.

In the first row of Figure 3, the comparative statics with respect to m_{12} show that, *ceteris paribus*, covariance and volatility hedging demands increase with m_{12} : for a value of $m_{12} = 1.18$, that is, one standard deviation above the GMM estimate, the covariance (volatility) hedging component increases to 8% (14%) for stocks and 23% (14%) for bonds.

Insert Figure 3 about here

This effect is due to the higher persistence of the variance-covariance process caused by an increase in m_{12} , which implies a greater absolute marginal utility sensitivity to all variance-covariance risk factors.

The plots in the second row of Figure 3 present comparative statics with respect to q_{11} . As q_{11} increases, the first risky asset becomes a better hedging instrument against its volatility, and the second risky asset becomes a better hedging instrument against covariance risk. Parameter q_{11} also has an effect on the marginal utility sensitivities A_{ij} . We find that the higher variability of variance-covariance shocks implied by a higher parameter q_{11} lowers all absolute sensitivities $|A_{ij}|$, $1 \leq i, j \leq 2$. However, this effect is considerably smaller than the one implied by the change in the leverage structure of asset returns. Consequently, as q_{11} increases we obtain a decreasing (increasing) covariance hedging demand for the S&P500 futures (30-year Treasury futures), but also an increasing (decreasing) volatility hedging component.

The comparative statics with respect to parameters $\bar{\rho}_1$ and $\bar{\rho}_2$ are given in the third and fourth rows of Figure 3. As $\bar{\rho}_1$ decreases, all assets become better hedging instruments against volatility and correlation risk. At the same time, the variance-covariance process under the minimax measure becomes more persistent, increasing each absolute sensitivity coefficient $|A_{ij}|$. These two effects go in the same direction, but the effect on leverage is proportionally greater, and increases all volatility and covariance hedging demands. As $\bar{\rho}_2$ decreases, we observe almost no variation in volatility and covariance hedging demands. This follows from the fact that the leverage coefficients (18) and the minimax variance-covariance

dynamics depend on $\bar{\rho}_2$ with a weight that is proportional to the parameters q_{12} and q_{22} . According to our GMM estimates, these parameters are much smaller than q_{11} and q_{21} .

B.3. Time Horizon

An important question addressed by the literature is how the optimal allocation in risky assets varies with respect to the investment horizon. For instance, Kim and Omberg (1996) show that for the investor with utility over terminal wealth, and for $\gamma > 1$, the optimal allocation increases with the investment horizon as long as the risk premium is positive. Wachter (2002) extends this result to the case of utility over intertemporal consumption.

When covariances are stochastic, it is reasonable that the optimal demand for hedging covariance risk could mitigate, or strengthen, the speculative components. Internet Appendix D reports estimated hedging demands for the S&P500 Index futures and the 30-year Treasury Futures, as a function of the investment horizon. The total hedging demand of an investor with risk aversion $\gamma > 1$ increases with investment horizons of up to five years, where it approximately reaches a steady-state level. The reason for such a convergence is the stationarity of the Wishart process implied at our parameter estimates: shocks in the variance-covariance matrix do not seem to affect the transition density of the estimated variance-covariance process over horizons longer than five years. At very short horizons, for example, three months, hedging demands are small. For investment horizons of five years and higher, the total hedging demand is approximately 20% (25%) of the myopic portfolio for the S&P500 (Treasury) futures contracts. The covariance hedging demand for the 30-year Treasury futures increases quite quickly with the investment horizon, and it reaches a steady-state level of approximately 16% of the myopic demand. The covariance hedging demand for the S&P500 futures reaches a steady state of approximately 6.5% as the investment horizon increases.

B.4. Higher-dimensional Portfolio Choice Settings

For simplicity we have so far focused on a portfolio choice setting with only two risky assets. However, it is important to gain some intuition on the relevance of volatility and covariance hedging when more risky assets are available for investment. The complexity of the portfolio setting increases as more volatility and covariance factors affect returns, which makes general statements and conclusions difficult. On the one hand, given that the number of covariance factors increases quadratically with the dimension of the investment universe, but the number of volatility factors increases only linearly, one could expect covariance hedging to become proportionally more important as the investment dimension rises. On the other hand, as the number of assets rises, one could also argue that covariance risk could become less important than volatility risk because the potential for portfolio diversification

increases. The final effect depends on the extent to which shocks to the different covariance and volatility processes are diversifiable across assets.

We study quantitatively these issues in a concrete portfolio setting that includes also the Nikkei225 Index futures contract in the previous opportunity set, consisting of the S&P500 futures and the 30-year Treasury futures contracts. We estimate by this three-dimensional version of model (1) to (4) using GMM. Estimated hedging demands for covariance and volatility hedging are given in Internet Appendix D.

For illustration purposes, consider a relative risk aversion of $\gamma = 6$ and an investment horizon of $T = 5$ years. The covariance hedging demand for the S&P500 futures is now approximately 12.5%, almost twice the hedging demand of 6.5% estimated in the two risky assets setting. The inclusion of the Nikkei225 futures sensibly lowers the covariance hedging demand for 30-year Treasury futures, which drops from 17% in the two-asset case to 6.1%. The covariance hedging demand for the Nikkei225 futures is approximately 13%. On average, these results imply a covariance hedging demand of about 11%. The intuition for these findings is simple: as the dimension of the investment opportunity set increases, the relative importance of covariance shocks to the squared Sharpe ratio of the optimal portfolio increases. The Nikkei225 provides a good opportunity to diversify domestic equity risk, under the assumption that covariances do not change. At the parameter estimates, an increased investment in equity becomes increasingly coupled with a greater demand for hedging potential changes in these covariances. The volatility hedging demands for each asset are 9%, 6.1%, and 7.6%, respectively, and imply an average volatility hedging of about 7.5%. In the model with two risky assets, the average covariance hedging demand is about 11% and the average volatility hedging demand is about 10.5%.

Overall, these findings support the intuition that covariance hedging can become proportionally more important than volatility hedging as the dimension of the investment opportunity set rises.

III. Robustness and Extensions

In this section, we study the robustness of our findings, for example, with respect to a discrete-time solution of the portfolio choice problem or the inclusion of different portfolio constraints.

A. Risk Aversion

Our main findings do not depend on the choice of the relative risk aversion parameter used. Internet Appendix D contains a plot of hedging portfolios as a function of risk aversion. Hedging demands as a percentage of the myopic portfolio are monotonically increasing in the relative risk aversion coefficient, although the increase is small for relative risk aversion

parameters above six. Average covariance hedging demands as a percentage of the myopic portfolio are typically higher than average volatility hedging demands. For instance, the average covariance (volatility) hedging demand for a relative risk aversion of 10 is approximately 12% (10.5%) of the myopic portfolio. Thus, although we assume a constant relative risk aversion utility function to preserve closed-form optimal portfolios, our findings are likely to be even stronger in a setting with intertemporal consumption and Epstein-Zin recursive preferences, since the risk aversion can be calibrated at a higher level without generating undesirable properties for the elasticity of intertemporal substitution.

B. Discrete-time Solution and Portfolio Constraints

In our model, the optimal dynamic trading strategy is given by a portfolio that must be rebalanced continuously over time. In practice, this can at best be an approximation, because trading is only possible at discrete trading dates. Moreover, transaction costs, liquidity constraints, or policy disclosure considerations might further prevent investors from frequent portfolio rebalancing. Even if we do not model these frictions explicitly in our setting, it is instructive to analyze the impact of discrete trading on the optimal hedging strategy in the context of our model. We address this issue in Section I of Internet Appendix C. At a daily frequency, the hedging demands in the discrete-time model are virtually indistinguishable from the continuous-time hedging demands. The discrete-time optimal hedging demands for the monthly frequency are close to the hedging demands computed from the continuous time model. These findings suggest that the main implications derived from the continuous time multivariate portfolio choice solutions are realistic even in the context of monthly rebalancing.

Portfolio constraints are useful to avoid unrealistic portfolio weights, which can potentially arise due to extreme assumptions on expected returns, volatilities, and correlations, or from inaccurate point estimates of the model parameters. Intuitively, constraints on short selling or the portfolio VaR tend to constrain the investor from selecting optimal portfolios that are excessively levered. Therefore, it is interesting to study such constraints and their impact on the volatility and covariance hedging demands in our setting. Section II of Internet Appendix C provides a thorough analysis of this issue. We find that a VaR constraint has a significant effect on the optimal portfolios of investors with low risk aversion, which are those with the largest exposure to risky assets in the unconstrained setting. The VaR-constrained investor tends to reduce the size of the myopic demand. Furthermore, since changes in covariances have a first-order impact on the VaR of the portfolio, the investor can even increase the covariance hedging demand, to exploit the spanning properties of the risky assets. Thus, in this setting, which is relevant for institutions subject to capital requirements or for asset managers with self-imposed risk management constraints, the impact of covariance risk can be economically very significant.

IV. Discussion and Conclusions

We develop a new multivariate framework for intertemporal portfolio choice, in which stochastic second moments of asset returns imply distinct motives for volatility and covariance hedging. The model is solved in closed form and is used to study volatility and covariance hedging in several realistic settings. We find that the multivariate nature of second moments has important consequences for optimal asset allocation: hedging demands are typically four to five times larger than those of models with constant correlations or single-factor stochastic volatility. They include a substantial correlation hedging component, which tends to increase with the persistence of variance-covariance shocks, the strength of leverage effects, and the dimension of the investment opportunity set. These findings also arise within discrete-time versions of our model with short selling or VaR constraints.

The hedging demands due to variance-covariance risk are typically smaller than those found in the literature on intertemporal hedging with returns predictability. This feature follows mainly from the fact that returns span shocks in their covariance matrix much less than they typically do for shocks to the predictive variables. We do not explicitly incorporate Bayesian learning about model parameters. In continuous time, it is hard to motivate learning behavior about second moments of returns, because they are typically observable from the quadratic variation of returns. In discrete time, Bayesian learning about second moments can be more naturally considered. However, it is difficult to obtain tractable solutions for portfolio choice without a simple structure for the variance-covariance process. Barberis (2000), among others, studies estimation risk about the parameters of a predictive equation in a model with homoskedastic returns, and finds that parameter uncertainty dramatically reduces the exposure to stocks over longer horizons. Our model is very different from that of Barberis. However, one might try to conclude by analogy that learning could also substantially reduce hedging demands in our case. Interesting evidence on this issue is given in Brandt et al. (2005). They develop a dynamic programming algorithm to efficiently solve the portfolio problem with predictability. When learning is considered, they find hedging demands that are comparable to those found in our paper. When learning is neglected, these policies are much higher than ours. Interestingly, the hedging demand reduction is almost entirely due to learning about the predictability equation: learning about the variance-covariance matrix has a small influence on optimal portfolios. An interesting direction for future research could use the discrete-time Wishart process to study more systematically the portfolio implications of learning about the covariance matrix of returns.

APPENDIX

Proof of Proposition 1: The dynamics of the correlation process implied by the Wishart covariance matrix diffusion (4) is computed using Itô's Lemma. Let

$$\rho(t) = \frac{\Sigma_{12}(t)}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \quad (\text{A1})$$

be the instantaneous correlation between the returns of the first and second risky assets and denote by σ_{ij} , q_{ij} , and ω_{ij} the ij^{th} component of the volatility matrix $\Sigma^{1/2}$, the matrix Q , and the matrix $\Omega'\Omega = kQ'Q$ in equation (4), respectively. Applying Itô's Lemma to (A1) and using the dynamics for Σ_{11} , Σ_{22} , and Σ_{12} , implied by (4), it follows that

$$\begin{aligned} d\rho = & \left[\frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - \frac{\rho}{2\Sigma_{11}}\omega_{11} - \frac{\rho}{2\Sigma_{22}}\omega_{22} + \frac{\rho}{2} \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}} \right) \right. \\ & \left. + (\rho^2 - 2) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} + (1 - \rho^2) \frac{m_{21}\Sigma_{11} + m_{12}\Sigma_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dt \\ & - \left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}} (\Sigma_{22}\sigma_{11}q_{11} + \Sigma_{11}\sigma_{12}q_{12}) - \frac{\sigma_{12}q_{11} + \sigma_{11}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{11} \\ & - \left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}} (\Sigma_{11}\sigma_{22}q_{12} + \Sigma_{22}\sigma_{21}q_{11}) - \frac{\sigma_{22}q_{11} + \sigma_{21}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{21} \\ & - \left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}} (\Sigma_{22}\sigma_{11}q_{21} + \Sigma_{11}\sigma_{12}q_{22}) - \frac{\sigma_{11}q_{22} + \sigma_{12}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{12} \\ & - \left[\frac{\rho}{2\Sigma_{11}\Sigma_{22}} (\Sigma_{11}\sigma_{22}q_{22} + \Sigma_{22}\sigma_{21}q_{21}) - \frac{\sigma_{21}q_{22} + \sigma_{22}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{22}. \end{aligned} \quad (\text{A2})$$

$B_{ij}(t)$, $i, j = 1, 2$, are the entries of the 2×2 matrix of Brownian motions in (4). Therefore, the instantaneous drift of the correlation process is a quadratic polynomial with state-dependent coefficients:

$$\mathbb{E}[d\rho(t) | \mathcal{F}_t] = [E_1(t) \rho(t)^2 + E_2(t) \rho(t) + E_3(t)] dt, \quad (\text{A3})$$

where the coefficients $E_1(t)$, $E_2(t)$, and $E_3(t)$ are given by

$$E_1(t) = \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} - m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} - m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}, \quad (\text{A4})$$

$$E_2(t) = -\frac{\omega_{11}}{2\Sigma_{11}} - \frac{\omega_{22}}{2\Sigma_{22}} + \frac{1}{2} \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} \right), \quad (\text{A5})$$

$$E_3(t) = \frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - 2 \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} + m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} + m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}. \quad (\text{A6})$$

The instantaneous conditional variance of the correlation process is easily obtained from equation (A2) and is a third-order polynomial with state dependent coefficients:

$$\mathbb{E}[d\rho(t)^2 | \mathcal{F}_t] = \left[(1 - \rho^2(t)) \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} - 2\rho(t) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \right) \right] dt.$$

This concludes the proof. \square

Proof of Proposition 2: Since markets are incomplete, we follow He and Pearson (1991) and represent any market price of risk as the sum of two orthogonal components, one of which is spanned by the asset returns. Since Brownian motion W can be rewritten as $W = B\bar{\rho} + Z\sqrt{1 - \bar{\rho}'\bar{\rho}}$, for a standard bivariate Brownian motion Z independent of B , we rewrite the innovation component of the opportunity set dynamics as $\Sigma^{1/2}[Z, B]L$, with $L = [\sqrt{1 - \bar{\rho}'\bar{\rho}}, \bar{\rho}_1, \bar{\rho}_2]'$. Let Θ_ν be the matrix-valued extension of Θ that prices the

matrix of Brownian motions $\mathcal{B} = [Z, B]$. By definition of the market price of risk, we have

$$\Sigma^{1/2} \Theta_\nu L = \Sigma \lambda, \quad (\text{A7})$$

from which

$$\Theta_\nu = \Sigma^{1/2} \lambda L' + \Sigma^{1/2} \nu \quad (\text{A8})$$

for any 2×3 matrix valued process ν such that $\Sigma \nu L = 0_{2 \times 1}$. Since Σ is nonsingular, it follows that ν must be of the form $\nu = [-\bar{\nu} \frac{\rho}{\sqrt{1-\bar{\rho}'\bar{\rho}}}, \bar{\nu}]$. $\bar{\nu}$ is a 2×2 -matrix that prices the shocks that drive the variance-covariance matrix process.

