

Continuous-Time Asset Pricing with Ambiguity Aversion

Paolo Porchia

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Dissertation Committee

Prof. Fabio Trojani (thesis supervisor), USI, Lugano

Prof. Patrick Gagliardini, USI, Lugano

Prof. Giovanni Barone-Adesi, USI, Lugano

Prof. Domenico Cuoco, Wharton School, University of Pennsylvania

Grade Awarded: *summa cum laude*

Preface

Academic literature on financial modeling typically assumes that the probability law governing future realizations of key economic quantities is known to decision makers; this implies that the relevant source of uncertainty in the model is *risk*, that is, uncertainty deriving from the randomness of future contingencies with known odds. Knight first and Keynes successively emphasized the importance of the distinction between risk and *ambiguity*, the latter being defined as the situation in which agents do not rely on a single probability law for assessing future events. Ambiguity aversion means that investors penalize this unprecise probabilistic description in their preference orderings.

As will be discussed later in more details, the academic community has realized that ambiguity aversion is a plausible explanation for many financial phenomena. The interest in such a modeling device was indeed enhanced by the strong support found in experimental data: the famous Ellsberg's paradox (Ellsberg (1961)), in particular, confirmed the relevance of the distinction between risk and ambiguity for the economic behavior of decision makers. Four bets based on a draw from two urns are to be ranked: the urns contain either red or black balls, but the number of red balls is known only for the first urn (half of the balls). The first (second) bet is won if the ball drawn from the first urn is red (black), the third (fourth) bet is won if the ball drawn from the second urn is red (black). The prize won is constant across bets. Ellsberg noticed that decision makers are significantly indifferent between the first and the second bet and between the third and the fourth bet, but strictly prefer either the first or the second to either the third or the fourth. No probability distribution of balls in the second urn can give rise to a Savage expected utility representation of this preference ordering, therefore this paradigm is violated.

Gilboa and Schmeidler's (1989) seminal contribution relaxes the independence axioms to introduce in a static setting an axiomatic theory of choice coherent with ambiguity aversion, where preferences are represented by a Max-Min expected utility over a set of probability measures. Inspired by this contribution, several authors have attempted to embed ambiguity aversion in an intertemporal context: Epstein and Wang's (1994) make use in a discrete-time framework of a max-min recursive expected utility criterion over a set of distributions, called Recursive Multiple Prior Utility, later axiomatized by Epstein and Schneider (2003), who show the dynamic consistency of this approach. Independently of this approach, a second attempt to an ambiguity averse theory of dynamic decision making was made by Hansen, Sargent and coauthors (Hansen, Sargent and Tallarini (1999) and Anderson, Hansen and Sargent (2003) among other contributions) who built on Robust Control techniques to come up with an alternative form of Max-Min expected utility.

Several recent papers focused on asset pricing have relied on ambiguity aversion to successfully address stylized facts considered as 'puzzles' according to the standard Savage expected utility modelling approach. Among these contributions we recall Uppal and Wang (2003), Epstein and Miao (2003), for the home-bias 'puzzle' and underdiversification, Anderson, Hansen and Sargent (2000), Chen and Epstein (2002), Maenhout (2001) and Sbuelz and Trojani (2002) for the equity premium 'puzzle' and Leippold, Trojani and Vanini (2004) for the equity premium and interest rate 'puzzles'. Dow and Werlang (1992) and Trojani and Vanini (2004) generate endogenous limited stock market participation as a consequence of agents' optimizing behavior in the absence of market

frictions, whereas Liu, Pan and Wang (2003) are able to mimic the typical ‘smirk’ shape of options’ implied volatilities.

The axiomatic framework of Knox (2004), Leippold, Trojani and Vanini (2004), Epstein and Schneider (2002) and Hansen, Sargent and Wang (2002) tackle the issue of learning under ambiguity aversion, and, in addition to generating economic predictions consistent with the empirical evidence, emphasize that learning about an unknown parameter fails to resolve ambiguity asymptotically.

This Thesis is structured in two Chapters, each aimed at contributing to the existing literature by exploring the effects of ambiguity aversion on two classical equilibrium asset pricing problems: the term structure of interest rates and two-agents equilibrium. In both cases, ambiguity aversion is modeled by means of a Max-Min expected utility representation that falls within the Recursive Multiple Priors class - therefore delivering dynamic consistency of the optimal policies of the agents - but was originally adopted by the Robust Control school of Hansen, Sargent et al. The set of likelihood used in the preference orderings representation is identified by means of a bound on the maximum ‘distance’ between admissible probability measures and a reference one, interpreted as approximate description of the true data generating model.

To the end of analyzing the impact of aversion for *ambiguity* on agents’ consumption investment choices and on assets pricing in a rational expectation equilibrium context, the first Chapter of the Thesis, *A General Treatment of Equilibrium under Ambiguity* (joint with Fabio Trojani) considers a continuous-time pure exchange economy populated by two agents, whose decisions rely on a whole set of possible contaminations of a reference probabilistic model. As already pointed out implicitly, given that they adopt a form of max-min expected utility representation, they select the worst-case model among those considered as relevant. The methodology applied in order to characterize equilibrium equity premia and stock returns volatility is based on a weak notion of aggregation of the single agents into a representative agent, whose preferences depend on an additional state variable acting as a proxy for the stochastic shifts of the cross-sectional wealth distribution due to the different beliefs selected in equilibrium by agents. This methodology first appeared in Cuoco and He (1994). Closed form solutions for key equilibrium quantities are detailed for markovian specifications of the stochastic opportunity set. In accordance with the literature on ambiguity aversion (Epstein and Wang (1994), Anderson, Hansen and Sargent (2000), Trojani and Vanini, 2004; Trojani and Sbuelz, 2002) we find that this modelling framework suggests a possible explanation of the equity premium puzzle; what is more, endogenous cycles of restricted stock market participation are obtained, without imposing exogenous policy restrictions on agents: the agents who refrain from investing in risky assets select a consumption plan characterized by negligible correlation with aggregate consumption, and since the entire variability of the latter then falls on those whose risky position is not null, then the equilibrium risk premium is proportional to the the risk aversion of these agents rather than to the risk aversion of the representative agent. Therefore, cycles of limited stock market participation do indeed help overcome the inability of Lucas-type models to generate realistic equity premia for reasonable values of risk aversion. This result has been achieved in the literature by means of models in which the constraint of limited participation is assumed ex-ante (Cuoco and Basak (1998) for instance), and not derived as a consequence of agent’s optimizing

behavior.

The second Chapter of the Thesis, *Ambiguity aversion, bond pricing, and the non robustness of some affine term structures*, (joint work with Patrick Gagliardini and Fabio Trojani) is inspired by a simple consideration emphasized in the first chapter as well as in previous contributions (Trojani and Sbuelz, 2002): the impact of ambiguity aversion is more prominent on equity premia than price levels, therefore equilibrium models of the term structure taking into account ambiguity aversion should allow for a wide variety of implications. In light of this, we develop a continuous time general equilibrium model of the term structure of interest rates where economic agents are averse to model uncertainty and consider the possibility of a misspecified dynamic model for the latent random factors driving interest rates. Aversion to *ambiguity* is parameterized through the same form of Knightian uncertainty used in the first chapter, and this form may induce first order risk aversion effects in equilibrium if a suitable specification is selected. We find that a small concern for *ambiguity* significantly affects the implied term structures in equilibrium and drives the prices of common derivative securities toward the patterns observed in fixed income markets. Indeed, equilibrium risk premia and interest rates have a different functional form than in the standard model, due to an ambiguity aversion premium. Moreover, otherwise unpriced factors in the standard model receive a premium for model uncertainty which is of a particularly rich structure in the multiple factors setting. All these features induce in equilibrium term structure levels and shapes that are very different from those generated by the standard model. Examples of the impact of *ambiguity aversion* on popular factor models of the term structure are derived, both in cases for which the ‘level of concern’ *ambiguity* is time-varying and in cases for which it is time invariant. Furthermore, we analyze a form of ‘robustness’ property of some classes, that is, the functional differences between equilibrium quantities and their counterparts arising in an economy with standard Savage-type preferences.

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Chapter 1

A General Treatment of Equilibrium Under Ambiguity

In a continuous-time, pure exchange economy on a finite horizon financial agents display ambiguity aversion for a neighborhood of indistinguishable model specifications that are constrained in their relative entropy from a given reference model. We characterize equilibrium optimal consumption-portfolio choices under a general (possibly non Markovian) stochastic opportunity set by means of martingale methods and, once we restrict to the case of most common Markovian stochastic volatility models, we identify closed-form solutions. In an environment populated by multiple heterogeneous agents who derive utility from intertemporal consumption and terminal wealth, we investigate the impact of ambiguity aversion on asset prices from a rational expectations equilibrium perspective, using a notion of state-dependent representative agent introduced in Cuoco and He (1994). Both equilibrium interest rate and equity premium point towards a possible rationalization of the equity premium puzzle. This result is partly driven by the ability of the model to generate endogenous cycles of restricted stock market participation, achieved without imposing any a-priori trading constraint.

1.1 Introduction

This paper studies the influence of ambiguity aversion on equilibrium asset prices and interest rates in a continuous time two agent endowment economy with possibly incomplete markets. Ambiguity refers to a situation in which the probabilistic description of the dynamics of key economic factors is not known with certainty. Ambiguity aversion implies that the preference ordering representation of the agents penalizes the impreciseness of this knowledge. Although the distinction between ambiguity and standard risk aversion had been present in the literature at least since Knight (1921), its behavioral and economic relevance has been acknowledged mainly after Ellsberg (1961) and the literature inspired by this contribution.

Indeed recent academic papers have shown that ambiguity aversion may help to rationalize stylized facts that are considered ‘puzzles’ according to the Savage expected utility paradigm. Among these we recall Uppal and Wang (2003), Epstein and Miao (2003), for the home-bias ‘puzzle’ and underdiversification, Anderson, Hansen and Sargent (2000), Chen and Epstein (2002), Maenhout (2001) and Sbuelz and Trojani (2002) for the equity premium ‘puzzle’. Dow and Werlang (1992) and Trojani and Vanini (2004) generate endogenous limited stock market participation as a consequence of agents’ optimizing behavior in the absence of market frictions, whereas Liu, Pan and Wang (2003) are able to mimic the typical ‘smirk’ shape of options’ implied volatilities.

In an otherwise standard two-agent pure exchange general equilibrium economy, we allow for ambiguity aversion by assuming that agents’ preference orderings are characterized by the Recursive Multiple Prior representation (RMPU) introduced by Epstein and Wang (1994) and axiomatized in Epstein and Schneider (2001), generalizing to a dynamical context of the max-min expected utility representation pioneered by Gilboa and Schmeidler (1989). In this seminal contribution, the single likelihood involved in the standard Savage expected utility model is replaced by a whole set of probability measures that the agent regards as possible data generating processes.

The distinction between different models that incorporating ambiguity aversion within the RMPU paradigm is essentially one about different selections of the set of likelihoods used in the representation. In accordance with the intuition that an ‘admissible’ probability measure should belong to some ‘neighborhood’ of an approximate description - the reference belief - of the true data generating model, we identify this set of likelihoods by means of an upper bound on the ‘distance’ between from the reference belief, where the distance is defined according to a statistically sound metric.

Due to the presence of incomplete markets and the additional layer of heterogeneity implied by possibly different ambiguity aversions, the methodology applied to characterize key economic indicators in equilibrium adopts a weak notion of aggregation into a state dependent representative agent having preferences that depend on an additional state variable. Such state variable is a stochastic weighting process that acts as a proxy for the stochastic shifts of the cross-sectional wealth distribution due to the different beliefs (and min-max martingale measures¹) selected in equilibrium by the agents because of ambiguity and market incompleteness. Cuoco and He (1994) were the first to point out this line of reasoning for the case of incomplete markets. Several additional

¹See He and Pearson (1991) for the notion of min-max martingale measure and the corresponding static characterization of the dynamic consumption investment problem of an agent with incomplete markets.

contributions (Basak and Cuoco (1998), Basak (2000), Detemple and Murthy (1994), Epstein and Miao (2003), to quote just a few) exploit the same idea whenever some form of heterogeneity in the representations of the state prices faced by the agents arises. In accordance with these contributions, equilibrium quantities are driven by the aggregate endowment the state variable that affects the coefficients of the aggregate endowment, and the stochastic weighting process we have mentioned. As somewhat noted above, such a stochastic weighting process is really a form of disagreement process meant to address heterogeneities in the representations of the financial investment opportunities. In our setting such heterogeneities that in the present context are due to heterogeneous ambiguity aversions and preference-dependent representations of the state price process, as a consequence of market incompleteness. Contrarily to those frameworks in which different bayesian estimates of the aggregate endowment's conditional mean imply an essentially exogenous disagreement process, in the present setting the individual rationality constraint of the agents directly affects the equilibrium determination of the stochastic weighting process, giving rise to an endogenous disagreement.

The academic literature concerned with two-agent equilibrium characterizations is extensive. Focusing on contributions that embed this problem in an ambiguity averse framework, notably Trojani and Vanini (2004) carried out a thorough investigation of the distinctive equilibrium implications of competing frameworks of ambiguity. The present paper is more similar in spirit to Epstein and Miao (2003) who, in a complete markets economy populated by agents with logarithmic felicities displaying a higher concern for ambiguity over foreign stocks, generate predictions suggesting a possible rationalization of the home-bias puzzle. Our paper differs from the latter in that it adopts a different set of relevant likelihoods in the preference orderings representation, it allows for incomplete markets and it also studies felicity functions different from the logarithmic one. Such a generalization allows us to analyze also equilibrium investment policies that hedge against future changes of the investment opportunity set caused by the endogenous shifts in the 'disagreement' between the two agents.

In accordance with the literature on ambiguity aversion we find that the present modeling framework generates lower interest rates and higher equity premia. But, in addition to the existing literature the availability of exact solutions in a stochastic opportunity set environment allows for a wider set of implications. In particular, the equilibrium intertemporal hedging policy of a CRRA agent due to the additional state variable mentioned above is illustrated in details. With the aid of Malliavin calculus, the impact of ambiguity aversion on the equilibrium volatility of the stock is also clarified. What is more, endogenous cycles of restricted stock market participation are obtained without the imposition of exogenous policy restrictions or other market frictions. The agent who refrain from investing in the risky asset selects a consumption plan characterized by a negligible correlation with the aggregate endowment process. Since the entire variability of the latter then falls on the agent whose risky position is not null, the equilibrium risk premium is proportional to the risk and ambiguity aversion of this agent. Therefore, cycles of limited stock market participation implied by a concern for ambiguity do indeed help overcome the inability of Lucas-type models to generate realistic equity premia at reasonable values of risk aversions. This appealing feature of ambiguity aversion, first pointed out in a static framework by Dow and Werlang (1992), mimics the similar results achieved in the literature by means of modeling frameworks in which restricted

participation is imposed ex-ante by means of policy constraints (Basak and Cuoco (1998)), instead of being derived as a consequence of agents' optimizing behavior.

The structure of the paper is as follows. The next Section describes the pure exchange economic environment, defines ambiguity and introduces the max-min expected utility representation we have opted for. Section 1.3 is about the individual consumption problem when the coefficients of the opportunity set are known functions of a vector of state variables. Due to the features peculiar to an equilibrium setting, this hypothesis is not fulfilled in general equilibrium and thus removed in Section 1.4, where a general characterization of equilibrium is given. In Section 1.5, equilibrium consumption-investment policies, equity premia, interest rates and equity volatilities are characterized for two different choices of risk aversions and heterogeneous ambiguity aversions. Finally two examples are worked out, inspired by specific dynamics chosen for the aggregate endowment process. Section 1.6 concludes, whereas all proofs are relegated to the Appendix.

1.2 Model Set-Up

We consider an endowment economy populated by two financial agents whose action does not affect equilibrium prices. They optimize a linear combination of the utilities arising from intertemporal and terminal consumption of a single perishable good (the numeraire). Their consumption plans are financed on a given time horizon $[0, T]$, $0 < T < \infty$, by continuously trading a locally riskless investment opportunity with return process $r(t)$ as well as a stock representing a claim to an exogenously given dividend process.

1.2.1 Reference Belief

Agents are provided with an approximate probabilistic description of this dynamic economic environment; they are characterized by an imprecise knowledge of the probability measure that governs the realization of different economic paths. The 'reference belief' P should really be regarded as an uncertain view on the future evolution of their opportunity set. Under this measure the dividend process $\varepsilon(t)$ is posited to follow the dynamics :

$$\begin{aligned} \frac{d\varepsilon(t)}{\varepsilon(t)} &= \mu_\varepsilon(Y)dt + \sigma_\varepsilon(Y)dw(t) \\ dY(t) &= [\Lambda(\omega, t)dt + \Xi(\omega, t)dw(t)] \end{aligned} \tag{1.1}$$

The riskless asset is in zero net supply, whilst the number of shares outstanding is normalized to one. In expression (1.1) $\sigma_\varepsilon(x)$ is a $[1 \times (k + 1)]$ -vector valued function of the k -dimensional vector of driving state variables Y . $\mu_\varepsilon(x)$ is a deterministic scalar function of the same argument. Both σ_ε and μ_ε satisfy appropriate integrability conditions, which will be detailed in the sequel. $\Lambda(\omega, t)$ and $\Xi(\omega, t)$ are adapted $(k \times 1)$ and $k \times (k + 1)$ vector and matrix valued processes, respectively, satisfying the suitable conditions for the opportunity set Ito process Y to be well defined. $w(s)$ is a $(k + 1)$ -dimensional standard Wiener process on a complete probability space (Ω, \mathcal{F}, P) , endowed with the augmentation by P -null sets of the natural filtration of $w(s)$, $(\mathcal{F}_t, 0 \leq t \leq T)$, with $\mathcal{F} = \mathcal{F}_T$.

All stochastic processes to appear in the sequel are progressively measurable with respect to this filtration, equalities involving random variables are understood to hold P -a.s. and those involving stochastic processes $dP \otimes dt$ -a.s.

Given the the dynamics (1.1) for the dividend process, the cum-dividend stock price is accordingly posited to admit the Ito representation:

$$\frac{dS(t)}{S(t)} = \mu(\omega, t)dt + \sigma(\omega, t)dw(t) \quad (1.2)$$

This conjecture will be fulfilled in the equilibrium analysis to be pursued, where the interest rate process $r(\omega, t)$ and the coefficients of the stock price process will be determined endogenously².

In order to slightly simplify the analysis, and in line with our purpose of emphasizing the equilibrium influence of ambiguity aversion, we look for equilibria in which, possibly after suitable rescaling, the rows of the $k \times (k+1)$ -dimensional matrix-valued process $\Xi(\cdot)$ are orthonormal vectors spanning the kernel of $\sigma(\omega, t)$; so that, if $\Sigma(\omega, t)$ denotes the volatility matrix

$$\Sigma(\omega, t) = \begin{bmatrix} \sigma(\omega, t) \\ \Xi(\omega, t) \end{bmatrix}_{1 \times (k+1) \atop k \times (k+1)}$$

then:

$$\Sigma(\omega, t)^{-1} = [\sigma(\omega, t)'(\sigma(\omega, t)\sigma(\omega, t)')^{-1} \quad \Xi(\omega, t)']$$

We will show that for the matrix $\Xi(\omega, t)$ to have the above property in equilibrium, one needs the same matrix to span the Kernel of the dividend's volatility $\sigma_\varepsilon(Y)$ as well.

1.2.2 Modelling Ambiguity

We pointed out that the measure P under which the dynamics (1.1)-(1.2) have been specified plays the role of a reference belief, supposed to describe with some approximation the data generating process of the primitive state variables. Agents are assumed to display aversion towards the ambiguous specification of this model by considering a whole set of relevant misspecifications, which take the form of equivalent changes of probability measure. The requirement of absolute continuity of 'contaminated' measures with respect to the reference belief seems to yield no loss of generality: it is somewhat natural to expect a higher level of confidence on those events regarded as impossible. Such a requirement is then strengthened to equivalence in order to leave aside technical issues that would come at the expense of tractability, without affecting substantially the economic analysis. Given an \mathbb{R}^{k+1} -valued adapted process $\kappa(t)$ satisfying suitable integrability conditions³ the exponential

²Assumed to satisfy regularity conditions similar to those of μ_ε and σ_ε .

³In particular we assume that a Novikov condition is satisfied, namely that the following inequality holds

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T (\Sigma(s)^{-1} \kappa(s))' \cdot (\Sigma(s)^{-1} \kappa(s)) ds \right) \right] < \infty$$

martingale

$$Z(\kappa, t) = \exp \left\{ \int_0^t (\Sigma(\omega, s)^{-1} \kappa(\omega, s))' \cdot dw(s) - \frac{1}{2} \int_0^t |\Sigma(\omega, s)^{-1} \kappa(\omega, s)|^2 ds \right\} \quad (1.3)$$

is the density process for a ‘contaminated’ probability measure

$$P^\kappa(A) = \mathbb{E} [Z(\kappa, T) \mathbf{1}_A] , \quad A \in \mathcal{F}_T$$

and $w_\kappa(t) = w(t) - \int_0^t \Sigma(\omega, s)^{-1} \kappa(\omega, s) ds$ is a P^κ -Brownian motion. It will be convenient to decompose the vector κ by expliciting its first and remaining components: $\kappa = [\kappa_1 \quad \bar{\kappa}]$.

Notice that under the probability measure P^κ the following representation holds for the stochastic opportunity set of the economy⁴:

$$\frac{dS(t)}{S(t)} = [\mu(\omega, t) + \kappa_1(t)] dt + \sigma(\omega, t) dw_\kappa(t) \quad (1.4)$$

and

$$\begin{aligned} \frac{d\varepsilon(t)}{\varepsilon(t)} &= [\mu_\varepsilon(Y) + \sigma_\varepsilon(Y) \sigma(\omega, t)' (\sigma(\omega, t) \sigma(\omega, t)')^{-1} \kappa_1(t)] dt + \sigma_\varepsilon(Y) dw_\kappa(t) \\ dY(t) &= [\Lambda(\omega, t) + \bar{\kappa}(t)] dt + \Xi(\omega, t) dw_\kappa(t) \end{aligned} \quad (1.5)$$

Ambiguity has a ‘local’ nature in what follows, that is, the range of relevant mispecifications induced is limited by requiring the corresponding density processes to satisfy instantaneously a (possibly state dependent) upper bound constraint on the growth rate of their conditional relative entropy with respect to the reference belief. Namely, given the relative entropy of measure $\tilde{\Upsilon}$ with respect to measure Υ (both defined on the measurable space (Ω, \mathcal{F}))

$$\mathcal{H}(\tilde{\Upsilon} \parallel \Upsilon) := \begin{cases} \mathbb{E} \left[\frac{d\tilde{\Upsilon}}{d\Upsilon} \log \frac{d\tilde{\Upsilon}}{d\Upsilon} \right] & \text{if } \tilde{\Upsilon} \ll \Upsilon \\ +\infty & \text{otherwise} \end{cases}$$

the *continuation* relative entropy corresponding to the contaminated probability measure P^κ is the stochastic process $\mathcal{H}_t(\cdot)$ defined by

$$\mathcal{H}_t(P^\kappa \parallel P) := \mathbb{E} \left[\frac{Z(\kappa, T)}{Z(\kappa, t)} \log \frac{Z(\kappa, T)}{Z(\kappa, t)} \middle| \mathcal{F}_t \right] := \mathbb{E}^\kappa \left[\log \frac{Z(\kappa, T)}{Z(\kappa, t)} \middle| \mathcal{F}_t \right]$$

$t \in [0, T]$, where $\mathbb{E}^\kappa[\cdot]$ denotes expectation with respect to the measure P^κ . Since

$$\mathcal{H}_t(P^\kappa \parallel P) + \log Z(\kappa, t) = \mathbb{E}^\kappa [\log Z(\kappa, T) | \mathcal{F}_t]$$

⁴We remind that $\sigma_\varepsilon(Y) \Xi(\omega, t)' = 0 \bar{\Gamma}_k'$.

or

$$\begin{aligned} \mathcal{H}_t(P^\kappa \| P) + \int_0^t (\Sigma(\omega, s)^{-1} \kappa(\omega, s))' \cdot dw_\kappa(s) + \frac{1}{2} \int_0^t |\Sigma(\omega, s)^{-1} \kappa(\omega, s)|^2 ds = \\ \mathbb{E}^\kappa \left[\int_0^T (\Sigma(\omega, s)^{-1} \kappa(\omega, s))' \cdot dw_\kappa(s) + \frac{1}{2} \int_0^T |\Sigma(\omega, s)^{-1} \kappa(\omega, s)|^2 ds \middle| \mathcal{F}_t \right] \end{aligned}$$

we have:⁵

$$\mathcal{H}_t(P^\kappa \| P) = \frac{1}{2} \mathbb{E}^\kappa \left[\int_t^T |\Sigma(\omega, s)^{-1} \kappa(\omega, s)|^2 ds \middle| \mathcal{F}_t \right]$$

and

$$\begin{aligned} \frac{d}{dt} \mathcal{H}_t(P^\kappa \| P) &= \frac{\|\Sigma(\omega, t)^{-1} \kappa(\omega, t)\|^2}{2} \\ &= \frac{\kappa_1(\omega, t)^2 (\sigma(t) \sigma(t)')^{-1} + \bar{\kappa}(\omega, t)' \bar{\kappa}(\omega, t)}{2} \end{aligned} \quad (1.6)$$

A time varying bound on the time rate of change of the continuation relative entropy between an admissible likelihood and the reference belief is a convenient way to identify the set of probability measures appearing in the preference ordering representation of the ambiguity averse agent. It delivers time consistency of the dynamic decision making problems. Moreover, it captures the intuition that the more ambiguity averse agents are, the higher the tolerated maximal discrepancy between the reference belief and other candidate models. It is convenient to assume an upper bound on (1.6) by assuming an upper bound on the summands appearing in the last expression. Therefore, the following restriction on the drift change identifies the set of models regarded as relevant by the agents:

$$\kappa \in K \quad K := \left\{ \kappa(\omega, t) : \left(\frac{\kappa_1(\omega, t)^2 (\sigma(t) \sigma(t)')^{-1}}{2} \leq h_1(Y) \right) \cap \left(\frac{\bar{\kappa}(\omega, t)' \bar{\kappa}(\omega, t)}{2} \leq \bar{h}(Y) \right) \right\} \quad (1.7)$$

for given one dimensional non negative processes $h_1(Y)$ and $\bar{h}(Y)$.

1.2.3 Max-Min Expected Utility

Trading takes place continuously and there are no market frictions. Let $\pi(t)$ denote the proportion of wealth $W(t)$ that the agent invests in the stock. Then, given a nonnegative consumption rate process c with $\int_0^T c(u) du < \infty$, a trading strategy π that satisfies:

$$\int_0^T \left(|\pi(s) \mu(s)| + |\pi(s) \sigma(s)|^2 \right) ds < \infty \quad P - \text{a.s.}$$

⁵Possibly after a localization argument.

and a model ‘perturbation’ induced by some $\kappa \in K$, Girsanov theorem implies that $W(t)$ evolves under P^κ according to the dynamics⁶

$$\begin{aligned} W(t) = & x + \int_0^t [W(s)r(s) - c(s)] ds \\ & + \int_0^t W(s)\pi(s)(\mu(t) + \kappa_1(s) - r(s))ds + \int_0^t W(s)\pi(s)\sigma(t) \cdot dw_\kappa(s) \end{aligned} \quad (1.8)$$

where $\beta(t) = \exp(-\int_0^t r(s)ds)$ is the discounting process and $x = W(0)$. Agent i is provided with an initial endowment of η_i shares of the stock.⁷ Therefore, the initial condition $x_i = \eta_i S(0)$ holds throughout. A consumption process c is attainable if financed by an admissible trading strategy for which the budget equation (1.8) holds with $W(t) \geq 0, \forall t \in [0, T]$. Obviously $c(T) = W(T)$.

We capture agents’ concern for ambiguity by preference orderings that admit a Max-Min expected utility representation. We restrict both agents in our model to display an isoelastic felicity function with relative risk aversion $1 - \mathcal{R}$, $\mathcal{R} < 1$, the logarithmic case being recovered for $\mathcal{R} \rightarrow 0$, both for intertemporal consumption and terminal wealth. They are characterized by the same time horizon but possibly heterogeneous degrees of risk and ambiguity aversions, captured by different parameters h_1, \bar{h} and \mathcal{R} .

The preferences orderings of the agents over admissible consumption plans $\{c(t), W(T)\}_{0 \leq t < T}$, thus admit the lower expected utility representation:

$$\begin{aligned} U^i(c) &= \inf_{\kappa \in K} \mathbb{E} \left[A \int_0^T Z(\kappa, t) \left(\frac{c(t)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right) dt + (1 - A) Z(\kappa, T) \frac{W(T)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right] \\ &= \inf_{\kappa \in K} \mathbb{E}^\kappa \left[A \int_0^T \left(\frac{c(t)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right) dt + (1 - A) \frac{W(T)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right] \end{aligned} \quad (1.9)$$

$i = 1, 2$. It then follows that agent i solves the max-min program:

$$J^i(x) = \sup_{\pi, c} U^i(c) \quad (1.10)$$

Subject to (1.8) and the dynamics (1.1), (1.2).