Given Θ_ν , the associated martingale measure implies a process ξ_ν of stochastic discount factors, defined for $t \in [0, T]$ by¹⁸

$$\xi_\nu(t) = e^{-rt - tr(\int_0^t \Theta'_\nu(s) d\mathcal{B} + \frac{1}{2} \int_0^t \Theta'_\nu(s) \Theta_\nu(s) ds)}. \quad (\text{A9})$$

Our dynamic portfolio choice problem allows for an equivalent static representation by means of the following dual problem, as shown by He and Pearson (1991):

$$J(x, \Sigma_0) = \inf_\nu \sup_\pi \mathbb{E} \left[\frac{X(T)^{1-\gamma} - 1}{1-\gamma} \right], \quad (\text{A10})$$

$$\text{s.t. } \mathbb{E}[\xi_\nu(T) X(T)] \leq x, \quad (\text{A11})$$

where $X(0) = x$. In what follows, we focus on the solution of problem (A10) to (A11). The optimality conditions for the innermost maximization are

$$X(T) = (\psi \xi_\nu(T))^{-\frac{1}{\gamma}}, \quad (\text{A12})$$

where the Lagrange multiplier for the static budget constraint is

$$\psi = x^{-\gamma} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}} \right]^\gamma.$$

It then follows that

$$J(x, \Sigma_0) = x^{1-\gamma} \inf_\nu \frac{1}{1-\gamma} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}} \right]^\gamma - \frac{1}{1-\gamma}. \quad (\text{A13})$$

Using (A9) and (A13), one can see that the solution requires the computation of the expected value of the exponential of a stochastic integral. A simple change of measure reduces the problem to the calculation of the expectation of the exponential of a deterministic integral. Let P^γ be the probability measure defined by the following Radon-Nykodim derivative with respect to the physical measure P :

$$\frac{dP^\gamma}{dP} = e^{-tr\left(\frac{\gamma-1}{\gamma} \int_0^T \Theta'_\nu(s) d\mathcal{B}(s) + \frac{1}{2} \frac{(\gamma-1)^2}{\gamma^2} \int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds\right)}. \quad (\text{A14})$$

We denote expectations under P^γ by $\mathbb{E}^\gamma[\cdot]$. Then the minimizer of (A13) is the solution to the following problem:¹⁹

$$\begin{aligned} \hat{J}(0, \Sigma_0) &= \inf_\nu \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}} \right] \\ &= \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma-1}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} tr\left(\int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds\right)} \right] \\ &= \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma-1}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} tr\left(\int_0^T \Sigma(s) (\lambda \lambda' + \nu \nu') ds\right)} \right] \\ &= \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma-1}{\gamma} rT + \frac{1-\gamma}{2\gamma^2} tr\left(\int_0^T \Sigma(s) \left(\lambda \lambda' + \bar{\nu}' \bar{\nu} \left(I_2 + \frac{\bar{\rho} \bar{\rho}'}{1-\bar{\rho}'\bar{\rho}}\right) ds\right)\right)} \right]. \end{aligned} \quad (\text{A15})$$

¹⁸Remember that $W = \mathcal{B}L$.

¹⁹Strictly speaking, this holds for $\gamma \in (0, 1)$. For $\gamma > 1$, minimizations are replaced by maximizations and all formulas follow with the same type of arguments.

Notice that the expression in the exponential of the expectation in (A15) is affine in Σ . By the Girsanov Theorem, under the measure P^γ the stochastic process $\mathcal{B}^\gamma = [Z^\gamma, B^\gamma]$, defined as

$$\mathcal{B}^\gamma(t) = \mathcal{B}(t) + \frac{\gamma-1}{\gamma} \int_0^t \Theta_\nu(s) ds,$$

is a 2×3 matrix of standard Brownian motions. Therefore, the process (4) is an affine process also under the new probability measure P^γ :

$$\begin{aligned} d\Sigma(t) = & \left[\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma} Q'(\bar{\rho}\lambda' + \bar{\nu}') \right) \Sigma(t) + \Sigma(t) \left(M - \frac{\gamma-1}{\gamma} Q'(\bar{\rho}\lambda' + \bar{\nu}') \right)' \right] dt \\ & + \Sigma^{1/2}(t) dB^\gamma(t) Q + Q' dB(t)^{\gamma'} \Sigma^{1/2}(t). \end{aligned} \quad (\text{A16})$$

Using the Feynman Kac formula, it is known that if the optimal ν and \hat{J} solve the probabilistic problem (A15), then they must also be a solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\bar{\nu}} \left\{ \mathcal{A}\hat{J} + \hat{J} \left[-\frac{\gamma-1}{\gamma} r + \frac{1-\gamma}{2\gamma^2} \text{tr} \left(\Sigma \left(\lambda\lambda' + \bar{\nu}'\bar{\nu} \left(I_2 + \frac{\bar{\rho}\bar{\rho}'}{1-\bar{\rho}'\bar{\rho}} \right) \right) \right) \right] \right\}, \quad (\text{A17})$$

subject to the terminal condition $\hat{J}(T, \Sigma) = 1$, where \mathcal{A} is the infinitesimal generator of the matrix-valued diffusion (A16), which is given by

$$\begin{aligned} \mathcal{A} = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma} Q'(\bar{\rho}\lambda' + \bar{\nu}') \right) \Sigma + \Sigma \left(M - \frac{\gamma-1}{\gamma} Q'(\bar{\rho}\lambda' + \bar{\nu}') \right)' \right) \mathcal{D} \right) \\ & + \text{tr}(2\Sigma\mathcal{D}Q'Q\mathcal{D}), \end{aligned} \quad (\text{A18})$$

where

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix}. \quad (\text{A19})$$

The generator is affine in Σ . The optimality condition for the optimal control ν , implied by HJB equation (A17), is

$$-\frac{1}{\gamma} \Sigma \bar{\nu} \left(I_2 + \frac{\bar{\rho}\bar{\rho}'}{1-\bar{\rho}'\bar{\rho}} \right) = \frac{\partial}{\partial \bar{\nu}} \text{tr} \left((Q'\bar{\nu}'\Sigma + \Sigma\bar{\nu}Q) \frac{\mathcal{D}\hat{J}}{\hat{J}} \right) = \frac{\partial}{\partial \bar{\nu}} \text{tr} \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} Q'\bar{\nu}'\Sigma + \Sigma\bar{\nu}Q \frac{\mathcal{D}\hat{J}}{\hat{J}} \right).$$

Applying rules for the derivative of trace operators, the right-hand side can be written as $\Sigma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}'} \right) Q'$. It follows that

$$\bar{\nu} = -\gamma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}'} \right) Q' \left(I_2 + \frac{\bar{\rho}\bar{\rho}'}{1-\bar{\rho}'\bar{\rho}} \right)^{-1}. \quad (\text{A20})$$

Note that $\left(I_2 + \frac{\bar{\rho}\bar{\rho}'}{1-\bar{\rho}'\bar{\rho}} \right)^{-1} = I_2 - \bar{\rho}\bar{\rho}'$. Substituting the expression for ν in equation (A18), we obtain

the generator

$$\begin{aligned}
\mathcal{A} &= \text{tr} \left(\left(\Omega \Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right)' \right) \mathcal{D} + 2 \Sigma \mathcal{D} Q' Q \mathcal{D} \right) \\
&\quad + (\gamma-1) \hat{J} \text{tr} \left((I_2 - \bar{\rho} \rho') \left(Q' Q \left(\frac{\mathcal{D} \hat{J}}{\hat{J}} + \frac{\mathcal{D} \hat{J}'}{\hat{J}} \right) \Sigma + \Sigma \left(\frac{\mathcal{D} \hat{J}}{\hat{J}} + \frac{\mathcal{D} \hat{J}'}{\hat{J}} \right) Q' Q \right) \mathcal{D} \right) \\
&= \text{tr} \left(\left(\Omega \Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right)' \right) \mathcal{D} + 2 \Sigma \mathcal{D} Q' Q \mathcal{D} \right) \\
&\quad - (1-\gamma) \hat{J} \text{tr} \left((I_2 - \bar{\rho} \rho') \Sigma \left(\frac{\mathcal{D} \hat{J}}{\hat{J}} + \frac{\mathcal{D} \hat{J}'}{\hat{J}} \right) Q' Q \left(\frac{\mathcal{D}}{\hat{J}} + \frac{\mathcal{D}'}{\hat{J}} \right)' \right).
\end{aligned}$$

Substitution of the last expression for \mathcal{A} into the HJB equation (A17) yields the following partial differential equation for \hat{J} :

$$\begin{aligned}
-\frac{\partial \hat{J}}{\partial t} &= \text{tr} \left(\left(\Omega \Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \right)' \right) \mathcal{D} + 2 \Sigma \mathcal{D} Q' Q \mathcal{D} \right) \hat{J} \\
&\quad + \frac{\gamma-1}{\gamma} \hat{J} \left(-r - \frac{\text{tr}(\Sigma \lambda \lambda')}{2\gamma} \right) - \frac{1-\gamma}{2} \hat{J} \text{tr} \left((I_2 - \bar{\rho} \rho') \Sigma \left(\frac{\mathcal{D} \hat{J}}{\hat{J}} + \frac{\mathcal{D} \hat{J}'}{\hat{J}} \right) Q' Q \left(\frac{\mathcal{D} \hat{J}}{\hat{J}} + \frac{\mathcal{D} \hat{J}'}{\hat{J}} \right)' \right),
\end{aligned}$$

subject to the boundary condition $\hat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution,

$$\hat{J}(t, \Sigma) = \exp(B(t, T) + \text{tr}(A(t, T)\Sigma)),$$

for some state-independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for \hat{J} , the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$\begin{aligned}
-\frac{dB}{dt} &= \text{tr}(A \Omega \Omega') - \frac{\gamma-1}{\gamma} r, \\
-\text{tr} \left(\frac{dA}{dt} \Sigma \right) &= \text{tr} \left(\Gamma' A \Sigma + A \Gamma \Sigma + 2 A Q' Q A \Sigma - \frac{1-\gamma}{2} (A' + A) Q' (I_2 - \bar{\rho} \rho') Q (A' + A) \Sigma + C \Sigma \right),
\end{aligned}$$

with terminal conditions $B(T, T) = 0$ and $A(T, T) = 0_{2 \times 2}$, where

$$\Gamma = M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \lambda' \quad (\text{A21})$$

$$C = \frac{1-\gamma}{2\gamma^2} \lambda \lambda'. \quad (\text{A22})$$

For a symmetric matrix function A , the second differential equation implies the following matrix Riccati equation:

$$0_{2 \times 2} = \frac{dA}{dt} + \Gamma' A + A \Gamma + 2 A \Lambda A + C, \quad (\text{A23})$$

where

$$\Lambda = Q' (I_2 \gamma + (1-\gamma) \bar{\rho} \rho') Q. \quad (\text{A24})$$

This is the system of matrix Riccati equations in the statement of Proposition 2. These differential equations are completely integrable, so that closed-form expressions for \hat{J} (and hence for J) can be computed. For convenience, we consider coefficients A and B parameterized by $\tau = T - t$. This change of variable implies the following simple modification of the above system of equations:

$$\frac{dB}{d\tau} = \text{tr}(A\Omega\Omega') - \frac{\gamma-1}{\gamma}r, \quad (\text{A25})$$

$$\frac{dA}{d\tau} = \Gamma'A + A\Gamma + 2A\Lambda A + C, \quad (\text{A26})$$

subject to initial conditions $A(0) = 0_{2 \times 2}$ and $B(0) = 0$. Given a solution for A , function B is obtained by integration:

$$B(\tau) = \text{tr} \left(\int_0^\tau A(s)\Omega\Omega' ds \right) - \frac{\gamma-1}{\gamma} r \tau.$$

To solve equation (A26), we use Radon's Lemma. Let us represent the function $A(\tau)$ as

$$A(\tau) = H(\tau)^{-1}K(\tau), \quad (\text{A27})$$

where $H(\tau)$ and $K(\tau)$ are square matrices, with $H(\tau)$ invertible. Pre-multiplying (A26) by $H(\tau)$, we obtain

$$H \frac{dA}{d\tau} = H\Gamma'A + HA\Gamma + 2HA\Lambda A + HC. \quad (\text{A28})$$

Where no confusion may arise, we suppress the argument τ for brevity. On the other hand, in light of (A27), differentiation of

$$HA = K \quad (\text{A29})$$

results in

$$H \frac{dA}{d\tau} = \frac{d}{d\tau}(HA) - \frac{dH}{d\tau}A, \quad (\text{A30})$$

and

$$\frac{d}{d\tau}(HA) = \frac{dK}{d\tau}. \quad (\text{A31})$$

Substituting (A29), (A30), and (A31) into (A28) we get

$$\frac{dK}{d\tau} - \frac{dH}{d\tau}A = H\Gamma'A + K\Gamma + 2K\Lambda A + HC.$$

After collecting coefficients of A , we conclude that the last equation is equivalent to the following matrix system of ODEs:

$$\frac{dK}{d\tau} = K\Gamma + HC, \quad (\text{A32})$$

$$\frac{dH}{d\tau} = -2K\Lambda - H\Gamma', \quad (\text{A33})$$

or

$$\frac{d}{d\tau} \begin{pmatrix} K & H \end{pmatrix} = \begin{pmatrix} K & H \end{pmatrix} \begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix}.$$