1.3 Individual Consumption Problem

If the stock price process is supposed to be an Ito Process, with coefficients $\mu(\omega, t)$ and $\sigma(\omega, t)$ generic stochastic processes adapted to the filtration \mathcal{F}_t , then it is not obvious how the consumption-investment problem of an ambiguity averse agent should be implemented in a partial equilibrium framework. That is, when the dynamics of the opportunity set of the economy are taken as exogenously given. In light of this indeterminacy, we depart from the specification (1.4)-(1.5) and

⁶We omit the functional argument ω in order to simplify notation.

⁷Given the normalization in place on the aggregate supply of the stock, we have $\sum_i \eta_i = 1$.

postulate the following system for the stochastic opportunity set of economy under a probability measure P^κ

$$\begin{aligned}\frac{dS(t)}{S(t)} &= [\mu(Y) + \kappa_1(t)]dt + \sigma(Y)dw_\kappa(t) \\ \frac{d\varepsilon(t)}{\varepsilon(t)} &= \mu_\varepsilon(Y) + \sigma_\varepsilon(Y)dw_\kappa(t) \\ dY(t) &= [\Lambda(\omega, t) + \bar{\kappa}(t)]dt + \Xi(\omega, t)dw_\kappa(t)\end{aligned}\tag{1.11}$$

with $\mu(\cdot)$ and $\sigma(\cdot)$ deterministic functions of the state Y , so that the influence of the Girsanov kernel κ (and the Kuhn-Tucker multiplier used to address market incompleteness⁸) can be clearly recognized as an absolutely continuous change of probability measure on the space of sample paths of the state variable that drives these coefficients. Such a specification still encompasses most of the models investigated by the literature so far. In the next Section, it will become clear that this restriction does not hold in equilibrium. Nevertheless we will be able to achieve a characterization of equilibrium quantities under the general dynamics (1.2), while still assuming the exogenous dividend dynamics (1.1). If no Markovian assumption about the process Y can be stated, dynamic programming techniques cannot be implemented to characterize optimal individual policies, contrary to what has been customary in the literature on uncertainty aversion as of today.⁹ The following analysis therefore relies on the martingale approach developed by Pliska (1986), Cox and Huang (1989) and Karatzas *et al.* (1987) for the case of complete markets, and extended by He and Pearson (1991), Karatzas *et al.* (1991) and Cuoco (1997) to the case of incomplete markets. A Markovian specification of the model suitable for computational purposes will be detailed in the sequel.

The limited analytical tractability of the optimization problem formulated in (1.10) suggests an inversion of the sequence of sup and inf optimizations, by virtue of a suitable Min-Max theorem.¹⁰ Therefore we analyze the equivalent problem :

$$\begin{aligned}J^i(x) &= \inf_{\kappa \in K} \sup_c \mathbb{E}^\kappa \left[A \int_0^T \left(\frac{c(t)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right) dt + (1 - A) \frac{W(T)^{\mathcal{R}_i} - 1}{\mathcal{R}_i} \right] \\ &\text{s.t.} \quad (1.2), (1.1)(1.8)\end{aligned}\tag{1.12}$$

The consumption-investment problem laid out within the infimum over model mispecifications in (1.12) is a standard one - with market incompleteness arising because of the dimensionality of the Brownian innovation that generates the uncertainty of the economy - and it can be handled by means of the above mentioned techniques. Being solutions qualitatively similar across agents, in what follows we omit the index i). The selection of the optimal Girsanov kernel κ^* that completes the characterization of the individual consumption-investment problem under ambiguity aversion is then addressed by means of methods borrowed from the martingale approach to stochastic control (Davis (1979), Elliot (1982)). The following Proposition summarizes the results of both steps:

Proposition 1 *Let the consumption-investment problem of the ambiguity averse agent be given by program (1.10) subject to the opportunity set dynamics (1.2). Let the probability measure $Q^\theta(\cdot)$ be*

⁸See the Appendix.

⁹see Epstein and Miao (2001) for an exception.

¹⁰whose assumptions are easily seen to be satisfied, see for instance Ky-Fan (1953) or Sion (1958))

defined by $Q_\theta(\cdot) = \mathbb{E}^\kappa \left[\mathcal{E} \left(\frac{\mathcal{R}}{\mathcal{R}-1} \int \theta_{\kappa+\lambda} dw_\kappa \right) \mathbf{1}(\cdot) \right]$, where $\mathcal{E}(\cdot)$ denotes stochastic exponentiation and $\theta_{\kappa+\lambda}(s)$ is an adapted process that satisfies the Novikov condition and will be detailed below. Define the random variable

$$f(t) = \exp \left\{ -\frac{\mathcal{R}}{\mathcal{R}-1} \int_0^t \left(r(s) - \frac{|\theta_{\kappa+\lambda}(s)|^2}{2(\mathcal{R}-1)} \right) ds \right\}$$

Then the optimal consumption policy is given by

$$c^*(t) = W^*(t) \text{mpc}(t)$$

for a marginal propensity to consume given by the stochastic process

$$\text{mpc}(t) = \frac{1}{\mathbb{E}_{\theta^*} \left[\int_t^T \frac{f(s)}{f(t)} ds + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right]}$$

where $\mathbb{E}_{\theta^*}[\cdot]$ denotes expectation with respect to the probability measure Q_{θ^*} . The optimally invested wealth process is given by

$$W^*(t) = x \frac{\xi(\kappa^* + \lambda^*, t)^{\frac{1}{\mathcal{R}-1}}}{f(t)} \frac{\mathbb{E}_{\theta^*} \left[\int_t^T f(s) ds + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \middle| \mathcal{F}_t \right]}{\mathbb{E}_{\theta^*} \left[\int_0^T f(s) ds + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \right]}$$

The shadow market price of risk and ambiguity that identifies the min-max martingale measure under ambiguity aversion is given by

$$\theta^*(t) := \theta_{\kappa^* + \lambda^*}(t) = \sigma(t)'(\sigma(t)\sigma(t))^{-1}(\mu(t) - r(t) + \kappa_1^*(t)) - \frac{(\mathcal{R}-1)a(t)}{\tilde{f}(t, T) - \int_0^t f(s)ds}$$

where $\tilde{f}(t, T)$ denotes the Levy-martingale

$$\tilde{f}(t, T) = \mathbb{E}_{\theta^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s)ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \middle| \mathcal{F}_t \right]$$

and $a(t)$ is the predictable integrand process in its stochastic integral representation

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t a(s) \cdot dw_{\theta^*}(s)$$

Furthermore, the optimal Girsanov kernel κ^* is given by

$$\kappa_1^* = \begin{cases} -(\mu - r) & \text{if } -\sqrt{2h_1(Y)} < \frac{(\mu - r)}{\sqrt{\sigma\sigma'}} < \sqrt{2h_1(Y)} \\ -\text{sgn}(\mu - r)\sqrt{2(\sigma\sigma')h_1(Y)} & \text{otherwise} \end{cases}$$

$$\bar{\kappa}^*(t) = -\sqrt{2\bar{h}(Y)} \frac{a(t)'}{\sqrt{a(t)a(t)'}}$$

The optimal investment policy in the stock is

$$\pi^*(t) = \frac{1}{1-\mathcal{R}} (\sigma(t)\sigma(t)')^{-1} [\mu(t) - r(t) + \kappa_1^*(t)] + \sigma(t)(\sigma(t)\sigma(t)')^{-1} \frac{a(t)}{\tilde{f}(t, T) - \int_0^t f(s)ds}$$

Certainly, a significant difficulty of the characterization above lies in the stochastic integral representation of the martingale $\tilde{f}(t, T)$, evaluated at the optimal Girsanov kernels κ^* . An attempt to a slightly more explicit characterization, based on the solution of a Backward Stochastic differential equations¹¹ is given in the next Corollary.

Corollary 1 *If there exists a unique pair of square integrable adapted processes (p, b) , solution of the following Backward Stochastic differential equation*

$$\begin{aligned} dp(t) &= \left[p(t) \left(\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2} b(t)b(t)' - \sqrt{2\bar{h}(Y)} \sqrt{b(t)b(t)'} \right) - \right. \\ &\quad \left. \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \right] dt + p(t)b(t)dw_\kappa(t) \\ p(T) &= \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \end{aligned} \quad (1.13)$$

then the value function (1.12) is given by

$$J(x) = \left(p(0) - \mathbb{E} \left[\int_0^T \xi(\kappa^* + \lambda^*, s)^{\frac{\mathcal{R}}{\mathcal{R}-1}} p(s) \left(\sqrt{2\bar{h}(Y)} \sqrt{b(s)b(s)'} + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} \tilde{\theta}_0(\kappa_1^*)' \tilde{\theta}_0(\kappa_1^*) \right) ds \right] \right)^{1-\mathcal{R}}$$

where

$$\tilde{\theta}_0(\kappa_1^*) = \begin{cases} 0 & \text{if } -\sqrt{2h_1(Y)} < \frac{\mu(t) - r(t)}{\sqrt{\sigma(t)\sigma(t)'}} < \sqrt{2h_1(Y)} \\ (\sigma(t)\sigma(t)')^{-1} \left[\mu(t) - r(t) - \text{sgn}(\mu(t) - r(t)) \sqrt{2\sigma(t)\sigma(t)'h_1(Y)} \right]^2 & \text{otherwise} \end{cases}$$

The shadow market price of risk that identifies the min-max martingale measure under ambiguity aversion is given by

$$\theta^*(t) = \sigma(t)'(\sigma(t)\sigma(t))^{-1} [\mu(t) - r(t) + \kappa_1^*(t)] - (\mathcal{R}-1)b(t)$$

and the optimal Girsanov kernel is $\kappa^*(t) = -\sqrt{2\bar{h}(Y)} \frac{b(t)'}{\sqrt{b(t)b(t)'}}$. Furthermore, the optimal portfolio policy is

$$\pi^*(t) = \frac{1}{1-\mathcal{R}} (\sigma(t)\sigma(t)')^{-1} [\mu(t) - r(t) + \kappa_1^*(t)] + \sigma(t)(\sigma(t)\sigma(t)')^{-1} \frac{b(t)}{p(t)}$$

¹¹See El Karoui, Peng and Quenez (1997)

where

$$\kappa_1^*(t) = \begin{cases} -(\mu(t) - r(t)) & \text{if } -\sqrt{2h_1(Y)} < \frac{\mu(t) - r(t)}{\sqrt{\sigma(t)\sigma(t)'}} < \sqrt{2h_1(Y)} \\ -\text{sgn}(\mu(t) - r(t))\sqrt{2\sigma(t)\sigma(t)'h_1(Y)} & \text{otherwise} \end{cases}$$

Corollary 7 identifies as a sufficient condition for the existence of a solution of the consumption-investment problem under ambiguity aversion the solvability of the BSDE (1.13).¹² It also characterizes the optimal policies in terms of the solution (p, b) of the latter. If the stochastic opportunity set (1.2) is postulated to be a Markovian system, that is, the processes $\mu(\cdot)$, $\sigma(\cdot)$, $\lambda(\cdot)$, $r(\cdot)$ and $\Xi(\cdot)$ are deterministic functions of time and the current realization of the state variable, then dynamic programming techniques can be applied to obtain a special case of Proposition 1:

Corollary 2 *Let the stochastic coefficients of the opportunity set process (1.2) be $\mathcal{C}^{1,2}$ deterministic functions of time and Y , respectively. Then the value function $J(x)$ of the optimization problem (1.12) is given by $J(x) = G^{\frac{1}{1-\bar{\kappa}}}$, where G is solution of the Hamilton-Bellman-Jacobi equation*

$$\begin{aligned} & \frac{\partial G}{\partial t} + \Lambda(Y)' \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 G}{\partial Y \partial Y'} \right] + \sqrt{2\bar{h}(Y)} \sqrt{\frac{\partial G'}{\partial Y} \frac{\partial G}{\partial Y}} \\ & + G \left[\mathcal{R} r(Y) - \frac{\mathcal{R}}{2(\mathcal{R} - 1)} (\sigma(Y) \sigma(Y)')^{-1} (\mu(Y) - r(Y) + \kappa_1^*(t))^2 \right] + (\mathcal{R} - 1) A^{\frac{1}{1-\bar{\kappa}}} G^{\frac{\mathcal{R}}{1-\bar{\kappa}}} = 0 \end{aligned} \quad (1.14)$$

with terminal condition $G(T, Y) = 1 - A$. The optimal Girsanov kernel is given by

$$\kappa_1^*(t) = \begin{cases} -(\mu(Y) - r(Y)) & \text{if } -\sqrt{2h_1(Y)} < -\frac{\mu(Y) - r(Y)}{\sqrt{\sigma(Y)\sigma(Y)'}} < \sqrt{2h_1(Y)} \\ -\text{sgn}(\mu(Y) - r(Y))\sqrt{2h_1(Y)} & \text{otherwise} \end{cases}$$

$$\bar{\kappa}^*(t) = -\sqrt{2\bar{h}(Y)} \frac{\frac{\partial G(t)}{\partial Y}}{\sqrt{\frac{\partial G(t)}{\partial Y} \frac{\partial G(t)}{\partial Y}}}$$

Optimal consumption-investment policies in the Markovian case can be inferred from their counterparts reported in Proposition 1 once we remind the equality $a(\kappa, t) = \frac{\partial J(t)}{\partial Y}$.

Remark 1. For expositional purposes, and with the treatment of general equilibrium in mind, we point out that in the simpler case of an agent maximizing utility of terminal wealth alone ($A = 0$), the partial differential equation to be solved in order to sort out the function G , hence the indirect utility of the problem, becomes

$$\begin{aligned} & \frac{\partial G}{\partial t} + \Lambda(Y)' \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 G}{\partial Y \partial Y'} \right] + \sqrt{2\bar{h}(Y)} \sqrt{\frac{\partial G'}{\partial Y} \frac{\partial G}{\partial Y}} \\ & + G \left[\mathcal{R} r(Y) - \frac{\mathcal{R}}{2(\mathcal{R} - 1)} (\sigma(Y) \sigma(Y)')^{-1} (\mu(Y) - r(Y) + \kappa_1^*(t))^2 \right] = 0 \end{aligned}$$

with terminal condition $G(Y, T) = 1$. Feymann-Kac theorem implies that this problem coincides with

¹²See references for the important issue of the existence of adapted solutions of this nonlinear BSDE.

the HJB equation that would arise from the dynamic optimization problem

$$\inf_{\kappa \in K} \mathbb{E}^\kappa \left[e^{\int_t^T \mathcal{R} r(s) - \frac{\mathcal{R}}{2(\mathcal{R}-1)} (\sigma(Y)\sigma(Y)')^{-1} (\mu(Y) - r(Y) + \kappa_1(s))^2} ds \middle| \mathcal{F}_t \right]$$

$$\text{s.t. } dY = [\Lambda(Y) + \bar{\kappa}(t)]dt + \Xi(Y)dw_\kappa(t)$$

We remind that $\mathbb{E}^\kappa[\cdot]$ denotes expectation with respect to the probability measure P^κ , under which w_κ is a $(k+1)$ -dimensional Brownian motion under this measure. We notice in particular that the (innermost) value function of the program that identifies the min-max martingale measure is given by

$$\mathcal{V}(t) = \left(\mathbb{E}^\kappa \left[e^{\int_t^T \mathcal{R} r(Y) - \frac{\mathcal{R}}{2(\mathcal{R}-1)} (\sigma(Y)\sigma(Y)')^{-1} (\mu(Y) - r(Y) + \kappa_1)^2} ds \middle| \mathcal{F}_t \right] \right)^{\frac{1}{1-\mathcal{R}}}$$

from which one easily recovers the non ambiguity averse counterpart ($h_1(Y) = 0$, $\bar{h}(Y) = \bar{1}_k 0$ and $\kappa(t) = 0$ in (1.10). \square

Remark 2. There is not much hope of achieving a complete analytical characterization of the consumption investment problem of an ambiguity averse agent whose preferences ordering admits a max-min expected utility representation with CRRA felicity of intertemporal consumption. Let alone the additional layer of complication arising because of ambiguity aversion, the determination of the min-max martingale measure by means of the optimal Kuhn-Tucker multiplier λ^* does not seem to be an easy task. To see this, consider the non ambiguity averse counterpart of (1.14) with $A = 1$:¹³

$$\begin{aligned} \frac{\partial G}{\partial t} + \Lambda(Y)' \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 G}{\partial Y \partial Y'} \right] \\ + G \left[\mathcal{R} r(Y) - \frac{\mathcal{R}}{2(\mathcal{R}-1)} (\sigma(Y)\sigma(Y)')^{-1} (\mu(Y) - r(Y))^2 \right] + (\mathcal{R}-1)G^{\frac{\mathcal{R}}{1-\mathcal{R}}} = 0 \end{aligned}$$

$G(T, Y) = 0$. This problem is equivalent¹⁴ to the integral equation

$$G(t, Y) = (1 - \mathcal{R}) \int \mathcal{G}(t, Y, x) G(t, x)^{\frac{\mathcal{R}}{\mathcal{R}-1}} dx \quad (1.15)$$

where $\mathcal{G}(t, Y, x)$ is the Green's function of the linear differential operator

$$\begin{aligned} \mathcal{L}u = \frac{\partial u}{\partial t} + \Lambda(Y)' \frac{\partial u}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 u}{\partial Y \partial Y'} \right] \\ + u \left[\mathcal{R} r(t) - \frac{\mathcal{R}}{2(\mathcal{R}-1)} (\sigma(Y)\sigma(Y)')^{-1} (\mu(Y) - r(Y))^2 \right] \end{aligned}$$

Analytical solutions of this integral equation (1.15) are confined to some very peculiar case. The computation of the Green's function of the operator \mathcal{L} for interesting parameter specifications may likely prove an ambitious challenge. \square

¹³The case in which felicity of terminal wealth is optimized too is treated along the same lines

¹⁴Appropriate references

As often the case for optimal consumption-investment problems, the logarithmic felicity function constitutes an exception to the poor analytical tractability of the intertemporal consumption cases. With this choice, it is easily seen that the conditional version of the value function (1.10), $J(t, Y)$, reduces to the following control problem :¹⁵

$$J(t) = \inf_{\kappa \in K} \mathbb{E}^\kappa \left[\int_t^T \int_t^s r(u) + \frac{1}{2} (\sigma(u)\sigma(u)')^{-1} (\mu(u) - r(u) + \kappa_1(u))^2 du ds \middle| \mathcal{F}_t \right] \quad (1.16)$$

$$\text{s.t. } dY(t) = [\Lambda(Y) + \bar{\kappa}(t)] dt + \Xi(Y) dw_\kappa(t)$$

the optimal controls of which are easily seen to coincide with those highlighted in Corollary 2, and the value function J is solution of

$$\begin{aligned} \frac{\partial J}{\partial t} + \Lambda(Y)' \frac{\partial J}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 J}{\partial Y \partial Y'} \right] + \sqrt{2\bar{h}(Y)} \sqrt{\frac{\partial J'}{\partial Y} \frac{\partial J}{\partial Y}} \\ r(Y) + \frac{1}{2} (\sigma(Y)\sigma(Y)')^{-1} (\mu(Y) - r(Y) + \kappa_1^*)^2 = 0 \end{aligned}$$

with terminal condition $J(T, Y) = 0$. In light of this characterization, the portfolio policy of the ambiguity averse agent with a logarithmic felicity function inherits an interesting property, basically deriving from the myopic behavior typical of logarithmic felicities.

Corollary 3 (Endogenous stock market participation). *Let the preference orderings of a financial agent admit a max-min expected utility representation with a logarithmic felicity function, that is*

$$U(c) = \inf_{\kappa \in K} \mathbb{E}^\kappa \left[A \int_0^T \log c(t) dt + (1 - A) \log W(T) \right]$$

Then the optimal portfolio policy of the agent is

$$\pi^*(t) = \begin{cases} 0 & \text{if } -\sqrt{2h_1(Y)} < \frac{\mu(Y) - r(Y)}{\sqrt{\sigma(Y)\sigma(Y)'}} < \sqrt{2h_1(Y)} \\ (\sigma(Y)\sigma(Y)')^{-1} [\mu(Y) - r(Y) - \text{sgn}(\mu(Y) - r(Y)) \sqrt{2\sigma(Y)\sigma(Y)'h_1(Y)}] & \text{otherwise} \end{cases}$$

Limited stock market participation has been deemed as a possible explanation for the inability of consumption based models to capture the order of magnitude of the equity premium at reasonable levels of risk aversion. See Basak and Cuoco (1998)..... Several models appeared in the literature have exogenously imposed constraints of the ability of agents to fully invest in the opportunity set in order to enhance the asset pricing performance of equilibrium consumption-based models. It is therefore a desirable side-effect of ambiguity aversion that unconstrained agents find it optimal to avoid locally risky investments when the excess return on the stock falls within a given set determined by the instantaneous entropy bound. We emphasize that the equilibrium analysis pursued in the next section will highlight how these endogenous cycles of stock market participation are determined by the interplay of the agents. References to Dow and Werlang (...), Trojani and Vanini (...), Wang et al. (...), explain why we improve.

¹⁵See the Appendix for details.

Example. Consider the individual consumption investment problem of an agent with a CRRA felicity function that maximizes utility of terminal wealth, subject to a constant entropy bound (h_1, \bar{h}) and the simple Markovian (affine) specification:

$$\mu(Y) = \bar{\mu}Y \quad r(Y) = rY \quad \sigma(Y) = \bar{\sigma}$$

$$dY = [\alpha(m - Y) + \bar{\kappa}] dt + dz(t)$$

where all coefficients involved are positive constants, with $m > r$, and the stock is driven by a standard Brownian motion $w(t)$ independent of $z(t)$. After the provisions of Corollary 2, our task amounts to solving the boundary value problem

$$\frac{\partial G}{\partial t} + \alpha(m - Y) \frac{\partial G}{\partial Y} + \frac{1}{2} \frac{\partial^2 G}{\partial Y \partial Y'} + \sqrt{2\bar{h}} \frac{\partial G}{\partial Y} + G \left[\mathcal{R}rY - \frac{\mathcal{R}}{2(\mathcal{R} - 1)} \tilde{\theta}_0(\kappa_1^*)^2 \right] = 0$$

$G(Y, T) = 1$ where

$$\tilde{\theta}_0(\kappa_1^*) = \begin{cases} 0 & \text{if } -\sqrt{2h_1} < -\frac{(\bar{\mu}-r)Y}{\bar{\sigma}^2} < \sqrt{2h_1} \\ \bar{\sigma}^{-1}((\bar{\mu} - r)Y - \bar{\sigma} \operatorname{sgn}((\bar{\mu} - r)Y)\sqrt{2h_1}) & \text{otherwise} \end{cases}$$

Notice that the value function admits the following probabilistic representation:

$$G(t, Y) = \mathbb{E}^\kappa \left[e^{\int_t^T \left(-\frac{\mathcal{R}}{2(\mathcal{R}-1)} \bar{\sigma}^{-2} ((\bar{\mu}-r)Y - \bar{\sigma} \operatorname{sgn}((\bar{\mu}-r)Y)\sqrt{2h_1})^2 \mathbf{1}_B \right) ds} e^{\int_t^T \mathcal{R}rY ds} \middle| \mathcal{F}_t \right]$$

where B is the set involved in the definition of $\tilde{\theta}_0(\kappa_1^*)$ and the expectation is taken with respect to the density implied by the dynamics

$$dY = [\sqrt{2\bar{h}} + \alpha(m - Y)]dt + dz(t)$$

Therefore the optimal Girsanov kernel is $(\kappa_1, \bar{\kappa}) = (\bar{\sigma} \operatorname{sgn}((\bar{\mu} - r)Y)\sqrt{2h_1}, \sqrt{2\bar{h}})$ and the optimal portfolio policy is $\pi^* = \frac{(\bar{\mu}-r)Y + \bar{\sigma} \operatorname{sgn}((\bar{\mu}-r)Y)\sqrt{2h_1}}{(1-\mathcal{R})\bar{\sigma}^2} + \frac{1}{\bar{\sigma}G(t,Y)} \frac{\partial G(t,Y)}{\partial Y}$

1.4 Equilibrium

Our aim is to characterize equilibrium asset prices and consumption-investment policies in the context of the pure exchange economy we analyze. As a consequence of market incompleteness and ambiguity aversion, the conjecture $\mu(\omega, t) = \mu(Y)$, $\sigma(\omega, t) = \sigma(Y)$ - with $\mu(\cdot)$ and $\sigma(\cdot)$ deterministic functions of the state variable Y - is not fulfilled in equilibrium, therefore we revert to the general equity dynamics (1.2). For the sake of analytical simplicity, we assume that agents' preference orderings representations include a felicity function over either intertemporal consumption or terminal wealth, rather than the convex combination of the two used in the partial equilibrium setting. When utility over terminal wealth is considered, the stock is a claim to a final dividend payment. Clearing of the good market then mandates that the stock price coincides with the aggregate wealth of the economy. In light of the fact that $c(T) = W(T)$, the following analysis applies to either case.

We find it convenient to introduce the process

$$k_1 = \sigma'(\sigma\sigma')^{-1}\kappa_1$$

and consider it as control variable in place of κ_1 , without loss of generality. The (scalar) drift perturbation affecting the dynamics of the stock is then recovered from the above (vector) process by means of the immediate equality $\kappa_1 = \sigma k_1$. With this notation, the set K of Girsanov kernels that identifies the admissible likelihoods in the max-min expected utility representation may be rewritten as

$$K := \left\{ \kappa(\omega, t) : \left(\frac{k_1'(\omega, t) \cdot k_1(\omega, t)}{2} \leq h_1(Y) \right) \cap \left(\frac{\bar{\kappa}(\omega, t)' \cdot \bar{\kappa}(\omega, t)}{2} \leq \bar{h}(Y) \right) \right\} \quad (1.17)$$

An *equilibrium* is a pair of interest rate-stock price processes (S, r) (or, equivalently, a price system (μ, σ, r)) and a set of admissible policies (c_i, π_i, κ^i) , $i = 1, 2$, such that:

- (i) Individual rationality of agents holds: beliefs Q^{κ^i} are optimally selected; the consumption plan $(c_i(t))_{t \leq T}$, is optimal for U^i - given the dynamics (S, r) with respect to the measure Q^{κ^i} - and is attained by means of π_i .
- (ii) All markets clear:

$$c_1 + c_2 = \varepsilon \quad \pi_1 W_1^{c_1, \pi_1} + \pi_2 W_2^{c_2, \pi_2} = S \quad (1 - \pi_1) W_1^{c_1, \pi_1} + (1 - \pi_2) W_2^{c_2, \pi_2} = 0$$

where $W^{c, \pi}$ denotes the optimally invested wealth process corresponding to the policy (c, π) .

Let us consider first a model setting in which agents' preference orderings are characterized by generic felicity functions $u^i(c)$ in the CRRA family and by heterogeneous ambiguity aversion, indexed by parameters $(h_1^{(1)}, \overline{h^{(1)}})$ and $(h_1^{(2)}, \overline{h^{(2)}})$. $\kappa^{(i)} = (\kappa_1^{(i)}, \overline{\kappa^i})$, $i = 1, 2$, denote the optimal Girsanov kernels chosen by the agents. In a second step we will detail the relevant equilibrium quantities by restricting ourselves to specific risk aversion coefficients and dynamics of the driving state process Y .

$v_i(y)$, $t \leq T$, will henceforth denote the inverse marginal felicity function of agent i . If ψ_i is the Lagrange multiplier for the static budget constraint of agent i , then it follows from the individual optimal consumption policies of the agents that after the normalization $\gamma(t) = [\xi(t, \kappa^{(2)} + \lambda^2)] / [\xi(t, \kappa^{(1)} + \lambda^1)]$

the clearing condition for the good market

$$v_1(\psi_1 \xi(\kappa^{(1)} + \lambda^1, t)) + v_2(\psi_2 \xi(\kappa^{(2)} + \lambda^2, t)) = \varepsilon(t)$$

can be stated in the equivalent form:

$$v_2(\psi_1 \xi(\kappa^{(1)} + \lambda^1) \psi \gamma(t)) + v_1(\psi_1 \xi(\kappa^{(1)} + \lambda^1, t)) = \varepsilon(t)$$

where $\psi = \psi_2/\psi_1$ is the ratio of the static lagrange multipliers of the agents. The stochastic weighting process $\gamma(t)$ accounts for the heterogeneity in the perceived opportunity sets by directly modeling the intra-agents marginal rate of substitution and regarding it as a state variable the equilibrium state price density has to depend on, in excess of aggregate output.

If we define $\mathcal{U}'(t, \varepsilon, \gamma)$ to be the strictly decreasing inverse of the aggregate demand function:

$$x \longrightarrow v_1(x) + v_2(x \psi \gamma(t))$$

then we have, by construction

$$v_1(\mathcal{U}'(t, \varepsilon, \gamma)) + v_2(\psi \gamma(t) \mathcal{U}'(t, \varepsilon, \gamma)) = \varepsilon(t) \quad (1.18)$$

It is easily seen that the function $\mathcal{U}'(t, \cdot, \gamma)$ is the marginal utility of an aggregate, state-dependent welfare function describing a ‘representative agent’ who optimally consumes the endowment process, by suitably assigning a stochastic weight to the individual agents:

$$\mathcal{U}(t, \varepsilon, \gamma) \equiv \max_{c_1 + c_2 = \varepsilon} u^1(c_1(t)) + \psi \gamma u^2(c_2(t))$$

It should be noticed that the stochastic weight placed by the representative agent on the investors is meant to address two layers of heterogeneity, namely differing ‘worst case’ beliefs and differing (min-max) state price densities (He and Pearson (1991)) selected as a consequence of the market incompleteness implied by the opportunity set (1.4). Since the the choice of the min-max state price density is part of the equilibrium determination, we denote by $\bar{\lambda}(t)$ the k -dimensional dynamic Kuhn-Tucker multiplier meant to characterize the possible state-price densities of the model.¹⁶

The ‘market prices of risk and ambiguity’ that identify all possible martingale measures are then¹⁷

$$\theta_{\kappa^{(i)} + \lambda^i}(t) = \theta_0(\kappa_1^{(i)}) + \Xi(\omega, t)' \left(\bar{\lambda}^i(t) - \bar{1}_k r(t) + \overline{\kappa^{(i)}} \right) \quad i = 1, 2$$

¹⁶See, for instance, He and Pearson (1991) and the Appendix for more details.