The above ODE can be solved by exponentiation:

$$\begin{aligned} \begin{pmatrix} K(\tau) & H(\tau) \end{pmatrix} &= \begin{pmatrix} K(0) & H(0) \end{pmatrix} \exp \left[\tau \begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix} \right] \\ &= \begin{pmatrix} A(0) & I_2 \end{pmatrix} \exp \left[\tau \begin{pmatrix} \Gamma & -2\Lambda \\ C & -\Gamma' \end{pmatrix} \right] \\ &= \begin{pmatrix} A(0)F_{11}(\tau) + F_{21}(\tau) & A(0)F_{12}(\tau) + F_{22}(\tau) \end{pmatrix} \\ &= \begin{pmatrix} F_{21}(\tau) & F_{22}(\tau) \end{pmatrix}. \end{aligned}$$

We conclude from equation (A27) that the solution to (A26) is given by

$$A(\tau) = F_{22}(\tau)^{-1}F_{21}(\tau). \quad (\text{A34})$$

This concludes the proof. \square

Proof of Proposition 3: To recover the optimal portfolio policy, we have, from the proof of Proposition 2,

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)} \mathbb{E}[\xi_{\nu^*}(T) X^*(T) | \mathcal{F}_t] = \psi^{-\frac{1}{\gamma}} \xi_{\nu^*}(t)^{-\frac{1}{\gamma}} \widehat{J}(t, \Sigma(t)). \quad (\text{A35})$$

For the Wishart dynamics (4), Itô's Lemma applied to both sides of (A35) gives, for every state Σ ,

$$X^*(t) \text{tr} \left(\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Sigma^{1/2} dBL \right) = X^*(t) \text{tr} \left(\frac{1}{\gamma} \Theta'_{\nu^*} d\mathcal{B} + \frac{\mathcal{D}\widehat{J}}{\widehat{J}} \left(\Sigma^{1/2} dBUQ + Q'U' d\mathcal{B}' \Sigma^{1/2} \right) \right). \quad (\text{A36})$$

where matrix U is given by:

$$U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies

$$L \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \Sigma^{1/2} = \frac{1}{\gamma} (L\lambda' + \nu') \Sigma^{1/2} + 2UQA\Sigma^{1/2}.$$

Pre-multiplying both sides by L' , post-multiplying them by $\Sigma^{1/2}$, and recalling that $L'\nu'\Sigma = 0_{1 \times 2}$, we conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{\lambda}{\gamma} + 2AQ'\bar{\rho} = \frac{1}{\gamma} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + 2 \begin{bmatrix} (q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2)A_{11} + (q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2)A_{12} \\ (q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2)A_{22} + (q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2)A_{12} \end{bmatrix}. \quad (\text{A37})$$

This concludes the proof of the proposition. \square

Proof of Proposition 4: We apply the following lemma, similar to a result in Buraschi, Cieslak and Trojani (2007), to which we refer for a proof.

Lemma A1: *Consider the solution $A(\tau)$ of matrix Riccati equation (A26). If matrix C is negative semidefinite, then $A(\tau)$ is negative semidefinite and monotonically decreasing for any τ , that is, $A(\tau_2) - A(\tau_1)$ is a negative semidefinite matrix for any $\tau_2 > \tau_1$.*

Since $C = (1 - \gamma)/(2\gamma^2)\lambda\lambda'$, if $\gamma > 1$ then C is negative semidefinite. From Lemma IA.A1, $A(\tau)$ is also negative semidefinite. It follows that $A_{11}(\tau) \leq 0$ and $A_{22}(\tau) \leq 0$. Inequality $|A_{12}| \leq |A_{11} + A_{22}|/2$ follows from the properties of negative semidefinite matrices. Now consider a neighborhood of $\tau = 0$ of arbitrary small length ϵ . By the fundamental theorem of calculus, we have

$$A(\epsilon) = A(0) + \frac{dA(\tau)}{d\tau} \Big|_{\tau=0} \epsilon + o(\epsilon). \quad (\text{A38})$$

But $A(0) = 0$ and $\frac{dA(\tau)}{d\tau} \Big|_{\tau=0} = C$. If λ_1 and λ_2 agree in sign and $\gamma > 1$, then $C_{12} < 0$ and $A_{12}(\epsilon) < 0$. If, in addition, $|\lambda_1| > |\lambda_2|$, we have $\lambda_1^2 > \lambda_1\lambda_2 > \lambda_2^2$, that is, $|C_{11}| > |C_{12}| > |C_{22}|$. We conclude from (A38) that $|A_{11}| > |A_{12}| > |A_{22}|$. This concludes the proof of the proposition. \square

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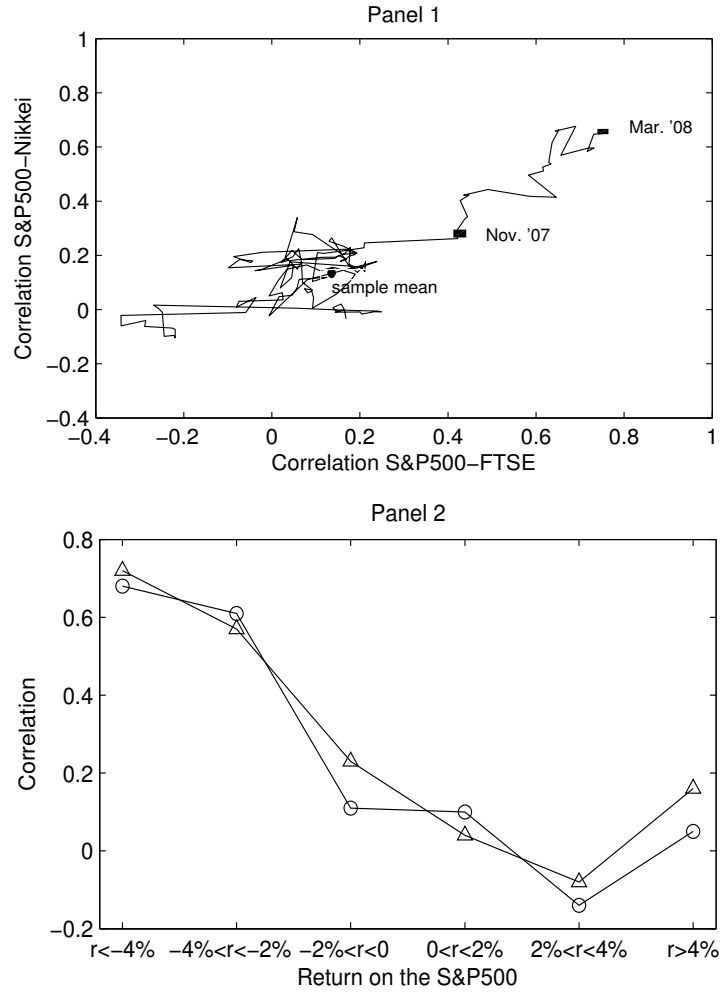


Figure 1. Joint correlation dynamics of stock index returns and leverage effect. Panel 1 shows sample correlations between the S&P500 and FTSE100 Index (x -axis) versus sample correlations between the S&P500 and Nikkei225 Index (y -axis). Each point in the graph represents couples of realized sample correlations between these stock indices. Sample correlations for the time period April 2005 to April 2008 are computed applying the methodology proposed by Andersen et al. (2003) on high frequency data to obtain realized volatilities and covariances from daily quadratic variations and covariations of log futures prices. Panel 2 shows monthly returns on the S&P500 index (from January 1987 to April 2008) have been divided into six equal size bins. The y -axis of this figure shows the average empirical correlations between the S&P500 and the Nikkei225 (circles), and between the S&P500 and the FTSE100 (triangles), given S&P500 return realizations. Return bins are reported on the x -axis.

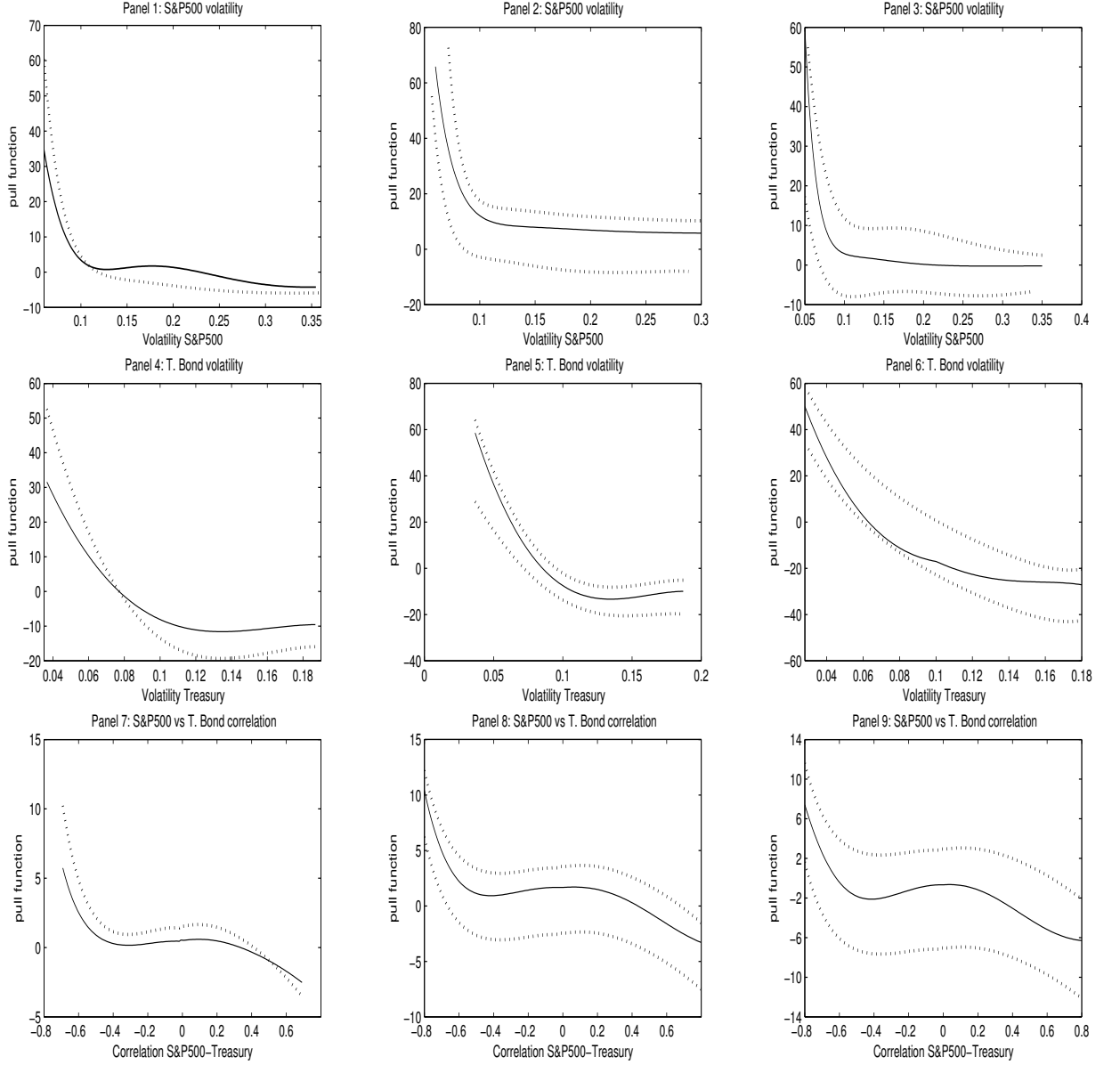


Figure 2. Pull function of the volatility and correlation processes. Panels 1, 4, and 7 show the nonparametric pull functions for weekly and monthly data frequencies (dotted and solid lines, respectively) based on the estimation procedure in Conley et al. (1997). Panels 1 and 4 present the pull function estimates for the conditional volatility of S&P500 Index and 30-year Treasury bond futures returns, respectively. Panel 7 plots the pull function estimate for their conditional correlation. Panels 2, 5, and 8 present the corresponding pull function estimates for volatilities and correlations, but based on a long time series of weekly volatilities and correlations simulated from the Wishart variance-covariance process (4) using the weekly parameter estimates in Table I. Panels 3, 6, and 9 present corresponding pull function estimates for volatilities and correlations, but based on a long time series of monthly volatilities and correlations simulated from the Wishart variance-covariance process (4) using the monthly parameter estimates in Table I. In Panels 2,3,5,6,8, and 9, the model implied pull functions are plotted together with a 95% confidence interval around the empirical pull functions of Panels 1, 4 and 7.

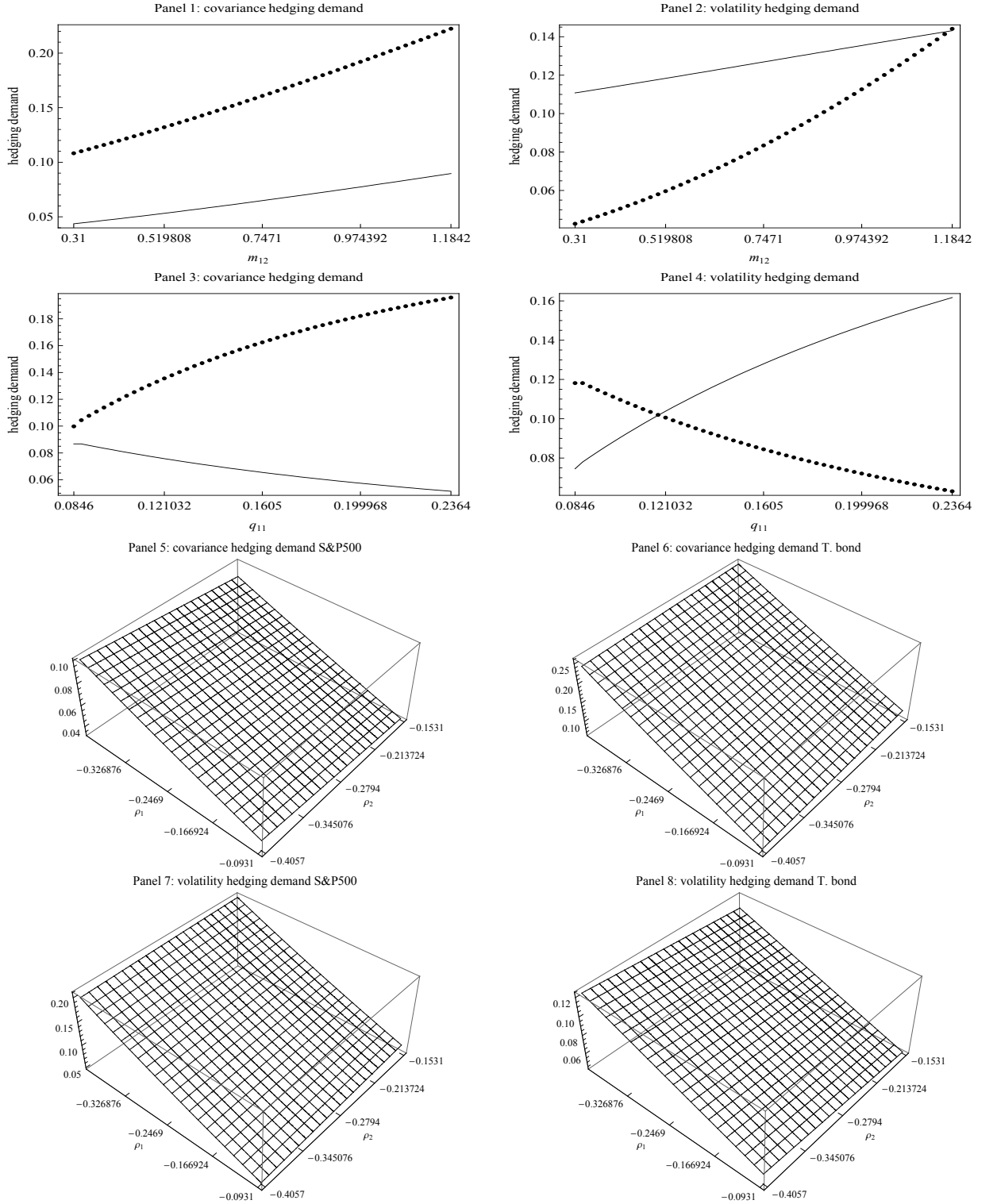


Figure 3. Comparative statics of optimal hedging demands. This figure shows comparative statics for optimal covariance hedging demands (Panels 1, 3, 5, and 6) and volatility hedging demands (Panels 2, 4, 7, and 8), obtained by letting the values of parameters m_{12} (Panels 1 and 2), q_{11} (Panel 3 and 4), $\bar{\rho}_1$, and $\bar{\rho}_2$ (Panels 7 to 10) vary in a confidence interval of one sample standard deviation around the monthly parameter estimates in Table I. m_{ij} and q_{ij} are the entries of parameter matrices M and Q , respectively, appearing in the Wishart covariance matrix dynamics:

$$d\Sigma(t) = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)\Sigma^{1/2}(t).$$