¹⁷The first component of the expression, $\theta_0(\kappa_1^{(i)})$, is given in terms of the new control variable \mathbf{k}_1 by

$$\theta_0(\kappa_1^{(i)}) = \sigma'(\sigma\sigma')^{-1}(\mu - r) + \mathbf{k}_1^{(i)}$$

See the Appendix for more details.

Therefore, in light of the dynamics

$$\frac{d\xi_{\kappa^{(i)}+\lambda^i}(t)}{\xi_{\kappa^{(i)}+\lambda^i}(t)} = -r(t)dt - \theta_{\kappa^{(i)}+\lambda^i}(t)dw_{\kappa^{(i)}}(t)$$

We have, by Ito's lemma, the following evolution of the stochastic weighting process $\gamma(t)$:

$$\begin{aligned} \frac{d\gamma(t)}{\gamma(t)} = & \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)'(\bar{\lambda}^1 - r\bar{\mathbf{1}}_k + \bar{\kappa}^1) \right] dt + \left[\left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right) + \right. \\ & \left. \Xi(t)' \left(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1 \right) \right]' dw_{\kappa^{(2)}}(t) \end{aligned} \quad (1.19)$$

The inversion procedure just outlined provides a direct method to handle the consumption good market clearing conditions by means of equilibrium state-price processes, as functions of the dividend $\varepsilon(t)$ supplied by the stock and of the stochastic weighting process $\gamma(t)$. Clearly, in equilibrium the agents' state-price densities have the form:

$$\xi(\kappa^{(1)} + \lambda^1, t) = \frac{\mathcal{U}'(t, \varepsilon, \gamma)}{\mathcal{U}'(0, \varepsilon, \gamma)} \quad \xi(\kappa^{(2)} + \lambda^2, t) = \frac{\mathcal{U}'(t, \varepsilon, \gamma)}{\mathcal{U}'(0, \varepsilon, \gamma)} \gamma(t) \quad (1.20)$$

and $\psi_1 = \mathcal{U}'(0, \varepsilon, \psi\gamma)$. We can now state the following proposition where, as customary in the two agents-equilibrium literature¹⁸, equilibrium quantities are characterized in terms of the inverse function $\mathcal{U}'(t, \varepsilon, \psi\gamma)$.

Proposition 2 *If the SDE (1.19) has a strong solution, then an equilibrium exists. Let ψ^* be the positive value of the parameter $\psi = (\psi_2/\psi_1)$ for which the budget constraint*

$$\begin{aligned} \mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \frac{\mathcal{U}'(t, \varepsilon, \psi\gamma)}{\mathcal{U}'(0, \varepsilon, \psi\gamma)} v_1(\mathcal{U}'(t, \varepsilon, \psi\gamma), t) dt + \right. \\ \left. (1 - A) \frac{\mathcal{U}'(T, \varepsilon, \psi\gamma)}{\mathcal{U}'(0, \varepsilon, \psi\gamma)} v_1(\mathcal{U}'(T, \varepsilon, \psi\gamma), T) \right] = \eta_1 S(0) \end{aligned} \quad (1.21)$$

is satisfied; the agents' state-price densities as in (1.20) and their consumption policies given by

$$c_1^*(t) = v_1(\mathcal{U}'(t, \varepsilon, \psi\gamma), t) \quad c_2^*(t) = v_2(\psi\gamma(t)\mathcal{U}'(t, \varepsilon, \psi\gamma), t) \quad (1.22)$$

The equilibrium stock price process is defined by the expression:

$$S(t) = \frac{1}{\mathcal{U}'(t, \varepsilon, \psi\gamma)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_t^T \mathcal{U}'(s, \varepsilon, \psi\gamma) \varepsilon(s) ds + (1 - A) \mathcal{U}'(T, \varepsilon, \psi\gamma) \varepsilon(T) \middle| \mathcal{F}_t \right] \quad (1.23)$$

¹⁸see Karatzas, Lehocsky, Shreve and Xu (1991), Cuoco and He (1994), Cuoco and Basak (1998), Basak (2000) to mention just a few.

The equilibrium interest rate is given by:

$$\begin{aligned}
r = & -\frac{\mathcal{U}'_\varepsilon}{\mathcal{U}'} \varepsilon \left(\mu_\varepsilon + \sigma_\varepsilon k_1^{(1)} \right) - \frac{\mathcal{U}'_\gamma}{\mathcal{U}'} \gamma \left[k_1^{(1)'} \cdot k_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)' (\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1) + \right. \\
& \left. (\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1)' (\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1) + (k_1^{(1)} - k_1^{(2)})' \theta_0(k_1^{(1)}) \right] - \frac{1}{2} \frac{\mathcal{U}'_{\varepsilon\varepsilon}}{\mathcal{U}'} \varepsilon^2 \sigma'_\varepsilon - \\
& \frac{1}{2} \frac{\mathcal{U}'_{\gamma\gamma}}{\mathcal{U}'} \gamma^2 \left[(k_1^{(2)} - k_1^{(1)})' (k_1^{(2)} - k_1^{(1)}) + (\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1)' (\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1) \right] - \\
& \frac{\mathcal{U}'_{\gamma\varepsilon}}{\mathcal{U}'} \gamma \varepsilon \sigma_\varepsilon \left(k_1^{(2)} - k_1^{(1)} \right) \quad (1.24)
\end{aligned}$$

where $\mu - r + \sigma \cdot k_1^{(1)}$ and $\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1$ are as reported below. The equilibrium excess return relative to the first agents selected measure $Q^{\kappa^{(1)}}$

$$\mu - r + \sigma \cdot k_1^{(1)} = -\frac{\mathcal{U}'_\varepsilon}{\mathcal{U}'} \varepsilon \sigma \sigma'_\varepsilon - \frac{\mathcal{U}'_\gamma}{\mathcal{U}'} \gamma \sigma (k_1^{(2)} - k_1^{(1)}) \quad (1.25)$$

and the following relation holds between the equilibrium Kuhn-Tucker multiplier and Girsanov kernels of the agents.

$$\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1 = -\frac{\mathcal{U}'_\gamma}{\mathcal{U}'} \gamma \left(\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1 \right) \quad (1.26)$$

Furthermore the volatility process of the stock is given in equilibrium by the following expression

$$\begin{aligned}
\sigma(t)' = & - \left(\frac{\mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_\gamma(s) \varepsilon(s) \gamma(s) ds \middle| \mathcal{F}_t \right]}{S(t) \mathcal{U}'(t)} + \frac{\mathcal{U}'_\gamma(t)}{\mathcal{U}'(t)} \gamma(t) \right) (k_1^{(2)}(t) - k_1^{(1)}(t)) + \\
& \sigma_\varepsilon(t)' - \frac{\mathcal{U}'_\varepsilon(t)}{\mathcal{U}'(t)} \sigma_\varepsilon(t)' \varepsilon(t) + \frac{\sigma_\varepsilon(t)'}{S(t) \mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_\varepsilon(s) \varepsilon(s)^2 ds \middle| \mathcal{F}_t \right] \quad (1.27)
\end{aligned}$$

with $S(t)$ as in (1.23). The equilibrium investment policy of the first agent is

$$\pi_1^* = (\sigma \sigma')^{-1}(t) \left(\mu^*(t) - r^*(t) + \sigma(t) \cdot k_1^{(1)*}(t) \right) + (\sigma(t) \sigma(t)')^{-1} \sigma(t) \cdot \delta(t)$$

where $\delta(t)$ is the predictable integrand process in the stochastic integral representation of the Levy martingale

$$\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \frac{\mathcal{U}'(s)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}}{\mathcal{U}'(0)} ds + (1-A) \frac{\mathcal{U}'(T)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}}{\mathcal{U}'(0)} \middle| \mathcal{F}_t \right]$$

This Proposition fully characterizes the equilibrium: determines the state price densities, the consumption allocations, excess return on equity and the interest rate. In light of the fact that the security S is in positive net supply and pays an exogenously given dividend, the analysis is able to sort out the equilibrium equity volatility process¹⁹.

Let us briefly turn our attention to the process $\gamma(t)$ to gain further insight into the stochastic weighting that arises as a consequence of the agents' facing different state price densities. Contrarily to

¹⁹This is in contrast to the indeterminacy that would arise in a pure exchange economy with an opportunity set in zero net supply that pays no dividends, as in He and Leland (1993), Karatzas et al. (1990)

several studies dealing with heterogeneous beliefs equilibria²⁰, whereby this ‘disagreement’ process is fully determined in terms of exogenously specified primitives due to consistency requirements, in the current framework expression (1.19) reveals that its dynamics will be endogenously determined by agents’ individual rationality constraint. In particular, as briefly pointed out above, the process $\gamma(t)$ addresses two layers of ‘disagreement’ between the agents, the first arising from market incompleteness and the different martingale measures selected individually (He and Pearson (1991), Cuoco (1997)), the second being due to heterogeneous ambiguity aversions and different beliefs implied by the Max-Min expected utility representation. Full insight on the dynamics of this additional state variable will be gained in the sequel, once the equilibrium controls of the agents are illustrated in the context of specific examples; for the time being we notice that, according to (1.19), the volatility of the ‘disagreement’ intuitively increases with the differences in beliefs (Girsanov kernels) between the agents, thus, at least in those states for which the instantaneous entropy constraint binds, increases with the dispersion of ambiguity aversion across agents, as parameterized by the entropy bound $(h_1^{(i)}, \bar{h}^i)$. The solution of (1.19) may be expressed under the reference belief²¹ P as

$$\begin{aligned} \gamma(t) = \exp \left(\frac{1}{2} \int_0^t \left[\left(k_1^{(2)'} \cdot k_1^{(2)} + \bar{\kappa}^{2'} \cdot \bar{\kappa}^2 \right) - \left(k_1^{(1)'} \cdot k_1^{(1)} + \bar{\kappa}^{1'} \cdot \bar{\kappa}^1 \right) \right] - \right. \\ \left. \left[\left(\bar{\lambda}^2 - r \bar{1}_k \right)' \left(\bar{\lambda}^2 - r \bar{1}_k \right) - \left(\bar{\lambda}^1 - r \bar{1}_k \right)' \left(\bar{\lambda}^1 - r \bar{1}_k \right) \right] ds - \right. \\ \left. \int_0^t \left[\left(k_1^{(2)} - k_1^{(1)} \right) + \Xi(t)' \left(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1 \right) \right]' dw(s) \right) \end{aligned}$$

The last expression allows for a sharp disentangling of the two distinguished sources of disagreement mentioned above. In particular, assuming $h_1^{(1)} < h_1^{(2)}$ and $\bar{h}^1 < \bar{h}^2$ (we will keep this assumption in the examples to be discussed), $\gamma(t)$ tends to be higher the higher the heterogeneity in ambiguity aversions over the past; nevertheless, this effect might be possibly mitigated by the past occurrence of negative shocks whenever that circumstance was verified.

Before turning our attention to the equilibrium determination of the optimal Girsanov kernels κ^i and dynamic multipliers λ^i , we provide in the next Corollary the analog of the equilibrium interest rate and risk premium for the case of agents’ heterogeneous felicity functions being of the CRRA type; in this case the function $\mathcal{U}'(\varepsilon(t), \psi\gamma(t))$ is not explicitly computed in general, and the expressions appearing in Proposition 2 are not useful for computational purposes.

Corollary 4 *Let the agents’ felicity function be of the CRRA type with coefficients of relative risk aversion \mathcal{R}_i , $i = 1, 2$, then the equilibrium excess return on the stock is given by the following expressions*

$$\mu - r + \sigma \cdot k_1^{(1)} = \frac{R_a^1 R_a^2}{R_a^1 + R_a^2} \varepsilon \sigma \sigma'_\varepsilon - \frac{R_a^1}{R_a^1 + R_a^2} \sigma \gamma \left(k_1^{(2)} - k_1^{(1)} \right) \quad (1.28)$$

The following relation holds in equilibrium between optimal Kuhn-Tucker multipliers λ and Girsanov

²⁰see, for instance, Basak (2000)

²¹The stochastic differential equation (1.19) has been specified under any admissible likelihood of the second agent, $Q^{\kappa^{(2)}}$ mainly with the equilibrium treatment in mind, as Proposition 3 will clarify.

kernels κ selected by the agents.

$$\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1 = \gamma \frac{R_a^1}{R_a^1 + R_a^2} \left(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1 \right) \quad (1.29)$$

Furthermore, the equilibrium interest rate is given by the following expression

$$\begin{aligned} r = & \bar{R}_a \varepsilon \left(\mu_\varepsilon + \sigma_\varepsilon \mathbf{k}_1^{(1)} \right) - \frac{1}{2} \bar{R}_a \bar{P}_a \left| \theta_0(\kappa_1^{(1)}) + \Xi'(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) \right|^2 - \\ & \frac{\bar{R}_a}{R_a^2} \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + \left(\bar{\lambda}^2 - \bar{\lambda}^1 \right)' \left(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1 \right) + (\mathbf{k}_1^{(1)'} - \mathbf{k}_1^{(2)'}) \theta_0(\kappa_1^{(1)}) + (\bar{\kappa}^1 - \bar{\kappa}^2)' (\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) \right] + \\ & \frac{\bar{R}_a}{R_a^2} \frac{\mathcal{R}_2}{2(\mathcal{R}_2 - 1)} \left| (\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)}) + \Xi'(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1) \right|^2 + \\ & \frac{\bar{R}_a}{R_a^2} \frac{1}{\mathcal{R}_2 - 1} \left[\theta_0(\kappa_1^{(1)}) + \Xi'(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) \right]' \left[(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)}) + \Xi'(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1) \right] \end{aligned} \quad (1.30)$$

where $R_a^i = -(\mathcal{R}_i - 1)/c_i$ is the coefficient of absolute risk aversion of agent i , $(\mu - r + \sigma \cdot \mathbf{k}_1^{(1)})$, $(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)$ are as in (1.28) and (1.29), respectively, $\bar{R}_a = \frac{R_a^2 R_a^1}{R_a^1 + R_a^2}$ and $\bar{P}_a = \frac{c_1 \mathcal{R}_1}{(\mathcal{R}_1 - 1)^2} + \frac{c_2 \mathcal{R}_2}{(\mathcal{R}_2 - 1)^2}$

A similar characterization holds regardless of the isoelastic form assumed for the felicity functions of the agents, with the definition $R_a^i = -(u_{cc}^i(c^*)/u_c^i(c^*))$, $i = 1, 2$. Expressions (1.28) and (1.29) characterize the risk premium on the stock, impose restrictions on the equilibrium form of the optimal Girsanov kernels and Kuhn-Tucker multipliers adopted by the two agents. They are the analog of the expression derived for the case of market incompleteness but no ambiguity by Cuoco and He (1994). The equilibrium risk premium as represented by agent 2 can be expressed as

$$\mu - r + \sigma \cdot \mathbf{k}_1^{(2)} = \frac{R_a^1 R_a^2}{R_a^1 + R_a^2} \varepsilon \sigma \sigma'_\varepsilon + \frac{R_a^2}{R_a^1 + R_a^2} \sigma \gamma \left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right)$$

therefore, in analogy with the literature on heterogeneous belief equilibria (see, for instance, Basak (2000)), one may write equivalently

$$\begin{aligned} \mu - r + \sigma \cdot \mathbf{k}_1^{(1)} &= \frac{R_a^1 R_a^2}{R_a^1 + R_a^2} \text{Cov}^{\kappa^{(1)}} \left[\frac{dS}{S}, d\varepsilon \right] - \frac{R_a^1}{R_a^1 + R_a^2} \text{Cov}^{\kappa^{(1)}} \left[\frac{dS}{S}, d\gamma \right] \\ \mu - r + \sigma \cdot \mathbf{k}_1^{(2)} &= \frac{R_a^1 R_a^2}{R_a^1 + R_a^2} \text{Cov}^{\kappa^{(2)}} \left[\frac{dS}{S}, d\varepsilon \right] + \frac{R_a^2}{R_a^1 + R_a^2} \text{Cov}^{\kappa^{(2)}} \left[\frac{dS}{S}, d\gamma \right] \end{aligned}$$

where each agents takes the relevant expectations under his optimal belief, although due to absolute continuity of the probability measures the instantaneous covariances above are really invariant to this measure change. As in the standard consumption CAPM, a risky's security premium is positively related to the covariance of its return with aggregate consumption. In the present context, however, an additional component is driven by the covariance of the asset's return with the evolution of the stochastic weighting process, and risk premia are increasing in this covariance only for the agent to whom positive innovations of the additional state variable are valuable, as they represent favorable shifts of the cross-sectional wealth distribution (agent 2). To the purpose of disentangling the

influence of the two components that drive the risk premia, additional insight may be gained by virtue of the following expression, counterpart of a representation that holds in the heterogeneous belief case²²:

$$\mu - r + \sigma \cdot \left(\frac{R_a^1}{R_a^1 + R_a^2} k_1^{(1)} + \frac{R_a^2}{R_a^1 + R_a^2} k_1^{(2)} \right) = \frac{R_a^1 R_a^2}{R_a^1 + R_a^2} \text{Cov}^{\kappa^{(1)}} \left[\frac{dS}{S}, d\varepsilon \right]$$

Therefore the aggregate endowment's variability entirely drives the risk premium as represented with respect to a fictitious belief, which corresponds to a Girsanov kernel that is a risk-aversion weighted average of the individual selections, functions of the respective ambiguity aversions. Once again we emphasize the additional layer of endogeneity that arises in the present context, whereby beliefs are optimally selected in equilibrium.

As in the economy with no concern for ambiguity, the equilibrium interest rate is positively related to the expected endowment growth rate and negatively related to the endowment risk, if the aggregate prudence is positive. Nevertheless, the influence of market incompleteness and especially ambiguity aversion on this classical component is yet apparent, as long as this growth rate and the equilibrium market price of risk are influenced by 'belief selection' problem of the first agent. On discussing equilibrium risk premia, we have seen that the variability of the market price of risk is increased by the dispersion of ambiguity aversions across agents, and that aggregate consumption is no more its only driving component; to the extent that aggregate prudence is positive, ambiguity enhances the impact of this negative term to compensate for agents' additional precautionary savings due to future risky shifts of the cross-sectional wealth distribution. As a result of model uncertainty, the interest rate is driven by three extra terms. The first is related to the discrepancy in agents' representations of the expected growths of the relevant state variables: if the expected consumption growth relative to the second agent's belief is higher than the same quantity represented with respect to agent's one belief²³, then the interest rate increases in order to provide adequate incentive to reduce the excessive saving demand. An opposite tendency arises in the other case. The fourth term in (1.30) decreases (for $\mathcal{R}_2 > 0$) the interest rate to compensate for the extra precautionary saving demand induced by the dispersion in beliefs due to ambiguity aversion. The last term implies that the higher the instantaneous covariance between the equilibrium state price density and the weighting process the higher the equilibrium interest rate.

Since the signs of the additional terms need to be identified on a case by case basis, the net effect of ambiguity on the equilibrium short rate is, in general, undetermined. If agents are highly prudent though, and ambiguity aversions are sufficiently heterogeneous, we may expect the present framework to generate lower term structure of interest rates than a model were ambiguity is of no concern.

As already pointed out, Proposition 2 highlights important restrictions about key economic quantities that have to hold in equilibrium, but leaves otherwise untouched the fundamental issue of determination of equilibrium levels of optimal Girsanov kernels κ^* and Kuhn-Tucker multipliers λ^* . Quite clearly the control problems that delivered these quantities in a partial equilibrium framework are now handled at equilibrium prices, that is, imposing the equilibrium relations dictated by Propo-

²²The expression is obtained by scaling the individual risk premia with the weights $\frac{R_a^i}{R_a^1 + R_a^2}$ and summing them up.

²³we remind that agent's one state price density is assumed to be the equilibrium state price density

sition 2. Notice that (1.24), (1.25) and (1.26) are path-wise restrictions that hold for each $\omega \in \Omega$ and cannot be modified by the action of each agent; furthermore, the competitive behavior of the agents suggests that each agent takes as exogenously given the policies of the other; in particular, agent 2 considers the equilibrium state price density $\xi_1 = \mathcal{U}'$ exogenous.

Proposition 3 *Let $A = 0$ or $A = 1$. In equilibrium, the optimal Girsanov kernels κ_i , $i = 1, 2$ and Kuhn-Tucker multipliers λ_i are solutions of the following control problems.*

$$\inf_{\lambda^1} \inf_{\kappa^{(1)} \in K^1} \mathbb{E}^{\kappa^{(1)}} \left[A \int_t^T \mathcal{U}'(s, \varepsilon, \hat{\gamma})^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}} ds + (1-A) \mathcal{U}'(T, \varepsilon, \hat{\gamma})^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}} \middle| \mathcal{F}_t \right] \quad (1.31)$$

$$\text{s.t.} \quad \frac{d\varepsilon(t)}{\varepsilon(t)} = (\mu_\varepsilon(Y) + \sigma_\varepsilon(Y) \cdot \mathbf{k}_1^{(1)}(t))dt + \sigma_\varepsilon(Y)dw_{\kappa^{(1)}}$$

$$dY = [\Lambda(t) + \overline{\kappa^1}(t)] dt + \Xi(t)dw_{\kappa^{(1)}}$$

and

$$\inf_{\lambda^2} \inf_{\kappa^{(2)} \in K^2} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T A \mathcal{U}'(s, \varepsilon, \hat{\gamma})^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} \gamma(s)^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} ds + (1-A) \mathcal{U}'(T, \varepsilon, \hat{\gamma})^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} \gamma(T)^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} \middle| \mathcal{F}_t \right] \quad (1.32)$$

$$\text{s.t.} \quad \frac{d\varepsilon(t)}{\varepsilon(t)} = (\mu_\varepsilon(Y) + \sigma_\varepsilon(Y) \cdot \mathbf{k}_1^{(2)}(t))dt + \sigma_\varepsilon(Y)dw_{\kappa^{(2)}}$$

$$dY = [\Lambda(t) + \overline{\kappa^2}] dt + \Xi(t)dw_{\kappa^{(2)}}$$

$$\begin{aligned} \frac{d\gamma(t)}{\gamma(t)} = & \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + (\overline{\lambda^2} - \overline{\lambda^1})'(\overline{\lambda^1} - r\overline{\mathbf{I}}_k + \overline{\kappa^1}) \right] dt + \left[\left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right) + \right. \\ & \left. \Xi(t)' \left(\overline{\lambda^2} - \overline{\lambda^1} + \overline{\kappa^2} - \overline{\kappa^1} \right) \right] dw_{\kappa^{(2)}}(t) \end{aligned}$$

for agent 1 and agent 2, respectively, where K^1 denotes the set $\{((\kappa_1^{(1)})^2 \leq 2h_1) \cap (\overline{\kappa^1}' \cdot \overline{\kappa^1} \leq 2\overline{h}_1)\}$ and K^2 is defined similarly.

The equilibrium state price density $\mathcal{U}'(s, \varepsilon, \gamma)$ reflects the impact of ambiguity aversion and market incompleteness on consumption choices, and depends on the interplay between the heterogeneous preference orderings of the agents. This effect feeds back into the individual choice problem of the agents and must be consistent with the consumption-investment policy selected, in a way that keeps into account the price-taking assumption. The notion of rational expectations equilibrium mandates that agents anticipate the influence of the additional state variable γ , to which the equilibrium control problems to be solved will thus be subject. In the next section we will investigate two explicit cases, the analytical tractability of which allows to overcome this additional difficulty.

1.5 Two Worked out Equilibrium Examples

Suppose that agents share the same felicity function over intertemporal consumption or terminal wealth. Their preference orderings still differ because of heterogeneity of the instantaneous entropy bounds that determine the set of priors in their utility representation and, as will be shown by means of the examples below, the general expressions discussed above assume can be clarified to a further extent. In what follows we assume that $h^1 \leq h^2$ and $\bar{h}^1 \leq \bar{h}^2$, so that

$$\sup_{\kappa^{(1)} \in K^1} \mathcal{H}_t \left(Q^{\kappa^{(1)}} \parallel P \right) \leq \sup_{\kappa^{(2)} \in K^2} \mathcal{H}_t \left(Q^{\kappa^{(2)}} \parallel P \right)$$

because $K^1 \subseteq K^2$, that is, the maximal continuation relative entropy between considered priors and the reference belief is higher for agent 2 than for agent 1, who believes in a narrower discrepancy between the reference model and the true data generating process and in this sense is less ambiguity averse.

We first deal with logarithmic felicity of intertemporal consumption and comment the interesting results arising yet at this level of tractability; we then pursue the more ambitious task of studying an equilibrium in which agent derive utility from terminal wealth but have a general CRRA felicity function. For the sake of clarity we adopt a markovian framework in the second example, that is, we let the coefficients of the dividend process μ_ε and σ_ε deterministic and well behaved functions of the current state Y . According to what we saw in the partial equilibrium framework, the relaxation of this hypothesis is not a formidable challenge.

1.5.1 Utility of intertemporal consumption. Logarithmic felicities.

Both agents adopt a logarithmic felicity function in their preference ordering representation and maximize utility of intertemporal consumption:

$$U^i(c) = \inf_{\kappa^{(i)} \in K^i} \mathbb{E}^{\kappa^{(i)}} \left[\int_0^T \log c(t) dt \right]$$

The next proposition characterizes the equilibrium quantities prevailing in this economy.

Proposition 4 *Let the economic agents be characterized by logarithmic felicity functions and ambiguity aversion parameters (h_1, \bar{h}_1) and (h_2, \bar{h}_2) , with $h_1 \leq h_2$ and $\bar{h}_1 \leq \bar{h}_2$. Then the optimal equilibrium Kuhn-Tucker multipliers and Girsanov kernels selected by the agents are given by*

$$\begin{aligned} \overline{\lambda^{1*}} &= r^* \bar{1}_k - \overline{\kappa^{1*}} & \overline{\lambda^{2*}} &= 2(\overline{\lambda^{1*}} + \overline{\kappa^{1*}}) - r^* \bar{1}_k \\ \kappa_1^{(1)*} &= -\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} & \overline{\kappa^{1*}} &= -\sqrt{2\bar{h}_1(Y)} \frac{\tilde{a}}{\sqrt{\tilde{a}' \tilde{a}}} \end{aligned} \quad (1.33)$$

$$k_1^{(2)*} = \begin{cases} -\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} - \sigma'_\varepsilon & \text{if} \\ & (-\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} - \sigma'_\varepsilon)' (-\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} - \sigma'_\varepsilon) \leq 2h_2(Y) \\ \sqrt{2h_2(Y)} \frac{k_1^{(1)} - \sigma'_\varepsilon}{\sqrt{(k_1^{(1)} - \sigma'_\varepsilon)'(k_1^{(1)} - \sigma'_\varepsilon)}} & \text{otherwise} \end{cases}$$

$$\overline{\kappa^{2*}} = -\sqrt{2h_2(Y)} \frac{a}{\sqrt{a'a}} \quad (1.34)$$

where the stochastic processes a and \tilde{a} are the predictable integrand processes in the stochastic integral representation of the Levy martingales (1.101) and (1.102), respectively, reported in the Appendix²⁴. The optimal consumption allocation is

$$c_1^*(t) = \frac{\varepsilon(t)\gamma(t)}{1 + \gamma(t)} \quad c_2^*(t) = \frac{\varepsilon(t)}{1 + \gamma(t)} \quad (1.35)$$

The excess return with respect to the reference probability measure P and the equilibrium interest rate are given by the following expressions:

$$r = \mu_\varepsilon - \sigma_\varepsilon \sigma'_\varepsilon - \sqrt{2h_1(Y)} \sqrt{\sigma_\varepsilon \sigma'_\varepsilon} + \frac{1}{(1 + \gamma)} \left(2h_1 - d(Y)' \theta_0(\kappa_1^{(1)}) \right) - \frac{1}{1 + \gamma} [\sigma_\varepsilon + d(Y)'] d(Y) \quad (1.36)$$

$$\mu^* - r^* = \sigma \left(\sigma'_\varepsilon + \frac{d(Y)}{1 + \gamma} + \sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma_\varepsilon \sigma'_\varepsilon}} \right) \quad (1.37)$$

where

$$d(Y) = \begin{cases} \sigma_\varepsilon & \text{if} \\ & (-\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} - \sigma'_\varepsilon)' (-\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} - \sigma'_\varepsilon) \leq 2h_2(Y) \\ \sqrt{2h_2(Y)} \frac{k_1^{(1)*} - \sigma'_\varepsilon}{\sqrt{(k_1^{(1)*} - \sigma'_\varepsilon)'(k_1^{(1)*} - \sigma'_\varepsilon)}} + \sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma_\varepsilon \sigma'_\varepsilon}} & \text{otherwise} \end{cases}$$

The equilibrium weighting process is

$$\gamma(t) = \exp \left(\int_0^t \left(2h_1(Y) - \frac{d(Y)' d(Y)}{2} \right) ds - \int_0^t d(Y)' dw_{\kappa(2)} \right) \quad (1.38)$$

and the equilibrium volatility process of the equity is given by

$$\sigma(t)' = \sigma_\varepsilon(t)' + \left(\frac{\mathbb{E}^{\kappa(1)} \left[\int_t^T \gamma(s)^{-1} ds \middle| \mathcal{F}_t \right]}{\mathbb{E}^{\kappa(1)} \left[\int_t^T (1 + \gamma(s)^{-1}) ds \middle| \mathcal{F}_t \right]} + \frac{1}{1 + \gamma(t)} \right) d(Y) \quad (1.39)$$

²⁴ Alternative characterizations of the stochastic process a , in particular a markovian characterization in terms of the gradient of the value function, are easily obtained by mimicking the line of reasoning followed in the partial equilibrium setting.