$\bar{\rho}_1$ and $\bar{\rho}_2$ are the entries of the vector $\bar{\rho}$, which controls leverage by means of the following relation:

$$dW(t) = dB(t)\bar{\rho} + \sqrt{1 - \bar{\rho}'\bar{\rho}}dZ(t),$$

where $W(t)$ is the Brownian motion driving asset returns. In Panels 1 to 4, hedging demands for the S&P500 index (30-year Treasury bond) futures are plotted with solid (dotted) lines. A relative risk aversion coefficient of five and an investment horizon of six years have been assumed.

Table I
Estimation Results for the Model with Two Risky Assets

This table shows estimated matrices M and Q and vectors λ and $\bar{\rho}$ for the returns dynamics (1) under the Wishart variance-covariance diffusion process

$$d\Sigma(t) = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t),$$

where we have set $\Omega\Omega' = kQ'Q$ and $k = 10$. Parameters are estimated by GMM using time series of returns and realized variance-covariance matrices for S&P500 Index and 30-year Treasury bond futures returns, computed at both a weekly and a monthly frequency. The detailed set of moment restrictions used for GMM estimation is given in Internet Appendix A. We report parameter estimates and their standard errors (in parentheses), together with the p -values for Hansen's J-test of overidentifying restrictions. An asterisk denotes estimated parameters that are *not* significant at the 5% significance level.

	M		Q		λ	$\bar{\rho}$
<i>weekly</i>	-1.210 (0.330)	0.491 (0.203)	0.167 (0.047)	0.033* (0.027)	4.722 (1.910)	-0.115 (0.058)
	0.329 (0.127)	-1.271 (0.363)	0.001* (0.020)	0.090 (0.044)	3.317 (1.442)	-0.549 (0.250)
<i>monthly</i>	-1.122 (0.410)	0.747 (0.368)	0.160 (0.063)	0.083 (0.0413)	4.612 (1.182)	-0.279 (0.139)
	0.884* (0.556)	-0.888 (0.235)	-0.021* (0.028)	0.009* (0.118)	2.891 (1.342)	-0.247 (0.086)
<i>p</i> -values for Hansen's J-test		<i>Weekly</i> : 0.14		<i>Monthly</i> : 0.38		

Table II
Optimal Hedging Demands for the Model with Two Risky Assets

This table shows optimal covariance and volatility hedging demands as a percentage of the myopic portfolio for different investment horizons and relative risk aversion parameters. The last column of each panel reports the myopic portfolio. We compute these demands for monthly parameter estimates reported in Table I. Each entry in the table is a vector with two components: the first component is the demand for the S&P500 Index futures and the second one is the demand for the 30-year Treasury futures.

	<i>RRR</i>	<i>T</i>	<i>3m</i>	<i>6m</i>	<i>1y</i>	<i>2y</i>	<i>5y</i>	<i>7y</i>	<i>10y</i>	Myopic demand
Covariance hedging (/ myopic)	2		0.0090 0.0221	0.0170 0.0420	0.0293 0.0728	0.0418 0.1040	0.0472 0.1172	0.0473 0.1175	0.0474 0.1175	2.3060 1.4455
			0.0150 0.0370	0.0281 0.0698	0.0472 0.1171	0.0625 0.1550	0.0665 0.1655	0.0670 0.1659	0.0670 0.1659	0.7687 0.4818
	6		0.0156 0.0388	0.0295 0.0733	0.0493 0.1225	0.0648 0.1610	0.0686 0.1705	0.0687 0.1705	0.0687 0.1705	0.5765 0.3641
			0.0162 0.0404	0.0307 0.0761	0.0511 0.1267	0.0666 0.1654	0.0704 0.1747	0.0705 0.1748	0.0705 0.1748	0.4193 0.2628
	11		0.0168 0.0417	0.0316 0.0785	0.0525 0.1303	0.0682 0.1692	0.0718 0.1782	0.0718 0.1783	0.0718 0.1784	0.2883 0.1807
			0.0170 0.0423	0.0321 0.0797	0.0532 0.1322	0.0690 0.1712	0.0726 0.1800	0.0727 0.1801	0.0727 0.1802	0.2196 0.1377
	16		0.0175 0.0434	0.0329 0.0816	0.0544 0.1350	0.0702 0.1742	0.0737 0.1829	0.0740 0.1832	0.0740 0.1832	0.1125 0.0705
	21									
Volatility hedging (/ myopic)	2		0.0205 0.0097	0.0360 0.0195	0.0594 0.0364	0.0800 0.0553	0.0883 0.0640	0.0883 0.0640	0.0883 0.0640	2.3060 1.4455
			0.0346 0.0160	0.0622 0.0317	0.0983 0.0568	0.1245 0.0789	0.1309 0.0852	0.1309 0.0852	0.1309 0.0852	0.7687 0.4818
	6		0.0364 0.0168	0.0654 0.0332	0.1031 0.0591	0.1297 0.0814	0.1360 0.0874	0.1360 0.0874	0.1360 0.0874	0.5765 0.3641
			0.0379 0.0174	0.0681 0.0345	0.1070 0.0610	0.1340 0.0834	0.1401 0.0892	0.1401 0.0892	0.1401 0.0892	0.4193 0.2628
	11		0.0391 0.0179	0.0703 0.0355	0.1103 0.0626	0.1376 0.0850	0.1434 0.0906	0.1434 0.0906	0.1434 0.0906	0.2883 0.1807
			0.0398 0.0182	0.0726 0.0360	0.1128 0.0634	0.1397 0.0858	0.1452 0.0913	0.1452 0.0913	0.1452 0.0913	0.2196 0.1377
	16		0.0408 0.0187	0.0733 0.0368	0.1147 0.0646	0.1423 0.0871	0.1479 0.0924	0.1480 0.0924	0.1480 0.0924	0.1125 0.0705
	21									

Internet Appendix

for

“Correlation Risk and Optimal Portfolio Choice”*

THIS INTERNET APPENDIX includes four subappendices, identified by roman letters from A to D. Internet Appendix A provides details about the estimation procedure that we use in the empirical application of Section II of the article. In particular, it derives in closed form the GMM moment restrictions. In Internet Appendix B we solve and discuss a model specification alternative to that considered in Section I of the article, namely a specification that comprises stochastic interest rates and constant risk premia. Internet Appendix C reports results for the discrete-time analog of the continuous-time specification of the article. In this discrete-time setting, we also discuss the implications of short selling and VaR-type constraints on portfolio allocations. Finally, Internet Appendix D reports tables and graphs in support of the robustness checks and extensions provided in the article.

Propositions, lemmas, and equation numbers are prefixed with the letter that identifies the appendix. Numbers without prefix refer to Propositions, lemmas, or equations in the main text. Tables and figures of this appendix are labeled as “Table (Figure) IA.LX”, where X denotes the number of the Table (Figure) and L is the letter that identifies the Appendix.

* Citation format: Buraschi, Andrea, Paolo Porchia, and Fabio Trojani, Internet Appendix to “Correlation Risk and Optimal Portfolio Choice”, *Journal of Finance* [vol #], [pages], [http://www.afajof.org/IA/\[year\]](http://www.afajof.org/IA/[year]). Please note: Wiley-Blackwell is not responsible for the content or functionality of any supporting information supplied by the authors. Any queries (other than missing material) should be directed to the authors of the article.

A. Moment Restrictions for the GMM Estimation

This Appendix provides detailed expressions for the moment conditions used in the GMM estimation of our model. The following computations make use of the closed-form expressions for the moments of the Wishart process, which can be found, for example, in the Appendix of Buraschi, Cieslak, and Trojani (2007).

Let τ denote data sampling frequency. We have $\tau = 5/250$ for weekly data and $\tau = 22/250$ for monthly data.

1) Unconditional risk premia of log returns.

The conditional risk premia of asset i 's logarithmic returns, at frequency τ , $i = 1, 2$, are given by

$$\mathbb{E}_t [\log S^i(t + \tau) - \log S^i(t)] - \int_t^{t+\tau} r ds = \mathbb{E}_t \left[\int_t^{t+\tau} e_i' \Sigma(s) \left(\lambda - \frac{1}{2} e_i \right) ds \right]. \quad (\text{IA.A1})$$

The unconditional risk premia are thus

$$M_1 = \left(\mathbb{E}[\Sigma(t)] \lambda - \frac{1}{2} \begin{bmatrix} e_1' \mathbb{E}[\Sigma(t)] e_1 \\ e_2' \mathbb{E}[\Sigma(t)] e_2 \end{bmatrix} \right) \tau. \quad (\text{IA.A2})$$

2) Unconditional mean of the realized variance-covariance matrix of log returns.

$$M_2 = \text{vech}(\mathbb{E}[\Sigma(t)]) \tau, \quad (\text{IA.A3})$$

where $\text{vech}(X)$ denotes the lower triangular vectorization of a square matrix X .

3) Unconditional second moment of the realized variance-covariance matrix of log-returns.

$$\begin{aligned} \mathbb{E} \left[\left(\int_t^{t+\tau} \text{vec}(\Sigma(s)) ds \right) \left(\int_t^{t+\tau} \text{vec}(\Sigma(s)) ds \right)' \right] &= 2 \int_0^\tau dr_2 \int_0^{r_2} dr_1 \mathbb{E} [\text{vec}(\Sigma(r_1)) \text{vec}(\Sigma(r_2))'] \quad (\text{IA.A4}) \\ &= 2 \int_0^\tau dr_2 \int_0^{r_2} dr_1 \mathbb{E} [\text{vec}(\Sigma(r_1)) \mathbb{E}_{r_1} [\text{vec}(\Sigma(r_2))']] \\ &= 2 \left(\mathbb{E} [\text{vec}(\Sigma(r_1)) \text{vec}(\Sigma(r_1))'] \int_0^\tau dr_2 \int_0^{r_2} dr_1 (\exp(M'(r_2 - r_1)) \otimes \exp(M'(r_2 - r_1))) \right. \\ &\quad \left. + \mathbb{E} [\text{vec}(\Sigma(r_1))] \int_0^\tau dr_2 \int_0^{r_2} dr_1 \text{vec} \left(\int_0^{r_2 - r_1} \exp(sM) k Q' Q \exp(sM') \right)' ds \right). \end{aligned}$$

Therefore,

$$M_3 = \text{vech} \left(\mathbb{E} \left[\left(\int_t^{t+\tau} \text{vec}(\Sigma(s)) ds \right) \left(\int_t^{t+\tau} \text{vec}(\Sigma(s)) ds \right)' \right] \right).$$

4) Unconditional covariance between assets' simple excess returns and the variance-covariance matrix of log returns.

For asset i , $i = 1, 2$, and $s > t$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}_t \left[\exp \left(\int_s^{s+\tau} e_i \Sigma(u) dW(u) + \int_s^{s+\tau} e_i \Sigma(u) \lambda du - \frac{1}{2} \int_s^{s+\tau} e_i \Sigma(u) e_i' du \right) \otimes \int_s^{s+\tau} \Sigma(u) du \right] &= \\ \exp \left(A_i(\tau) + \tilde{A}_i(\infty) \right) \left(\int_0^\tau \exp(\tilde{M}(u)u) \mathbb{E}[\Sigma(t)] \exp(\tilde{M}(u)'u) du + \int_0^\tau \int_0^s \exp(\tilde{M}(u)u) k Q' Q \exp(\tilde{M}(u)'u) du ds \right), \end{aligned} \quad (\text{IA.A5})$$

where

$$\begin{aligned} \tilde{M}(\tau) &= M + Q' \rho e_i' + Q' Q B_i(\tau) \\ A_i(\tau) &= k \int_0^\tau \text{tr}(B_i(s) Q' Q) ds \\ \tilde{A}_i(\infty) &= k \int_0^\infty \text{tr}(\tilde{B}_i(s) Q' Q) ds \\ B_i(t) &= B_{22}(t)^{-1} B_{21}(t) \\ \tilde{B}_i(t) &= (B_i(t) \tilde{B}_{12}(t) + \tilde{B}_{22}(t))^{-1} B_i(t) \tilde{B}_{11}(t) \end{aligned}$$

and

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} = \exp \left[t \begin{pmatrix} M + Q' \rho e'_i & -2Q'Q \\ \lambda e'_i & -(M + Q' \rho e'_i)' \end{pmatrix} \right]$$

$$\begin{pmatrix} \tilde{B}_{11}(t) & \tilde{B}_{12}(t) \\ \tilde{B}_{21}(t) & \tilde{B}_{22}(t) \end{pmatrix} = \exp \left[t \begin{pmatrix} M & -2Q'Q \\ 0 & -M' \end{pmatrix} \right].$$

The last set of moment conditions is therefore given by

$$M_{3+i} = \text{vech} \left(\exp \left(A_l(\tau) + \tilde{A}_l(\infty) \right) \times \left(\int_0^\tau \exp(\tilde{M}(u)u) \mathbb{E}[\Sigma(t)] \exp(\tilde{M}(u)'u) du + \int_0^\tau \int_0^s \exp(\tilde{M}(u)u) k Q' Q \exp(\tilde{M}(u)'u) du ds \right) \right) \quad (\text{IA.A6})$$

for $i = 1, 2$. Summarizing, the vector-valued function $\mu^\tau(M, Q, \lambda, \rho, k)$ of theoretical moment conditions, for sampling frequency τ , is given by

$$\mu^\tau(M, Q, \lambda, \rho, k) = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix}.$$

In our GMM estimation, this is compared to its empirical counterpart $\hat{\mu}^\tau$ based on historical returns, volatilities, and covariances.

B. Constant Risk Premia and Stochastic Interest Rate

A direct way to study pure variance-covariance hedging demands is by assuming a constant risk premium. For analytical purposes, this comes at the cost of specifying a Wishart state process for the precision matrix Σ^{-1} , which implies a less transparent interpretation of some model parameters. This can be achieved even in a setting with a stochastic interest rate, where the interest rate can also depend on some of the risk factors driving the covariance matrix of asset returns.¹

ASSUMPTION IA.B1: *Let the process Y satisfy the following Wishart dynamics:*

$$dY(t) = [\Omega\Omega' + MY(t) + Y(t)M']dt + Y^{1/2}(t)dBQ + Q'dB'Y^{1/2}(t), \quad (\text{IA.B1})$$

where matrices Ω , M , and Q are now of dimension 3×3 and where B is a 3×3 matrix of independent standard Brownian motions. We model Σ^{-1} as a projection of matrix Y :

$$\Sigma^{-1} = SY S',$$

where the 2×3 matrix S is such that $SS' = id_{2 \times 2}$.² The stochastic riskless rate $r(t)$ is defined by

$$r(t) = r_0 + \text{tr}(Y(t)D), \quad (\text{IA.B2})$$

where $r_0 > 0$ and D is a 3×3 matrix.

Notice that the nonnegativity of $r(t)$ can be easily ensured simply by assuming that matrix D is positive definite. Since $\Sigma^{-1} = SY S'$, we define $\Sigma^{-1/2}$ as the 2×3 matrix $SY^{-1/2}$. Since $\Sigma^{-1/2}\Sigma^{1/2'} = id_{2 \times 2}$, it is natural to define $\Sigma^{1/2}$ as the 2×3 matrix $SY^{1/2}$. We introduce the following process for asset returns:

$$dS(t) = I_S \left[\begin{pmatrix} r(t) + \mu_1^e \\ r(t) + \mu_2^e \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t) \right], \quad (\text{IA.B3})$$

where the excess return vector $\mu^e = (\mu_1^e, \mu_2^e)' \in \mathbb{R}^2$ is constant and $r(t)$ is given by equation (IA.B2). To model leverage effects, we define the standard Brownian motion W as

$$W(t) = \sqrt{1 - \bar{\rho}'\bar{\rho}}Z(t) + B(t)\bar{\rho}, \quad (\text{IA.B4})$$

where Z is a three-dimensional standard Brownian motion independent of B and $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)'$ is a vector of correlation parameters such that $\bar{\rho}_i \in [-1, 1]$ and $\bar{\rho}'\bar{\rho} \leq 1$.

This setting is effectively a six-factor model with some interest rate risk factors that might be linked to the covariance matrix of stock returns, depending on the form of the matrix D in equation (IA.B2). The squared Sharpe ratio in this model is affine in Y . Therefore, we can solve in closed form the dynamic portfolio choice problem in this extended dynamic setting as well.

PROPOSITION IA.B1: *The solution to the portfolio problem for the return dynamics (IA.B1) to (IA.B3) and under a stochastic interest rate (IA.B2) is*

$$J(X_0, Y_0) = \frac{X_0^{1-\gamma} \hat{J}(0, Y_0)^\gamma - 1}{1-\gamma},$$

where

$$\hat{J}(t, Y) = \exp(B(t, T) + \text{tr}(A(t, T)Y)),$$

with $B(t, T)$ and the symmetric matrix-valued function $A(t, T)$ solving in closed form the following system of matrix Riccati differential equations:

$$-\frac{dB}{dt} = -\frac{\gamma-1}{\gamma}r_0 + \text{tr}(A\Omega\Omega'), \quad (\text{IA.B5})$$

$$-\frac{dA}{dt} = \Gamma'A + A'\Gamma + 2A'\Lambda A + C, \quad (\text{IA.B6})$$

subject to $B(T, T) = 0$ and $A(T, T) = 0$. In these equations, the coefficients Γ , Λ , and C are given by

$$\begin{aligned}\Gamma &= M - \frac{\gamma}{\gamma-1} Q' \bar{\rho} \mu^{e'} S \\ \Lambda &= Q' (\gamma I_3 + (1-\gamma) \bar{\rho} \bar{\rho}') Q \\ C &= \frac{1-\gamma}{2\gamma^2} S' \mu^e \mu^{e'} S - \frac{\gamma-1}{\gamma} D.\end{aligned}$$

Finally, the optimal policy for this portfolio problem reads

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu^e + 2 \Sigma^{-1} S A Q' \bar{\rho}. \quad (\text{IA.B7})$$

Proof. Analogous to the proof of Proposition 2, we rewrite the innovation component of the opportunity set dynamics as $\Sigma^{1/2} [Z, B] L$, with $L = [\sqrt{1 - \bar{\rho}' \bar{\rho}}, \bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3]'$. By definition, the market price of risk Θ_ν satisfies

$$\Sigma^{1/2} \Theta_\nu L = \mu^e, \quad (\text{IA.B8})$$

from which

$$\Theta_\nu = \Sigma^{-1/2'} \mu^e L' + Y^{1/2} \nu, \quad (\text{IA.B9})$$

where $\Sigma^{-1/2} = S Y^{1/2}$ and ν is a 3×4 matrix-valued process such that $\nu L = 0_{3 \times 1}$, that is $\nu = [-\bar{\nu} \frac{\bar{\rho}}{\sqrt{1 - \bar{\rho}' \bar{\rho}}}, \bar{\nu}]$. $\bar{\nu}$ is a 3×3 matrix that prices the shocks that drive the Wishart state variable Y .

It turns out, that the value function can be written in the form:

$$J(x, Y_0) = x^\gamma \inf_\nu \frac{1}{1-\gamma} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}} \right]^\gamma - \frac{1}{1-\gamma} = \frac{x^{1-\gamma} \hat{J}(0, Y_0)^\gamma - 1}{1-\gamma},$$

where

$$\begin{aligned}\mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma-1}{\gamma}} \right] &= \mathbb{E}^\gamma \left[e^{-\frac{\gamma-1}{\gamma} \int_0^T r(s) ds + \frac{1-\gamma}{2\gamma^2} \text{tr}(\int_0^T \Sigma(s)^{-1} ds \mu^e \mu^{e'} + \int_0^T Y(s) ds \bar{\nu}' \bar{\nu} (I_3 + \frac{\bar{\rho} \bar{\rho}'}{1 - \bar{\rho}' \bar{\rho}}))} \right] \\ &= \mathbb{E}^\gamma \left[e^{-\frac{\gamma-1}{\gamma} (r_0 + \text{tr}(\int_0^T Y(s) ds D)) + \frac{1-\gamma}{2\gamma^2} \text{tr}(\int_0^T Y(s) ds (S' \mu^e \mu^{e'} S + \bar{\nu}' \bar{\nu} (I_3 + \frac{\bar{\rho} \bar{\rho}'}{1 - \bar{\rho}' \bar{\rho}})))} \right], \quad (\text{IA.B10})\end{aligned}$$

for a probability measure P^γ defined by the density

$$\frac{dP^\gamma}{dP} = e^{-\text{tr} \left(\frac{\gamma-1}{\gamma} \int_0^T \Theta'_\nu(s) dB + \frac{1}{2} \frac{(\gamma-1)^2}{\gamma^2} \int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds \right)}.$$

The dynamics of Y under the probability P^γ are

$$\begin{aligned}dY(t) &= \left[\Omega \Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' (\bar{\rho} \mu^{e'} S + \bar{\nu}') \right) Y(t) + Y(t) \left(M - \frac{\gamma-1}{\gamma} Q' (\bar{\rho} \mu^{e'} S + \bar{\nu}') \right)' \right] dt \\ &\quad + Y^{1/2}(t) dB^\gamma(t) Q + Q' dB(t) \gamma' Y^{1/2}(t).\end{aligned} \quad (\text{IA.B11})$$

These dynamics are affine in Y . It follows that the function \hat{J} is a solution of the following HJB equation:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\bar{\nu}} \left\{ \mathcal{A} \hat{J} + \hat{J} \left[-\frac{\gamma-1}{\gamma} (r_0 + \text{tr}(Y D)) + \frac{1-\gamma}{2\gamma^2} \text{tr} \left(Y \left(S' \mu^e \mu^{e'} S + \bar{\nu}' \bar{\nu} \left(I_3 + \frac{\bar{\rho} \bar{\rho}'}{1 - \bar{\rho}' \bar{\rho}} \right) \right) \right) \right] \right\}, \quad (\text{IA.B12})$$

subject to the terminal condition $\hat{J}(T, Y) = 1$, where \mathcal{A} is the infinitesimal generator of the matrix-valued diffusion (IA.B11), which is given by

$$\begin{aligned}\mathcal{A} &= \text{tr} \left(\left(\Omega \Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' (\bar{\rho} \mu^{e'} S + \bar{\nu}') \right) Y + Y \left(M - \frac{\gamma-1}{\gamma} Q' (\bar{\rho} \mu^{e'} S + \bar{\nu}') \right)' \right) \mathcal{D} \right) \\ &\quad + \text{tr}(2Y \mathcal{D} Q' Q \mathcal{D}).\end{aligned} \quad (\text{IA.B13})$$

The generator is affine in Y . As in the proof of Proposition 2, the optimality condition for the optimal control $\bar{\nu}$ yields

$$\bar{\nu} = -\gamma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q' \left(I_3 + \frac{\bar{\rho}\rho'}{1 - \bar{\rho}'\bar{\rho}} \right)^{-1}. \quad (\text{IA.B14})$$

Note that $\left(I_3 + \frac{\bar{\rho}\rho'}{1 - \bar{\rho}'\bar{\rho}} \right)^{-1} = I_3 - \bar{\rho}\rho'$. Substituting the expression for $\bar{\nu}$ into equation (IA.B12), we obtain the following partial differential equation for \hat{J} :

$$\begin{aligned} -\frac{\partial \hat{J}}{\partial t} = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \mu^{e'} S \right) Y + Y \left(M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \mu^{e'} S \right)' \right) \mathcal{D} + 2Y \mathcal{D} Q' Q \mathcal{D} \right) \hat{J} \\ & + \frac{\gamma-1}{\gamma} \hat{J} \left(-r_0 - \text{tr}(YD) - \frac{\text{tr}(Y S' \mu^e \mu^{e'} S)}{2\gamma} \right) \\ & - \frac{1-\gamma}{2} \hat{J} \text{tr} \left((I_3 - \bar{\rho}\rho') Y \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q' Q \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right)' \right), \end{aligned}$$

subject to the boundary condition $\hat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution:

$$\hat{J}(t, \Sigma) = \exp(B(t, T) + \text{tr}(A(t, T)Y),$$

for some state-independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for \hat{J} , the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$\begin{aligned} -\frac{dB}{dt} &= \text{tr}(A\Omega\Omega') - \frac{\gamma}{\gamma-1} r_0, \\ -\text{tr} \left(\frac{dA}{dt} Y \right) &= \text{tr} \left(\Gamma' A Y + A \Gamma Y + 2A Q' Q A Y - \frac{1-\gamma}{2} (A' + A) Q' (I_3 - \bar{\rho}\rho') Q (A' + A) Y + C Y \right), \end{aligned}$$

with terminal conditions $B(T, T) = 0$ and $A(T, T) = 0_{3 \times 3}$, where

$$\Gamma = M - \frac{\gamma-1}{\gamma} Q' \bar{\rho} \mu^{e'} S \quad (\text{IA.B15})$$

$$C = \frac{1-\gamma}{2\gamma^2} S' \mu^e \mu^{e'} S - \frac{1-\gamma}{\gamma} D. \quad (\text{IA.B16})$$

Explicit solutions for $B(t, T)$ and $A(t, T)$ are computed as in the proof of Proposition 2. By the same argument applied in the Proof of Proposition 3, the following equality must hold:

$$X^*(t) \text{tr} \left([\pi_1 \quad \pi_2] \Sigma^{1/2} d\mathcal{B} L \right) = X^*(t) \text{tr} \left(\frac{1}{\gamma} \Theta'_{\nu^*} d\mathcal{B} + \frac{\mathcal{D}\hat{J}}{\hat{J}} \left(Y^{1/2} d\mathcal{B} U Q + Q' U' d\mathcal{B}' Y \right) \right). \quad (\text{IA.B17})$$

where matrix U is given by

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This implies

$$L [\pi_1 \quad \pi_2] \Sigma^{1/2} = \frac{1}{\gamma} \left(L \mu^{e'} \Sigma^{-1/2} + \nu' Y^{1/2} \right) + 2U Q A Y^{1/2}.$$

Pre-multiplying both sides by L' , post-multiplying them by $\Sigma^{-1/2'}$, and recalling that $L' \nu' = 0_{1 \times 3}$ and $\Sigma^{-1/2} = S Y^{1/2}$, we conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu^e + 2 \Sigma^{-1} S A Q' \bar{\rho}.$$

This concludes the proof of Proposition IA.B1.: \square

The optimal policy (IA.B7) consists of a myopic and an intertemporal hedging portfolio, which are both proportional to the stochastic inverse covariance matrix. As noted by Chacko and Viceira (2005), in the univariate setting the relative size of the hedging and myopic demands is independent of the current level of volatility. This property also holds in the multivariate case, in the sense that both policies are proportional to the inverse covariance matrix Σ^{-1} .

We investigate the empirical implications of this specification in a scenario where, for simplicity, a constant interest rate ($D = 0$) has been assumed. This setting is the exact multivariate extension of the univariate model considered in Chacko and Viceira (2005). We use the same basic GMM estimation procedure and the same data used for the empirical application in the main text,³ but we now apply it to the information matrix Σ^{-1} . The GMM moment restrictions for the variance-covariance matrix process are replaced by those for the precision process, which is assumed to follow a Wishart diffusion process. Table IA.BI, Panel A, presents estimation results for the model with a constant risk premium. Panel B summarizes the estimated hedging demands.