The stock price is given by the following expression

$$S(t) = \frac{\varepsilon(t)\gamma(t)}{1 + \gamma(t)} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \left(1 + \frac{1}{\gamma(s)} \right) ds \middle| \mathcal{F}_t \right] \quad (1.40)$$

Finally, the optimal investment policies in the stock are given by :

$$\pi_1^*(t) = \begin{cases} 0 & \text{if } -\sqrt{2h_1(Y)} < (\sigma\sigma')^{-1/2}\sigma \left(\sigma'_\varepsilon + \frac{d(Y)}{1+\gamma} + \sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma_\varepsilon\sigma'_\varepsilon}} \right) < \sqrt{2h_1(Y)} \\ (\sigma\sigma')^{-1}\sigma \left(\sigma'_\varepsilon + \frac{d(Y)}{1+\gamma} \right) & \text{otherwise} \end{cases} \quad (1.41)$$

$$\pi_2^*(t) = \frac{\gamma(t)}{T} \left(\mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \gamma(s)^{-1} ds \middle| \mathcal{F}_t \right] - T(\pi_1^*(t) - 1) \right) \quad (1.42)$$

Expressions (1.36) and expressions (1.37) unambiguously show that in this simple logarithmic framework, for a positive difference of the equilibrium Girsanov kernels, $d(Y)^{25}$, a concern for ambiguity delivers lower interest rates and higher equity premia compared to an economy with no such concern. The higher the stochastic weighting process, the smaller the effect induced by ambiguity, although the component due to the equilibrium representation of expected consumption growth is independent of the latter state variable.

As shown in the Appendix, the derivation of the equilibrium equity volatility process (1.39) is simplified by the requirement that the volatility matrix of the state variable vector, $\Xi(t)$, be in its orthogonal complement. This assumption allows nevertheless for a sharp identification of the direct effect of ambiguity²⁶. Expression (1.39) reveals that the higher the heterogeneity of Girsanov kernels selected by the agents, as captured by the magnitude of the process $d(Y)$, the higher the influence of ambiguity aversion; notice, however, that this effect is compensated by the occurrence of positive innovations of the process $\gamma(t)$, to whose magnitude the equity volatility is inversely proportional. Quite interestingly, a forward looking effect is captured by a concern for future realizations of the integrated stochastic weighting process. To see this, let us make use of (1.40) and rewrite the expression as

$$\sigma(t)' = \sigma_\varepsilon(t)' + \left(\frac{\varepsilon(t)}{S(t)} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \left(\frac{\gamma(s)}{\gamma(t)} \right)^{-1} ds \middle| \mathcal{F}_t \right] + 1 \right) \frac{d(Y)}{1 + \gamma(t)}$$

The equity volatility is then proportional to the dividend-price ratio scaled by the expected cumulated future contingencies of the inverse stochastic weighting process. Whether the net influence of ambiguity is positive or negative depends on the sign of the difference of the optimal Girsanov kernels, whereas the quantitative impact needs to be addressed once the dynamics of the aggregate endowment and the state variable are specified.

²⁵Notice that the particular sign of this difference that generates the effect is a consequence of having assumed the state price density of the first agent as equilibrium state price density and entails no loss of generality.

²⁶Indeed an additional, indirect effect, would arise in the general case as a consequence of ambiguity aversion's impact on the tangent process of the state variable Y .

1.5.2 Utility of terminal wealth. General CRRA felicity functions.

Agents adopt an homogeneous felicity function of the CRRA type with relative risk aversion coefficient \mathcal{R} . They maximize utility of terminal wealth, so that

$$U^i(c(T)) = \inf_{\kappa^{(i)} \in K^i} \mathbb{E}^{\kappa^{(i)}} \left[\frac{W(T)^{\mathcal{R}} - 1}{\mathcal{R}} \right]$$

The exogenous dividend process' coefficients are deterministic functions of the state Y , and the process Y is assumed to have the strong Markov property. The equilibrium characterization is clearly more complicated than its analog prevailing in the logarithmic case because of the intertemporal hedging behavior of the agents. The next Proposition illustrates the relevant equilibrium quantities.

Proposition 5 *Let the economic agents maximize utility of terminal wealth, be characterized by homogeneous CRRA felicity functions and ambiguity aversion parameters (h_1, \bar{h}_1) and (h_2, \bar{h}_2) , with $h_1 \leq h_2$ and $\bar{h}_1 \leq \bar{h}_2$. Then the optimal terminal consumption allocations are*

$$W_1^*(T) = \frac{\varepsilon(t)}{1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}}} \quad W_2^*(T) = \frac{\varepsilon(t)}{1 + \gamma(t)^{\frac{1}{1-\mathcal{R}}}} \quad (1.43)$$

and the equilibrium Girsanov kernels are

$$k_1^{(1)*} = -\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} \quad (1.44)$$

$$\bar{\kappa}^{1*} = -\sqrt{2\bar{h}_1(Y)} \frac{V_Y}{\sqrt{V'_Y V_Y}} \quad (1.45)$$

$$k_1^{(2)*} = \begin{cases} k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon & \text{if} \\ (k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon)'(k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon) \leq 2h_2(Y) \\ -\sqrt{2h_2(Y)} \frac{k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon}{\sqrt{(k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon)'(k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\sigma'_\varepsilon)}} & \text{otherwise} \end{cases} \quad (1.46)$$

$$\bar{\kappa}^{2*} = -\sqrt{2\bar{h}_2(Y)} \frac{\frac{G}{\alpha} \frac{\mathcal{R}}{\mathcal{R}-1} (\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) - G_Y}{\sqrt{\left(\frac{G}{\alpha} \frac{\mathcal{R}}{\mathcal{R}-1} (\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) - G_Y \right)' \left(\frac{G}{\alpha} \frac{\mathcal{R}}{\mathcal{R}-1} (\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) - G_Y \right)}} \quad (1.47)$$

where $G(Y)$ and $V(Y)$ solve the HBJ equations (1.107) and (1.103), respectively, and $\alpha = \frac{(\mathcal{R}-1)^2}{\mathcal{R} + (\mathcal{R}-1)^2}$. The equilibrium Kuhn-Tucker multipliers selected by the agent are

$$\bar{\lambda}^{1*} = f + r^* \bar{1}_k - \bar{\kappa}^{1*} \quad (1.48)$$

$$\bar{\lambda}^{2*} = \bar{\lambda}^{1*} - (\bar{\kappa}^{2*} - \bar{\kappa}^{1*}) - (\mathcal{R} - 1) \left(\bar{\lambda}^{1*} - r^* \bar{1}_k + \bar{\kappa}^{1*} \right) + \alpha \frac{G_Y}{G} \quad (1.49)$$

where f is the solution of the following equation²⁷ with respect to $(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)$

$$\left[1 + \frac{\mathcal{R} - 1}{1 + \gamma^{\frac{1}{1-\mathcal{R}}}}\right] (\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1) = \frac{1}{1 + \gamma^{\frac{1}{1-\mathcal{R}}}} \alpha \frac{G_Y(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)}{G(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)} \quad (1.50)$$

The equilibrium interest rate process and excess return with respect to the reference belief are given, respectively, by the following expressions

$$\begin{aligned} r = (1 - \mathcal{R}) \left(\mu_\varepsilon - \sqrt{2h_1(Y)} \sqrt{\sigma_\varepsilon \cdot \sigma_\varepsilon} \right) + \\ \frac{\gamma^{\frac{1}{\mathcal{R}-1}}}{1 + \gamma^{\frac{1}{\mathcal{R}-1}}} \left[2h_1 + \left(\bar{\lambda}^{2*} - \bar{\lambda}^{1*} \right)' f + \left((1 - \mathcal{R})f + \alpha \frac{G_Y}{G} \right)' f - d(Y)\theta_0(\kappa_1^{(1)}) \right] - \frac{1}{2} (\mathcal{R} - 1) (\mathcal{R} - 2) \sigma_\varepsilon \cdot \sigma_\varepsilon' + \\ \frac{1}{2} \left[\gamma^{\frac{1}{\mathcal{R}-1}} \left(\frac{1}{\mathcal{R} - 1} - 1 \right) \left(1 + \gamma^{\frac{1}{\mathcal{R}-1}} \right)^{-1} - \mathcal{R} \left(1 + \gamma^{\frac{1}{\mathcal{R}-1}} \right)^{-2} \gamma^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right] \times \\ \left[d(Y)' d(Y) + \left((1 - \mathcal{R})f + \alpha \frac{G_Y}{G} \right)' \left((1 - \mathcal{R})f + \alpha \frac{G_Y}{G} \right) \right] + (\mathcal{R} - 1) \frac{\gamma^{\frac{1}{\mathcal{R}-1}}}{1 + \gamma^{\frac{1}{\mathcal{R}-1}}} \sigma_\varepsilon \cdot d(Y) \quad (1.51) \end{aligned}$$

$$\mu^* - r^* = (1 - \mathcal{R}) \sigma \cdot \sigma'_\varepsilon + \frac{\gamma^{\frac{1}{1-\mathcal{R}}}}{1 + \gamma^{\frac{1}{1-\mathcal{R}}}} \sigma \cdot d(Y) - \sigma \cdot \mathbf{k}_1^{(1)*} \quad (1.52)$$

where $d(Y) = \mathbf{k}_1^{(2)*} - \mathbf{k}_1^{(1)*}$. The following expressions provide, respectively, the equilibrium volatility process of the stock and the equilibrium stock price process

$$\sigma(t)' = \sigma_\varepsilon(t)' + \left(\frac{\mathbb{E}^{\kappa^{(1)}} \left[\varepsilon(T)^\mathcal{R} \left(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}} \right)^{-\mathcal{R}} \gamma(T)^{\frac{1}{\mathcal{R}-1}} \middle| \mathcal{F}_t \right]}{\mathbb{E}^{\kappa^{(1)}} \left[\varepsilon(T)^\mathcal{R} \left(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}} \right)^{1-\mathcal{R}} \middle| \mathcal{F}_t \right]} + \frac{1}{1 + \gamma(t)^{\frac{1}{1-\mathcal{R}}}} \right) (\mathbf{k}_1^{(2)*} - \mathbf{k}_1^{(1)*}) \quad (1.53)$$

$$S(t) = \frac{1}{\mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} [\mathcal{U}'(T) \varepsilon(T) | \mathcal{F}_t] = \frac{(1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}-1}}{\varepsilon(t)^{\mathcal{R}-1}} \mathbb{E}^{\kappa^{(1)}} \left[\varepsilon(T)^\mathcal{R} (1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{1-\mathcal{R}} \middle| \mathcal{F}_t \right] \quad (1.54)$$

Finally, the equilibrium portfolio policies are

$$\pi_1^*(t) = (\sigma \sigma')^{-1} \left((\mathcal{R} - 1) \sigma \cdot \sigma'_\varepsilon + \frac{1}{1 + \gamma^{\frac{1}{1-\mathcal{R}}}} \sigma \cdot (\mathbf{k}_1^{(2)*} - \mathbf{k}_1^{(1)*}) \right) - \quad (1.55)$$

$$(\sigma \sigma')^{-1} \sigma \cdot (\mathbf{k}_1^{(2)*} - \mathbf{k}_1^{(1)*}) \frac{\mathcal{R}}{\mathcal{R} - 1} \mathbb{E}^{\kappa^{(1)}} \left[\frac{\gamma(T)^{\frac{1}{\mathcal{R}-1}} \varepsilon(T)^\mathcal{R}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}+1}} \middle| \mathcal{F}_t \right]$$

$$\pi_2^*(t) = \frac{(1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}-1}}{\varepsilon(t)^\mathcal{R}} \mathbb{E}^{\kappa^{(1)}} \left[\frac{\varepsilon(T)^\mathcal{R}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}-1}} \left(1 - \frac{\pi_1^*(t)}{1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}} \right) \middle| \mathcal{F}_t \right] \quad (1.56)$$

²⁷ $G(\cdot)$ is the solution of (1.107) with the functional dependence on $(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)$ made explicit.

where the equilibrium weighting process $\gamma(t)$ is obtained by substituting the equilibrium Girsanov kernels and Kuhn-Tucker multipliers into (1.19).

1.5.3 Explicit Dynamics

In this section we will specify in detail the opportunity set process Y and the form of the coefficients $\mu_\varepsilon(Y)$ and $\sigma_\varepsilon(Y)$ of the dividend process; in the theoretical framework of the Propositions above, we will then work out explicitly the equilibrium quantities.

Constant opportunity set: geometric brownian motion

We will consider a simple specification in which the market is complete and the dividend process evolves as a geometric brownian motion with respect to the reference measure P , namely:

$$\frac{d\varepsilon(t)}{\varepsilon(t)} = \bar{\mu}_\varepsilon + \bar{\sigma}_\varepsilon dw(t)$$

with constants $\bar{\mu}_\varepsilon$, $\bar{\sigma}_\varepsilon$ and $w(t)$ a scalar P -brownian motion. Agents are characterized by ambiguity aversion parameters h_i , $i = 1, 2$, with $h_1 \leq h_2$, so that the first agent is considered less ambiguity averse than the second. The available opportunity set dynamics reduce then to the following stock price process

$$dS = S(t) \left(\mu(t) + \sigma k_1^{(i)} \right) dt + S(t) \sigma(t) dw_{\kappa(i)}(t)$$

where $w_{\kappa(i)}$ is a standard brownian motion under the measure Q^{κ^i} , with $(k_1^{(i)})^2 \leq 2h_i$. It is straightforward to see that the form of the respective state-price densities implies the following dynamics for the weighting process under any admissible likelihood of the second agent, $Q^{\kappa^{(2)}}$:

$$\frac{d\gamma(t)}{\gamma(t)} = k_1^{(1)} \cdot k_1^{(1)} dt + \left(k_1^{(2)} - k_1^{(1)} \right) dw_{\kappa^{(2)}}(t) \quad (1.57)$$

An easy adaptation of the arguments that inspired Proposition 4 leads us to the equilibrium quantities described in the next Corollary, the proof of which is immediate, after we represent the equilibrium stochastic weighting process (1.57) with respect and to the measure $Q^{\kappa^{(1)}}$ and notice that its transition density reduces to a lognormal.