Insert Table IA.BI about here

The myopic portfolio is time varying, via the variation of the inverse covariance matrix Σ^{-1} . This time variation is also partly reflected in the time variation of hedging demands. All in all, the absolute size of total hedging demands is comparable to that obtained in the main text for a constant market price of variance-covariance risk. For example, for a risk aversion parameter $\gamma = 6$ and an investment horizon of $T = 5$ years, the average hedging demand is approximately 23% of the myopic portfolio. Similar demands obtain for higher risk aversions and investment horizons.

C. Discrete-time Solution and Portfolio Constraints

C.1 Discrete-time Solution

In our model, the optimal dynamic trading strategy is given by a portfolio that must be rebalanced continuously over time. In practice, this can at best be an approximation, because trading is only possible at discrete trading dates. Moreover, transaction costs, liquidity constraints, or policy disclosure considerations might further constrain investors from frequent portfolio rebalancing. Even if we do not model these frictions explicitly in our setting, it is interesting to study the impact of discrete trading on the optimal hedging strategy in the context of our model.

Several studies have found that, as long as the investment opportunity set does not contain derivatives, the gains/losses of the optimal discrete-time portfolio policy with respect to a naively discretized continuous-time policy are small. See, for instance, Campbell et al. (2004) and Branger, Breuer, and Schlag (2006). We study whether similar conclusions hold in our multivariate portfolio choice setting. We consider the exact discrete-time process implied by the continuous-time model (1) to (4) of the main text, in which observations are generated at fixed, evenly spaced, points in time. The parameters of the continuous-time model have been estimated by GMM using the exact discrete-time moments of this process. The moments are easily obtainable in closed form for each sampling frequency because the Wishart process allows for aggregation over time. By construction, the estimated parameters are then consistent with the discrete time transition density of the process, which is the one relevant to study optimal portfolio choice in discrete-time.

The discrete-time portfolio choice problem does not allow for closed-form solutions. Therefore, we rely on standard numerical methods to compute the optimal portfolio strategies. Table IA.CI presents the total hedging demands in S&P500 Futures (π_1) and Treasury Futures (π_2), as fractions of the myopic demand. The transition density used for the discrete time portfolio optimization is the one implied by the estimated continuous time model with monthly returns, realized volatilities, and realized correlations.

Insert Table IA.CI about here

We focus on optimal portfolios that can be rebalanced monthly, but we also compute optimal strategies using a weekly and daily rebalancing frequency in order to verify the convergence of our numerical solution to the continuous-time portfolio problem solution. At a daily frequency, the hedging demands in the discrete-time model are virtually indistinguishable from the continuous-time hedging demands reported in Table III of the main text. Consistent with the findings in the literature, the discrete-time optimal hedging demands for the monthly frequency are close to the hedging demands computed from the continuous-time model: the mean absolute difference between the hedging demands using daily and monthly rebalancing is less than 10% of the hedging demand implied by a monthly rebalancing frequency. These findings suggest that the main implications derived from the continuous-time multivariate portfolio choice solutions are realistic even in the context of monthly rebalancing.

C.2 Portfolio Constraints

Portfolio constraints are useful to avoid unrealistic portfolio weights, which can potentially arise due to some extreme assumptions on expected returns, volatilities, and correlations, or from inaccurate point estimates of the model parameters. The empirical results of the previous sections can imply, for instance, levered portfolios in settings of low risk aversion. For instance, for a relative risk aversion of $\gamma = 2$, the optimal portfolio of an investor with horizon $T = 5$ years implies an investment of approximately 260% of the total wealth in stocks and 170% in bonds. Intuitively, constraints on short selling or on the portfolio VaR tend to constrain the investor from selecting optimal portfolios that are excessively levered. Therefore, it is interesting to study these types of portfolio constraints and their impact on the volatility and correlation hedging demands in our setting. We solve the discrete-time portfolio choice problem in the last section and additionally impose, in two separate steps, short-selling and VaR constraints. In order to quantify the correlation and volatility hedging components, we numerically compute the projection of the total hedging demand on the implied elasticity of the indirect marginal utility of wealth with respect to volatilities and covariances.

In the first exercise, we consider state-independent constraints on the optimal portfolio weights. For every fraction π_i of total wealth invested in the risky asset i , we first enforce a short-selling constraint $\pi_i \geq 0$. In a second step, we also consider a less severe position limit $\pi_i \geq -1$. Table IA.CII presents the

optimal volatility and covariance hedging demands implied by these two settings. Note that even in cases where the current constraint might not be binding, the optimal hedging strategy is different from the one implied by the unconstrained solution. This feature exists because the future opportunity set is restricted by the fact that the constraint might be binding, with some probability, in the future. The indirect marginal utility of wealth in the constrained problem depends on the strength of this effect. Therefore, the optimal intertemporal hedging demand is different.

Insert Table IA.CII about here

Table IA.CII shows that the more severe the constraint is, the smaller are the absolute demands for volatility and covariance hedging as a percent of the myopic portfolio. However, the impact of the constraint is quite moderate, even in the short-selling case, and does not greatly influence the relative size of the hedging demands against volatility and covariance risk across assets. For instance, for an investment horizon of $T = 10$ years and a risk aversion of $\gamma = 2$, the average covariance (volatility) hedging demand is 10.5% (7%) in the unconstrained case and 8.5% (6.5%) in the setting with short selling constraints. For a higher risk aversion of $\gamma = 8$, the average covariance (volatility) hedging demand is 13.25% (10.25%) in the unconstrained case and 10.75% (9%) in the setting with short-selling constraints. These findings are consistent with the state-independent nature of the constraint used, which is not a function of the conditional covariance matrix of returns. The slightly larger percentage decrease in the hedging demands of low risk-aversion investors in the constrained case is mainly due to their large myopic demands in the unconstrained portfolio problem.

The results are different when we study the effects of (state-dependent) VaR constraints. At each trading date, we impose a constant upper bound on the VaR of the optimally invested wealth at the next trading date. We use a VaR at a confidence level of 99%. Since the VaR is computed for a monthly rebalancing frequency and investment horizons longer than one month, the VaR constraint is dynamically updated, as in Cuoco, He, and Isaenko (2008). Table IA.CIII summarizes our findings for the optimal VaR-constrained portfolios. For computational tractability of our numerical solutions, we focus on investment horizons up to $T = 2$ years.

Insert Table IA.CIII about here

The VaR constraint has a more significant effect on the optimal portfolios of investors with low risk aversion, which are those with the largest exposure to risky assets in the unconstrained setting. For instance, for a risk aversion coefficient of $\gamma = 2$ and an investment horizon of $T = 2$ years, the mean total allocation to stocks (bonds) shrinks from approximately 250% (160%) to about 175% (115%) of the total wealth. At the same time, the relative importance of the covariance hedging demand increases: even for a moderate investment horizon of $T = 2$ years and a low risk aversion of $\gamma = 2$, the correlation and volatility hedging demands are on average 11% and 7% of the myopic portfolio, respectively. With the same choice of parameters, the corresponding hedging demands in the unconstrained case are 7.7% and 10.7%, respectively. For a higher risk aversion of $\gamma = 8$ and the same investment horizon, the covariance hedging demand is on average about 11% of the myopic portfolio both in the VaR-constrained and VaR-unconstrained cases.

The VaR-constrained investor dislikes more volatile or extreme portfolio values than the unconstrained agent does, since (ceteris paribus) the VaR constraint becomes more restrictive when the volatility on the optimally invested portfolio increases. It follows that the investor is more concerned about the total volatility of the portfolio, which can cause the VaR constraint to be hit with a probability that is too large. Therefore, the VaR-constrained investor reduces the size of the myopic demand. Furthermore, since changes in correlation have a first-order impact on the VaR of the portfolio, the investor increases the covariance hedging demand, exploiting the spanning properties of the risky assets. Thus, in this setting, which is relevant for institutions subject to capital requirement or for asset managers with self-imposed risk management constraints, the impact of covariance risk is economically significant.

D. Additional Empirical Results

Figure IA.D1. The Effect of the Investment Horizon

Figure IA.D1 reports intertemporal hedging demands for the S&P500 Index futures and the 30-year Treasury futures, as functions of the investment horizon, using the GMM parameter estimates for the underlying opportunity set dynamics reported in Table I of the main text.

Figure IA.D2. The Effect of the Risk Aversion Parameter

Figure IA.D2 plots hedging demands as a function of the coefficient of Relative Risk Aversion.

Table IA.DI. Estimation Results for Univariate Stochastic Volatility Models

We compare the portfolio implications of our setting with those of univariate portfolio choice models with stochastic volatility; see Heston (1993) and Liu (2001), among others. These models are nested in our setting in the special case in which the dimension of the investment opportunity set is set equal to one. For each risky asset in our data set, we estimate these univariate stochastic volatility models by GMM. The moment restrictions employed are the univariate counterpart of the moment conditions used in the estimation of the multivariate model. Panel A of Table IA.DI presents parameter estimates, whereas Panel B reports the estimated volatility hedging demands as a percentage of the myopic portfolio.

Table IA.DII. Estimation Results for Model with Three Risky Assets

Using GMM we estimate the three-dimensional version of model (1) to (4) in the main text obtained by including also the Nikkei225 Index futures contract in the opportunity set consisting of the S&P500 futures and the 30-year Treasury futures contracts. We use monthly time series of returns, realized volatilities, and realized covariances for these three risky assets. GMM moment restrictions are obtained in closed form as for the bivariate case above using the properties of the Wishart process. It is also straightforward to extend the proofs of Propositions 2 and 3 in the main text to cover the general setting with n risky assets. With these results, we compute the estimated optimal portfolios for the model with three risky assets. Table IA.DII, Panel A, presents the results of our GMM model estimation. The implied hedging demands for covariance and pure volatility hedging on each asset are given in Panel B.

Table IA.DIII. Estimation Results for Model with Two Risky Assets Using Daily Data

Table IA.DIII reports estimates for the parameters of model (1) to (4) in the main text, obtained with daily data and using the GMM procedure discussed in Internet Appendix A.

Table IA.DIV. Optimal Hedging Demands with Two Risky Assets Using Weekly Data

Table IA.DIV reports estimated optimal covariance and volatility hedging demands obtained using the weekly parameter estimates reported in Table I of the main text.

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Notes

¹In this way, local asymmetries in the covariance matrix dynamics can be introduced in the model. To model asymmetric correlations across regimes, Ang and Bekaert (2002) use an i.i.d. regime-switching setting, in which one of the regimes is characterized by greater correlations and volatilities.

²A possible choice for S is a 2×3 selection matrix, for example,

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $SS' = id_{2 \times 2}$ and $SY S'$ is the 2×2 upper diagonal sub-block of Y .

³Discussed in Internet Appendix B.

Table IA.BI

Estimation Results and Hedging Demands for the Model with Constant Risk Premia

Panel A: We report parameter estimates, Hansen's statistics, and hedging demands for the following model specification:

$$\begin{aligned} dS(t) &= I_S \mu dt + I_S Y^{-1/2}(t)(dB(t)\bar{\rho} + \sqrt{1 - \bar{\rho}'\bar{\rho}} dZ(t)) \\ dY(t) &= [\Omega\Omega' + MY(t) + Y(t)M'] dt + Y^{1/2}(t)dB(t)Q + Q'dB(t)'Y^{1/2}(t), \end{aligned}$$

$S(t)$ is the two-dimensional vector of the prices of S&P500 Index and 30-year Treasury bond futures. μ is a bivariate vector of constants and the interest rate r is also constant. $Y(t)$ models the information matrix $\Sigma(t)^{-1}$ and follows a Wishart diffusion. $B(t)$ is a 2×2 matrix of standard Brownian motions and $Z(t)$ is a 2×1 vector of Brownian motions independent of $B(t)$. Vector $\bar{\rho}$ and matrices M and Q are the remaining model parameters. Parameters are estimated with the same GMM methodology outlined in Internet Appendix A, that is now applied to the information matrix $Y = \Sigma^{-1}$ sampled at a monthly frequency. An asterisk denotes parameter estimates that are not significant at the 5% significance level. Panel B: Optimal hedging demands in percentages of the myopic portfolio are given for different investment horizons and relative risk aversion parameters. Each entry of the array of Panel B is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index futures, while the second one is the hedging demand for the 30-year Treasury futures.

Panel A

	M		Q		$\bar{\rho}$	μ
<i>point estimates</i> (<i>standard errors</i>)	-0.149 (0.074)	0.114* (0.081)	0.706 (0.34)	0.494* (0.312)	0.381 (0.161)	0.0616 (0.008)
	0.070 (0.036)	-0.112 (0.055)	0.806 (0.371)	0.641* (0.591)	0.392 (0.189)	0.0114 (0.0009)
<i>p-value for</i> <i>Hansen's J-test</i>	0.254					

Panel B

RRA	T	$3m$	$6m$	$1y$	$2y$	$5y$	$7y$	$10y$
2		-0.034	-0.061	-0.111	-0.151	-0.172	-0.173	-0.174
		-0.036	-0.053	-0.095	-0.135	-0.144	-0.145	-0.146
6		-0.049	-0.105	-0.175	-0.240	-0.251	-0.252	-0.253
		-0.048	-0.090	-0.160	-0.207	-0.213	-0.214	-0.214
8		-0.057	-0.115	-0.182	-0.258	-0.262	-0.263	-0.264
		-0.053	-0.104	-0.164	-0.212	-0.224	-0.224	-0.224
11		-0.061	-0.119	-0.189	-0.260	-0.263	-0.264	-0.265
		-0.055	-0.108	-0.170	-0.219	-0.228	-0.229	-0.230
16		-0.063	-0.120	-0.190	-0.261	-0.270	-0.271	-0.271
		-0.063	-0.110	-0.171	-0.220	-0.230	-0.231	-0.231
21		-0.065	-0.121	-0.191	-0.262	-0.272	-0.273	-0.273
		-0.064	-0.111	-0.172	-0.221	-0.231	-0.232	-0.232
41		-0.065	-0.121	-0.192	-0.262	-0.273	-0.274	-0.275
		-0.065	-0.112	-0.173	-0.221	-0.233	-0.235	-0.236

Table IA.CI

Optimal Hedging Demands in the Discrete-Time Model

Using standard numerical dynamic programming methods, we compute optimal hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text. We compute optimal discrete-time hedging demands for a daily (d), a weekly (w), and a monthly (m) rebalancing frequency, and denote by π_1 and π_2 the hedging demands for the S&P500 Index and the 30-year Treasury bond futures, respectively.

RRA	T	$3m$		$6m$		$1y$		$2y$				
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2			
2	d	0.0304	0.0344	d	0.0541	0.0650	d	0.0910	0.1151	d	0.1250	0.1661
	w	0.0295	0.0401	w	0.0561	0.0702	w	0.0917	0.1189	w	0.1243	0.1715
	m	0.0291	0.0449	m	0.0570	0.0759	m	0.0918	0.1240	m	0.1248	0.1879
8	d	0.0525	0.0589	d	0.0972	0.1123	d	0.1550	0.1915	d	0.1975	0.2543
	w	0.0515	0.0605	w	0.1021	0.1162	w	0.1545	0.1955	w	0.1969	0.2636
	m	0.0525	0.0632	m	0.1078	0.1208	m	0.1533	0.1803	m	0.1966	0.2566
21	d	0.0573	0.0640	d	0.1069	0.1209	d	0.1686	0.2045	d	0.2111	0.2685
	w	0.0569	0.0641	w	0.1134	0.1259	w	0.1705	0.2078	w	0.2099	0.2645
	m	0.0580	0.0665	m	0.1266	0.1365	m	0.1761	0.1993	m	0.2076	0.2705

Table IA.CII

Optimal Hedging Demands in the Discrete-time Model with Short-selling Constraints

Using standard numerical dynamic programming methods, we compute optimal hedging demands as a percentage of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, when short-selling constraints are applied, for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text, and the rebalancing frequency is monthly. We denote by π_1 and π_2 the hedging demands for the S&P500 Index and the 30-year Treasury bond futures, respectively, and distinguish the cases u , c_1 , and c_2 corresponding to the unconstrained solution, the solution for a position limit of the form $\pi \geq -1$, and the solution in the short-selling constrained case ($\pi \geq 0$), respectively. Total hedging demands are decomposed into covariance and volatility hedging components by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of covariance and variances.

		Covariance hedging																
<i>RRA</i>	<i>T</i>	<i>3m</i>		<i>6m</i>		<i>1y</i>		<i>2y</i>		<i>5y</i>		<i>7y</i>		<i>10y</i>				
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
2	<i>u</i>	0.0071	0.0352	<i>u</i>	0.0168	0.0521	<i>u</i>	0.0319	0.0814	<i>u</i>	0.0406	0.1281	<i>u</i>	0.0512	0.1293	<i>u</i>	0.0562	0.1394
	<i>c</i> ₁	0.0081	0.0254	<i>c</i> ₁	0.0154	0.0448	<i>c</i> ₁	0.0298	0.0695	<i>c</i> ₁	0.0393	0.1280	<i>c</i> ₁	0.0515	0.1091	<i>c</i> ₁	0.0498	0.1121
	<i>c</i> ₂	0.0054	0.0253	<i>c</i> ₂	0.0136	0.0387	<i>c</i> ₂	0.0283	0.0591	<i>c</i> ₂	0.0385	0.0932	<i>c</i> ₂	0.0486	0.1043	<i>c</i> ₂	0.0514	0.1089
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
8	<i>u</i>	0.0140	0.0435	<i>u</i>	0.0298	0.0924	<i>u</i>	0.0452	0.1244	<i>u</i>	0.0619	0.1859	<i>u</i>	0.0603	0.1924	<i>u</i>	0.0612	0.1929
	<i>c</i> ₁	0.0100	0.0322	<i>c</i> ₁	0.0291	0.0812	<i>c</i> ₁	0.0425	0.1021	<i>c</i> ₁	0.0592	0.1635	<i>c</i> ₁	0.0532	0.1765	<i>c</i> ₁	0.0581	0.1819
	<i>c</i> ₂	0.0084	0.0316	<i>c</i> ₂	0.0215	0.0760	<i>c</i> ₂	0.0392	0.0951	<i>c</i> ₂	0.0496	0.1486	<i>c</i> ₂	0.0524	0.1522	<i>c</i> ₂	0.0525	0.1534
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
21	<i>u</i>	0.0165	0.0484	<i>u</i>	0.0371	0.0842	<i>u</i>	0.0510	0.1328	<i>u</i>	0.0671	0.1828	<i>u</i>	0.0651	0.1985	<i>u</i>	0.0679	0.2003
	<i>c</i> ₁	0.0139	0.0491	<i>c</i> ₁	0.0363	0.0715	<i>c</i> ₁	0.0481	0.1296	<i>c</i> ₁	0.0623	0.1768	<i>c</i> ₁	0.0632	0.1812	<i>c</i> ₁	0.0679	0.1858
	<i>c</i> ₂	0.0125	0.0367	<i>c</i> ₂	0.0342	0.0683	<i>c</i> ₂	0.0413	0.1198	<i>c</i> ₂	0.0556	0.1541	<i>c</i> ₂	0.0561	0.1599	<i>c</i> ₂	0.0588	0.1615
		Volatility hedging																
<i>RRA</i>	<i>T</i>	<i>3m</i>		<i>6m</i>		<i>1y</i>		<i>2y</i>		<i>5y</i>		<i>7y</i>		<i>10y</i>				
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
2	<i>u</i>	0.0198	0.0076	<i>u</i>	0.0381	0.0190	<i>u</i>	0.0578	0.0330	<i>u</i>	0.0790	0.0459	<i>u</i>	0.0801	0.0588	<i>u</i>	0.0803	0.0591
	<i>c</i> ₁	0.0171	0.0074	<i>c</i> ₁	0.0342	0.0185	<i>c</i> ₁	0.0538	0.0325	<i>c</i> ₁	0.0750	0.0455	<i>c</i> ₁	0.0765	0.0581	<i>c</i> ₁	0.0766	0.0583
	<i>c</i> ₂	0.0151	0.0067	<i>c</i> ₂	0.0284	0.0183	<i>c</i> ₂	0.0478	0.0318	<i>c</i> ₂	0.0690	0.0420	<i>c</i> ₂	0.0711	0.0571	<i>c</i> ₂	0.0712	0.0572
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
8	<i>u</i>	0.0354	0.0131	<i>u</i>	0.0755	0.0254	<i>u</i>	0.1031	0.0499	<i>u</i>	0.1287	0.0655	<i>u</i>	0.1344	0.0698	<i>u</i>	0.1345	0.0700
	<i>c</i> ₁	0.0321	0.0118	<i>c</i> ₁	0.0712	0.0252	<i>c</i> ₁	0.0945	0.0489	<i>c</i> ₁	0.1189	0.0631	<i>c</i> ₁	0.1255	0.0677	<i>c</i> ₁	0.1255	0.0677
	<i>c</i> ₂	0.0290	0.0111	<i>c</i> ₂	0.0673	0.0250	<i>c</i> ₂	0.0891	0.0451	<i>c</i> ₂	0.1121	0.0612	<i>c</i> ₂	0.1150	0.0633	<i>c</i> ₂	0.1150	0.0633
		π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2	
21	<i>u</i>	0.0389	0.0154	<i>u</i>	0.0711	0.0294	<i>u</i>	0.1201	0.0525	<i>u</i>	0.1375	0.0705	<i>u</i>	0.1439	0.0710	<i>u</i>	0.1439	0.0710
	<i>c</i> ₁	0.0366	0.0149	<i>c</i> ₁	0.0657	0.0291	<i>c</i> ₁	0.1088	0.0523	<i>c</i> ₁	0.1301	0.0701	<i>c</i> ₁	0.1354	0.0704	<i>c</i> ₁	0.1354	0.0704
	<i>c</i> ₂	0.0312	0.0143	<i>c</i> ₂	0.0621	0.0281	<i>c</i> ₂	0.1061	0.0511	<i>c</i> ₂	0.1145	0.0671	<i>c</i> ₂	0.1201	0.0691	<i>c</i> ₂	0.1201	0.0691

Table IA.CIII

Optimal Hedging Demands in the Discrete-time Model with VaR constraints

This table reports optimal VaR-constrained volatility and covariance hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1) to (4) of the main text, as a function of different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete-time transition density of the model are the monthly estimates in Table II of the main text, and the rebalancing frequency is monthly. As in Cuoco, He, and Isaenko (2008), the VaR constraint is updated at each trading date, by imposing a constant upper bound on the 99%-VaR of next-trading-date wealth. Total hedging demands are decomposed into covariance and volatility hedging components by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of variances and covariances. Each entry of the two arrays in the table is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index futures, while the second one is the hedging demand for the 30-year Treasury bond futures.

Volatility Hedging						Covariance Hedging					
<i>RRA</i>	<i>T</i>	<i>3m</i>	<i>6m</i>	<i>1y</i>	<i>2y</i>	<i>RRA</i>	<i>T</i>	<i>3m</i>	<i>6m</i>	<i>1y</i>	<i>2y</i>
2		0.023	0.048	0.064	0.086	2		0.019	0.035	0.048	0.069
		0.014	0.026	0.040	0.058			0.043	0.073	0.101	0.130
8		0.028	0.052	0.069	0.092	8		0.021	0.041	0.051	0.082
		0.019	0.031	0.045	0.061			0.046	0.081	0.102	0.138
21		0.031	0.055	0.072	0.095	21		0.032	0.052	0.072	0.082
		0.023	0.032	0.047	0.062			0.059	0.097	0.121	0.149

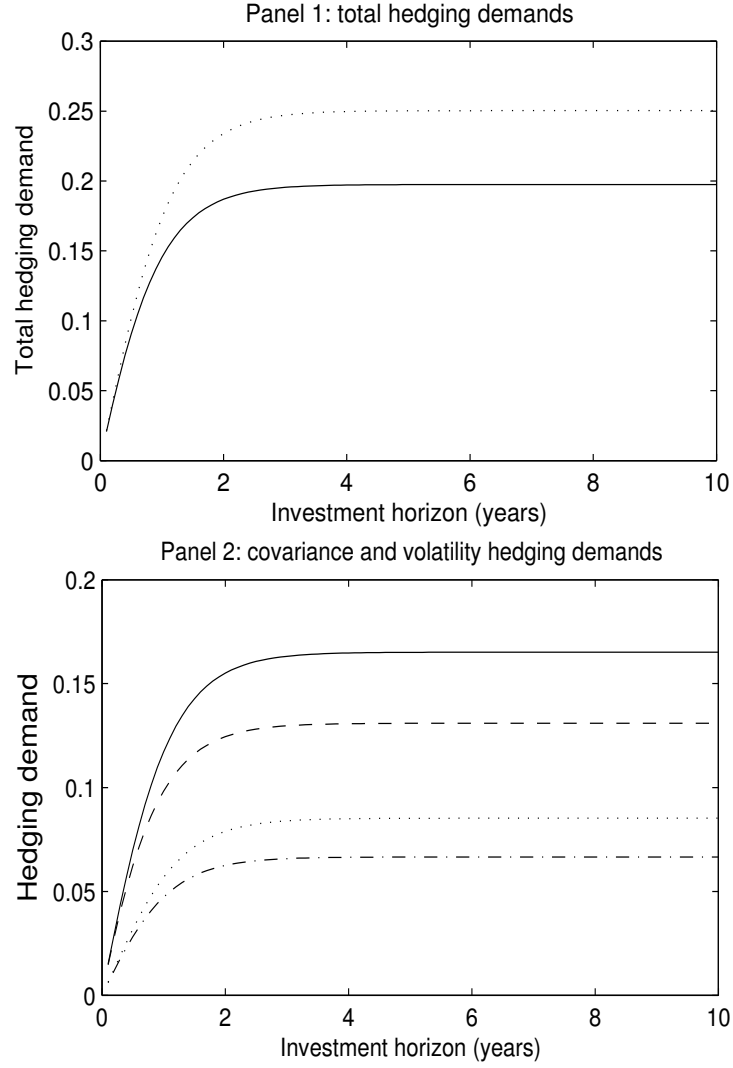


Figure IA.D1. The effect of the investment horizon. Panel 1: Total hedging demands for the S&P500 Index futures (solid line) and 30-year Treasury futures (dotted line) as a percentage of the Merton myopic portfolio are plotted as a function of the investment horizon (in years). These hedging demands are computed using the monthly parameter estimates in Table I of the main text, for a relative risk aversion parameter of $\gamma = 6$. Panel 2: Volatility hedging and covariance hedging demands for the 30-year Treasury bond futures (dotted and solid lines, respectively) and the S&P500 Index futures (dashed and dashed-dotted lines, respectively) are plotted as functions of the investment horizon (in years). Both hedging demands are expressed as a percentage of the Merton myopic portfolio. The same parameters as for Panel 1 are used to compute them.

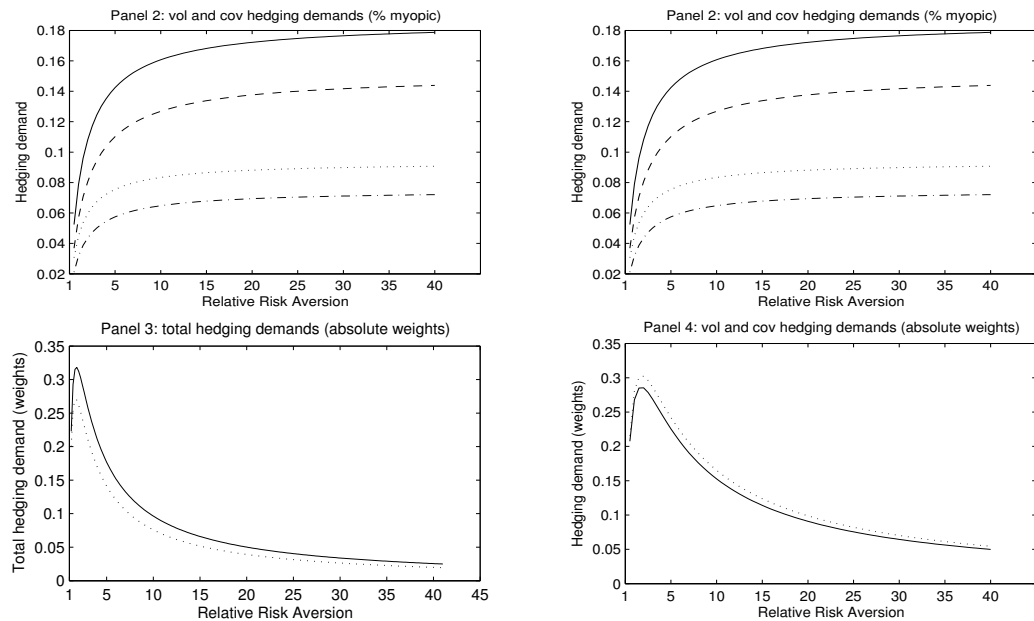


Figure IA.D2. The effect of the risk aversion parameter. Panel 1: Total hedging demands for the S&P500 Index futures (solid line) and 30-year Treasury futures (dotted line) as a percentage of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient for a fixed investment horizon of five years. To compute these policies, we use the monthly parameters estimates in Table I of the main text. Panel 2: Volatility hedging and covariance hedging demands for the 30-year Treasury bond futures (dotted and solid lines, respectively) and the S&P500 Index futures (dashed and dashed-dotted lines, respectively) as a percentages of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient. The same parameters as in Panel 1 are used to compute these policies. Panel 3: Same plots as in Panel 1, but with percentage hedging demands replaced by actual hedging portfolio weights. Panel 4: Total portfolio weights for covariance hedging (solid line) and for volatility hedging (dotted line), aggregated over risky assets, are plotted as functions of the Relative risk aversion parameter. The same parameters as in Panel 1 are used to compute these policies.

Table IA.DI

Estimation Results and Hedging Demands for Univariate Stochastic Volatility Models

Panel A: We report point estimates and standard errors (in parentheses) for the parameters of the following univariate stochastic volatility model:

$$\begin{aligned} dS_t &= S_t(r + \lambda\sigma_t^2)dt + \sigma_t(\rho dW_t + \sqrt{1 - \rho^2}dZ_t) \\ d\sigma_t^2 &= (k b^2 + 2 m \sigma_t^2)dt + 2 b \sigma_t dW_t. \end{aligned} \quad (1T)$$

S_t is the futures price of either the S&P500 futures, the 30-year Treasury bond futures, or the Nikkei 225 Index futures. σ_t is the stochastic volatility process of returns, modeled by a Heston (1993)-type model. W_t and Z_t are independent scalar Brownian motions and (k, λ, ρ, b, m) is the vector of parameters of interest. We estimate model (1T) by GMM using monthly time series of returns and realized volatilities for the S&P500 futures, 30-year Treasury bond futures, and Nikkei 225 Index futures returns. Panel B: We compute optimal (volatility) hedging demands for the univariate stochastic volatility model (1T), as a percentage of the myopic portfolio, using the parameter estimates in Panel A and for different investment horizons and relative risk aversion coefficients. The last column reports optimal myopic demands. The notation *S&P500*, *Trea*, and *Nik225* corresponds to the hedging demands in the univariate models for the S&P500 Index futures, the 30-year Treasury Bond futures, and the Nikkei225 Index futures, respectively.

Panel A

	k	m	b	ρ	λ
S&P500	1.18 (0.36)	-2.39 (0.42)	0.36 (0.08)	-0.88 (0.05)	0.72 (0.21)
Treasury	2.45 (0.84)	-2.10 (0.24)	0.29 (0.07)	-0.56 (0.04)	1.05 (0.34)
Nikkei	4.33 (1.16)	-2.82 (0.55)	-0.28 (0.08)	-0.67 (0.19)	0.64 (0.19)

Panel B

RRA	T	$6m$		$1y$		$5y$		$10y$		Myopic demand	
2	S&P500	0.022		S&P500	0.025	S&P500	0.027	S&P500	0.027	S&P500	0.360
	Trea	0.018		Trea	0.021	Trea	0.023	Trea	0.024	Trea	0.525
	Nik225	0.012		Nik225	0.019	Nik225	0.023	Nik225	0.025	Nik225	0.320
4	S&P500	0.034		S&P500	0.038	S&P500	0.041	S&P500	0.041	S&P500	0.180
	Trea	0.028		Trea	0.032	Trea	0.034	Trea	0.034	Trea	0.262
	Nik225	0.016		Nik225	0.021	Nik225	0.031	Nik225	0.033	Nik225	0.160
8	S&P500	0.041		S&P500	0.045	S&P500	0.048	S&P500	0.048	S&P500	0.090
	Trea	0.033		Trea	0.038	Trea	0.040	Trea	0.040	Trea	0.131
	Nik225	0.019		Nik225	0.025	Nik225	0.036	Nik225	0.038	Nik225	0.080

Estimation Results and Hedging Demands for the Model with 3 Risky Assets

We present parameter estimates, Hansen's statistics and optimal hedging demands for model (1)-(4) with 3 risky assets. *Panel A*: We report parameter estimates for M , Q , λ and \bar{p} (with standard errors in parentheses) in the returns dynamics (1)-(4), where $\Omega\Omega' = kQQ'$ for $k = 10$. The parameters are estimated using monthly returns, realized volatilities and correlations of S&P 500 index, 30-year US Treasury bond, and Nikkei 225 index future returns sampled at a monthly frequency. The GMM estimation procedure is similar to the one used to estimate the bivariate model and detailed moment restrictions are given in Internet Appendix A. Parameters that are not significant at the 5% significance level are marked with an asterisk. *Panel B*: We report optimal covariance and volatility hedging demands in percentage of the myopic portfolio. Each entry of the array in Panel B consists of three components, the first of which is the demand for the S&P500 Index Futures, the second one the demand for the 30-year Treasury bond Futures and the third one the demand for the Nikkei 225 Index Futures, respectively.

Panel A								
	M			Q			\bar{p}	λ
	-0.762 (0.293)	-0.251* (0.162)	0.390 (0.180)	0.005* (0.060)	0.064 (0.029)	0.069* (0.051)	-0.210 (0.090)	2.482 (0.380)
point estimates (standard errors)	0.511 (0.240)	-0.872 (0.281)	0.120* (0.110)	0.059* (0.048)	0.105* (0.074)	0.060* (0.100)	-0.230* (0.140)	2.327 (0.490)
	0.286* (0.153)	0.425 (0.212)	-0.968 (0.394)	0.070 (0.033)	0.055 (0.022)	0.004* (0.180)	-0.170 (0.075)	1.561 (0.160)
p-value for Hansen's J-test	0.115							

Panel B

Volatility Hedging										
RRA	T	3m	6m	1y	2y	5y	7y	10y	20y	Myopic demand
2		0.005	0.009	0.017	0.027	0.034	0.035	0.035	0.035	1.2410
		0.007	0.011	0.016	0.020	0.021	0.021	0.021	0.021	1.1635
		0.004	0.008	0.014	0.023	0.029	0.030	0.030	0.030	0.7805
6		0.014	0.028	0.051	0.077	0.090	0.091	0.091	0.091	0.4137
		0.019	0.033	0.050	0.059	0.061	0.061	0.061	0.061	0.3878
		0.013	0.025	0.044	0.065	0.076	0.077	0.077	0.077	0.2602
8		0.018	0.037	0.066	0.097	0.110	0.110	0.110	0.110	0.3103
		0.026	0.045	0.065	0.076	0.078	0.078	0.078	0.078	0.2909
		0.018	0.034	0.059	0.083	0.093	0.094	0.094	0.094	0.1951
11		0.025	0.050	0.088	0.123	0.134	0.134	0.135	0.135	0.2256
		0.036	0.061	0.087	0.100	0.101	0.101	0.101	0.101	0.2115
		0.025	0.046	0.077	0.105	0.114	0.114	0.115	0.115	0.1419
16		0.037	0.071	0.122	0.158	0.167	0.167	0.167	0.167	0.1551
		0.052	0.087	0.121	0.133	0.134	0.134	0.134	0.134	0.1454
		0.035	0.065	0.106	0.136	0.143	0.144	0.144	0.144	0.0976
21		0.048	0.092	0.150	0.187	0.194	0.194	0.194	0.194	0.1182
		0.068	0.113	0.152	0.163	0.164	0.164	0.164	0.164	0.1108
		0.047	0.085	0.132	0.161	0.167	0.167	0.167	0.167	0.0743
41		0.092	0.166	0.242	0.275	0.278	0.279	0.279	0.279	0.0605
		0.130	0.200	0.250	0.260	0.260	0.260	0.260	0.260	0.0568
		0.090	0.153	0.215	0.237	0.240	0.240	0.240	0.240	0.0381

Covariance Hedging

RRA	T	3m	6m	1y	2y	5y	7y	10y	20y	Myopic demand
2		0.010	0.017	0.030	0.041	0.046	0.048	0.048	0.049	1.2410
		0.004	0.008	0.013	0.019	0.021	0.022	0.022	0.022	1.1635
		0.008	0.016	0.029	0.042	0.048	0.048	0.049	0.049	0.7805
6		0.029	0.050	0.087	0.117	0.125	0.130	0.130	0.130	0.4137
		0.013	0.025	0.041	0.058	0.061	0.065	0.065	0.065	0.3878
		0.026	0.046	0.079	0.114	0.133	0.138	0.138	0.138	0.2602
8		0.041	0.068	0.119	0.151	0.163	0.163	0.164	0.164	0.3103
		0.016	0.031	0.051	0.068	0.072	0.073	0.073	0.073	0.2909
		0.031	0.061	0.102	0.148	0.159	0.159	0.160	0.160	0.1951
11		0.054	0.102	0.158	0.182	0.195	0.195	0.196	0.196	0.2256
		0.022	0.040	0.071	0.087	0.094	0.095	0.096	0.096	0.2115
		0.044	0.085	0.144	0.184	0.191	0.191	0.192	0.192	0.1419
16		0.078	0.138	0.211	0.253	0.265	0.267	0.267	0.268	0.1551
		0.031	0.062	0.098	0.125	0.128	0.130	0.130	0.132	0.1454
		0.063	0.122	0.196	0.247	0.247	0.268	0.268	0.270	0.0976
21		0.095	0.178	0.269	0.291	0.310	0.310	0.310	0.310	0.1182
		0.041	0.083	0.128	0.145	0.152	0.154	0.154	0.154	0.1108
		0.082	0.159	0.251	0.310	0.321	0.322	0.322	0.323	0.0743
41		0.172	0.331	0.422	0.450	0.451	0.451	0.451	0.451	0.0605
		0.081	0.159	0.193	0.205	0.206	0.206	0.206	0.206	0.0568
		0.167	0.271	0.391	0.431	0.432	0.432	0.432	0.432	0.0381

Table IA.DIII

Estimation Results for the Model with Two Risky Assets Using Daily Data

This table shows estimated matrices M and Q and vectors λ and $\bar{\rho}$ for the returns dynamics (1) in the main text, under the Wishart variance covariance diffusion process:

$$d\Sigma(t) = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t),$$

where $\Omega\Omega' = kQ'Q$ and $k = 10$. Parameters are estimated by GMM using time series of returns and realized variance-covariance matrices for S&P 500 Index and 30-year Treasury bond futures returns, computed for a daily frequency. The detailed set of moment restrictions used for GMM estimation is given in Internet Appendix A. We report parameter estimates and their standard errors (in parentheses), together with the p -values for Hansen's J-test of overidentifying restrictions.

	M		Q		λ	$\bar{\rho}$
<i>point estimates</i> (<i>p-values</i>)	-1.098 (0.0002)	0.42 (0.001)	-0.16 (0.01)	0.028 (0.3435)	4.89 (0.03)	0.1296 (0.0035)
	0.21 (0.002)	-1.58 (0.0035)	0.0049 (0.4534)	0.103 (0.024)	5.54 (0.04)	-0.24 (0.0121)
<i>p-value for</i> <i>Hansen's J-test</i>	0.03					

Table IA.DIV

Optimal Hedging Demands in the Model with Two Risky Assets Using Weekly Data

This table shows optimal covariance and volatility hedging demands as a percentage of the myopic portfolio, for different investment horizons and relative risk aversion parameters. The last column of each panel reports the myopic portfolio. We compute these demands for the weekly parameters estimates reported in Table I of the main text. Each entry in the table is a vector with two components, namely the demand for the S&P500 Index futures and the demand for the 30-year Treasury futures.

Covariance Hedging

<i>RRA</i>	<i>T</i>	<i>3m</i>	<i>6m</i>	<i>1y</i>	<i>2y</i>	<i>5y</i>	<i>7y</i>	<i>10y</i>	Myopic demand
2		0.0186	0.0310	0.0441	0.0512	0.0523	0.0523	0.0523	2.3610
		0.0121	0.0202	0.0290	0.0334	0.0342	0.0342	0.0342	1.6585
6		0.0311	0.0520	0.0732	0.0835	0.0848	0.0848	0.0848	0.7870
		0.0203	0.0340	0.0480	0.0541	0.0554	0.0555	0.0555	0.5528
8		0.0327	0.0545	0.0768	0.0874	0.0888	0.0888	0.0888	0.5903
		0.0214	0.0357	0.0502	0.0571	0.0580	0.0580	0.0580	0.4146
11		0.0340	0.0568	0.0797	0.0906	0.0920	0.0920	0.0920	0.4293
		0.0222	0.0371	0.0521	0.0592	0.0610	0.0610	0.0610	0.3015
16		0.0351	0.0586	0.0822	0.0933	0.0947	0.0947	0.0947	0.2951
		0.0229	0.0383	0.0537	0.0618	0.0621	0.0621	0.0621	0.2073
21		0.0359	0.0595	0.0835	0.0947	0.0961	0.0961	0.0961	0.2249
		0.0233	0.0389	0.0545	0.0623	0.0626	0.0626	0.0626	0.1580
41		0.0366	0.0610	0.0855	0.0969	0.0983	0.0985	0.0985	0.1152
		0.0239	0.0400	0.0556	0.0633	0.0642	0.0642	0.0642	0.0809

Volatility Hedging

<i>RRA</i>	<i>T</i>	<i>3m</i>	<i>6m</i>	<i>1y</i>	<i>2y</i>	<i>5y</i>	<i>7y</i>	<i>10y</i>	Myopic demand
2		0.0116	0.0188	0.0256	0.0289	0.0294	0.0294	0.0294	2.3610
		0.0194	0.0336	0.0500	0.0599	0.0617	0.0617	0.0617	1.6585
6		0.0195	0.0314	0.0425	0.0477	0.0477	0.0477	0.0477	0.7870
		0.0327	0.0564	0.0830	0.0977	0.0999	0.0999	0.0999	0.5528
8		0.0205	0.0330	0.0446	0.0494	0.0500	0.0500	0.0500	0.5903
		0.0343	0.0593	0.0871	0.1023	0.1046	0.1046	0.1046	0.4146
11		0.0213	0.0343	0.0463	0.0513	0.0518	0.0518	0.0518	0.4293
		0.0357	0.0617	0.0905	0.1061	0.1084	0.1084	0.1084	0.3015
16		0.0219	0.0354	0.0477	0.0528	0.0533	0.0533	0.0533	0.2951
		0.0368	0.0636	0.0933	0.1092	0.1105	0.1105	0.1105	0.2073
21		0.0223	0.0360	0.0484	0.0536	0.0541	0.0541	0.0541	0.2249
		0.0374	0.0647	0.0947	0.1108	0.1132	0.1132	0.1132	0.1580
41		0.0229	0.0369	0.0496	0.0548	0.0554	0.0554	0.0554	0.1152
		0.0384	0.0663	0.0970	0.1134	0.1157	0.1157	0.1157	0.0809