Corollary 5 *Let agents maximize utility of intertemporal consumption and have a logarithmic felicity function. Then, with the following choices of lagrange multipliers for the budget constraints*

$$\psi_1^* = \frac{1}{\varepsilon(0)} \left(1 + \frac{1}{\psi^*} \right) \quad \psi_2^* = \psi^* \psi_1^*$$

$\psi^* = \frac{\exp((-2h_1 + d^2 + d)(T-t))}{T(-2h_1 + d^2 + d)\left(\frac{1}{\eta_1} - 1\right)}$, the equilibrium Girsanov kernels selected by the agents are

$$k_1^{(1)*} = \sqrt{2h_1} \operatorname{sgn}(\bar{\sigma}_\varepsilon) \quad (1.58)$$

$$k_1^{(2)*} = \begin{cases} k_1^{(1)*} - \bar{\sigma}_\varepsilon & \text{if } -\sqrt{2h_2} < k_1^{(1)*} - \bar{\sigma}_\varepsilon < \sqrt{2h_2} \\ \operatorname{sgn}(k_1^{(1)*} - \bar{\sigma}_\varepsilon) \sqrt{2h_2} & \text{otherwise} \end{cases} \quad (1.59)$$

and $d = k_1^{(2)*} - k_1^{(1)*}$. The equilibrium consumption allocations are as in (1.35), with the stochastic weighting process (1.57) evaluated at the equilibrium Girsanov kernels. The equilibrium excess return on the stock and interest rate are given by

$$\mu^* - r^* = \sigma \left(\bar{\sigma}_\varepsilon + \frac{d}{1 + \psi^* \gamma} + \sqrt{2h_1} \operatorname{sgn}(\bar{\sigma}_\varepsilon) \right)$$

$$r = \frac{1 + \psi^* \gamma}{(1 + \psi^* \gamma) + d} \left(\bar{\mu}_\varepsilon + \sqrt{2h_1} \bar{\sigma}_\varepsilon \operatorname{sgn}(\bar{\sigma}_\varepsilon) \right) + \frac{1}{(1 + \psi^* \gamma) + d} \left((\sigma \sigma')^{-1} \mu + \sqrt{2h_1} \bar{\sigma}_\varepsilon \operatorname{sgn}(\bar{\sigma}_\varepsilon) d \right) - \frac{1 + \psi^* \gamma}{(1 + \psi^* \gamma) + d} \bar{\sigma}_\varepsilon^2 - \frac{1}{(1 + \psi^* \gamma) + d} [\bar{\sigma}_\varepsilon + d] d$$

The equilibrium stock price process is

$$S(t) = \frac{\psi^* \varepsilon(t) \gamma(t)}{1 + \psi^* \gamma(t)} \left((T-t) + \frac{e^{(T-t)(-2h_1 + d^2 + d)}}{-2h_1 + d^2 + d} \right)$$

and its volatility process is

$$\sigma = \bar{\sigma}_\varepsilon + d - \frac{d}{1 + \psi^* \gamma}$$

Finally, the equilibrium fractions of wealth held in the stock by the agents are

$$\pi_1^*(t) = \begin{cases} 0 & \text{if } -\sqrt{2h_1(Y)} < \left(\bar{\sigma}_\varepsilon + \frac{d}{1 + \psi^* \gamma} - \operatorname{sgn}(\bar{\sigma}_\varepsilon) \sqrt{2h_1(Y)} \right) < \sqrt{2h_1(Y)} \\ \sigma^{-1} \left(\bar{\sigma}_\varepsilon + \frac{d(Y)}{1 + \psi^* \gamma} \right) & \text{otherwise} \end{cases} \quad (1.60)$$

$$\pi_2^*(t) = \frac{\gamma(t)}{T} \left(\frac{e^{(T-t)(-2h_1 + d^2 + d)}}{-2h_1 + d^2 + d} - T(\pi_1^*(t) - 1) \right) \quad (1.61)$$

If we replace the logarithmic form of the felicity function with a general CRRA type and assume that agents maximize utility of terminal wealth, the investment policy of the agents, though not in line with endogenous cycles of absence of stock market participation, is characterized by an interesting hedging demand against future fluctuations of the opportunity set due to changes of the stochastic weighting process. But for the property we have just mentioned, this case is not more difficult to handle than the logarithmic one, due to the state independence of the coefficients of aggregate consumptions, and the relevant equilibrium quantities are not very dissimilar from those outlined in the last Corollary, modulo some scaling for the risk aversion parameter, as one may easily figure out from an adaptation of Proposition 5. The following Corollary illustrates the point by reporting

the equilibrium Girsanov kernels and portfolio policy that arise in the present framework.

Corollary 6 *Let agents maximize utility of terminal wealth and have a CRRA felicity function with risk aversion parameter \mathcal{R} , then the equilibrium Girsanov kernels selected by the agents are given by*

$$k_1^{(1)} = \sqrt{2h_1} \operatorname{sgn}(\bar{\sigma}_\varepsilon) \quad (1.62)$$

$$k_1^{(2)} = \begin{cases} k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\bar{\sigma}_\varepsilon & \text{if } -\sqrt{2h_2} < k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\bar{\sigma}_\varepsilon < \sqrt{2h_2} \\ -\sqrt{2h_2} \operatorname{sgn}(k_1^{(1)*} - (2\mathcal{R} - 1)(\mathcal{R} - 1)\bar{\sigma}_\varepsilon) & \text{otherwise} \end{cases} \quad (1.63)$$

and the optimal portfolio policy of the agents is

$$\pi_1^*(t) = \frac{1}{\sigma^2} \left((\mathcal{R} - 1)\sigma\bar{\sigma}_\varepsilon + \frac{1}{1 + \gamma^{\frac{1}{1-\mathcal{R}}}} \sigma(k_1^{(2)*} - k_1^{(1)*}) \right) - \quad (1.64)$$

$$\frac{1}{\sigma} (k_1^{(2)*} - k_1^{(1)*}) \frac{\mathcal{R}}{\mathcal{R} - 1} \mathbb{E}^{k^{(1)}} \left[\frac{\gamma(T)^{\frac{1}{\mathcal{R}-1}} \varepsilon(T)^\mathcal{R}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}+1}} \middle| \mathcal{F}_t \right]$$

$$\pi_2^*(t) = \frac{\left(1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}}\right)^{\mathcal{R}-1}}{\varepsilon(t)^\mathcal{R}} \mathbb{E}^{k^{(1)}} \left[\frac{\varepsilon(T)^\mathcal{R}}{\left(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}\right)^{\mathcal{R}-1}} \left(1 - \frac{\pi_1^*(t)}{1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}}\right) \middle| \mathcal{F}_t \right] \quad (1.65)$$

where $\log \varepsilon(t) \sim N(\mu_\varepsilon t - \frac{1}{2}\bar{\sigma}_\varepsilon t, \bar{\sigma}_\varepsilon^2 t)$ and $\log \gamma(t) \sim N((k_1^{(2)})^2 t - \frac{3}{2}d^2 t, d^2 t)$

1.6 Conclusions

A concern for an ‘ambiguous’ probabilistic description of the environment on which agent base their decision making process has been largely shown to be both economically and behaviorally relevant in terms of predictions on key economic indicators; we contribute to this strand of the literature by studying the influence of such a modelling framework on the classical the two-agent equilibrium problem. This set up allows us to clarify the impact of heterogeneous levels of ambiguity aversion on key economic indicators; in particular, we have emphasized that for significantly wide differences of ambiguity aversion between agents the model generates equity premia which are quantitatively in accordance with empirical evidence. Furthermore, both the precautionary saving and the speculative components of the equilibrium interest rate dynamics are significantly affected. An interesting feature that arises in a simple specification we discussed is the property of the model to generate endogenous stock market participation as a consequence of the optimizing behavior of the agents rather than exogenous policy restrictions.

Appendix

Proof of Proposition 1.

In the sequel we omit the index i for notational simplicity. Let $\lambda(t) = [0 \quad \bar{\lambda}(t)]'$ be an \mathbb{R}^{k+1} -valued processes ($\bar{\lambda}(t) \in \mathbb{R}^k$), satisfying the integrability condition

$$\mathbb{E} \left[\int_0^T |\lambda(t)|^2 dt \right] < \infty$$

and consider a complete-market economy characterized by identical preferences, unconstrained portfolio policies and the following modified opportunity set:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= [\mu(Y) + \kappa_1(t)]dt + \sigma(Y)dw_\kappa(t) \\ \frac{d\tilde{S}(t)}{\tilde{S}(t)} &= [\bar{\lambda}(t) + \bar{\kappa}(t)]dt + \vartheta(Y)dw_\kappa(t) \\ dY(t) &= [\Lambda(\omega, t) + \bar{\kappa}(t)]dt + \Xi(\omega, t)dw_\kappa(t) \end{aligned} \tag{1.66}$$

Following He and Pearson (1991) and Karatzas, Lehocsky, Shreve and Xu (1991) the program at hand can be equivalently stated as a consumption-investment problem where the market is fictitiously completed by means of a k -dimensional vector of stocks, with price vector $\tilde{S}(t)$ at time t , in which the agent is constrained not to invest. In this context the portfolio fraction π is interpreted as a $k+1$ -dimensional vector that takes values in the set $\{\pi : \pi \in \mathbb{R} \times \{0\}^k\}$. The optimal consumption-investment policy in the original market is then recovered as the infimum of the fictitious value function over the shadow martingale measures, indexed by dynamic Kuhn-Tucker multiplier $\lambda(t)$. Therefore, solving program (1.12) amounts to solving

$$J(x) = \inf_{\kappa \in K} \inf_{\lambda} \sup_c \mathbb{E}^\kappa \left[A \int_0^T \left(\frac{c(t)^\mathcal{R} - 1}{\mathcal{R}} \right) dt + (1 - A) \frac{W(T)^\mathcal{R} - 1}{\mathcal{R}} \right] \tag{1.67}$$

subject to the dynamics for invested wealth (to be outlined below) and the opportunity set process Y . Since the characterization achieved will be independent of the particular market completion, without loss of generality we assume once again that the rows of the $k \times (k+1)$ -dimensional matrix-valued process $\vartheta(\cdot)$ are orthonormal vectors spanning the kernel of $\sigma(Y)$: $\sigma(Y)\vartheta(Y)' = 0\bar{I}_k$ and $\vartheta(Y)\vartheta(Y)' = I_k$. Notice that in light of the same property shared by the volatility matrix of the state variable Y , $\Xi(\omega, t)$, we have $\vartheta(Y)\Xi(\omega, t) = I_k$. If $\Sigma(Y)$ denotes the volatility matrix

$$\Sigma(Y) = \begin{bmatrix} \sigma(Y) \\ \vartheta(Y) \end{bmatrix}_{\substack{1 \times (k+1) \\ k \times (k+1)}}$$

then:

$$\Sigma(Y)^{-1} = [\sigma(Y)' \|\sigma(Y)\|^{-2} \quad \vartheta(Y)']$$

It can be easily verified that the following dynamic budget constraint holds:

$$\begin{aligned} \beta(t)Z_0(\kappa + \lambda, t)W(t) + \int_0^t Z_0(\kappa + \lambda, s)\beta(s)c(s)ds = x \\ + \int_0^t W(s)\beta(s)Z_0(\kappa + \lambda, s) [\Sigma'\pi(s) - \theta_{\kappa+\lambda}(s)]' \cdot dw_\kappa(s) \end{aligned} \quad (1.68)$$

where is the discount factor $\beta(s) = \exp(-\int_0^s r(s)ds)$, π is now an \mathbb{R}^{k+1} -valued vector the components of which are portfolio proportions invested in the actual and the fictitious risky assets. The vector $\theta_{\kappa+\lambda}$ in (2.60) is defined by

$$\theta_{\kappa+\lambda} = \Sigma^{-1} \begin{pmatrix} \mu - r + \kappa_1 \\ \bar{\lambda} - r\bar{1}_k + \bar{\kappa} \end{pmatrix} = \theta_0(\kappa_1) + \vartheta'(\bar{\lambda} - r\bar{1}_k + \bar{\kappa})$$

where $\theta_0(\kappa_1) = \sigma'\|\sigma\|^{-2}(\mu - r + \kappa_1)$ and $Z_0(\kappa + \lambda, t)$ is the stochastic exponential

$$Z_0(\kappa + \lambda, t) = \exp \left(- \int_0^t \theta_{\kappa+\lambda}(s) \cdot dw_\kappa(s) - \frac{1}{2} \int_0^t \|\theta_{\kappa+\lambda}(s)\|^2 ds \right)$$

The l.h.s. of (2.60) is clearly a positive P^κ -local martingale²⁸, hence a P^κ -supermartingale. Therefore for every consumption plan $c(t)$ satisfying (2.60) (for some $\pi(t)$) in the fictitious economy the state prices of which are evaluated according to $\beta(t)Z_0(\kappa + \lambda, t)$, we have:

$$\begin{aligned} \mathbb{E}^\kappa \left[\int_0^T Z_0(\kappa + \lambda, s)\beta(s)c(s)ds \right] \leq \\ \mathbb{E}^\kappa \left[\beta(T)Z_0(\kappa + \lambda, T)W(T) + \int_0^T Z_0(\kappa + \lambda, s)\beta(s)c(s)ds \right] \leq x \end{aligned} \quad (1.69)$$

Conversely, it can be shown that if

$$\sup_\lambda \mathbb{E}^\kappa \left[\int_0^T Z_0(\kappa + \lambda, s)\beta(s)c(s)ds \right] = x$$

then there exists a (super)strategy $\pi(t)$ such that $c(t)$ satisfies (2.60). Therefore the dynamic budget constraint (2.60) admits the static formulation (2.61). The innermost optimization in (1.67) subject to (2.61) is an unconstrained consumption-portfolio optimization in a complete market framework. Lagrangian theory then dictates that the utility gradient at the optimal policies be proportional to the unique (fictitious) state-price density. Let

$$\xi(\kappa + \lambda, t) = Z_0(\kappa + \lambda, t)\beta(t)$$

denote such a state-price density. The optimal consumption and invested wealth process as well as the value function $V_\kappa(x, \lambda)$ of the innermost sup optimization program in (1.67) are then easily seen

²⁸Remember that for π to be admissible, the nonnegativity of the wealth process has to hold.

to be given by the following expressions, where we omit the functional arguments $\kappa + \lambda$ in ξ for brevity:

$$c^*(t) = [(\psi/A)\xi(t)]^{\frac{1}{\mathcal{R}-1}} \quad W^*(T) = [(\psi/(1-A))\xi(T)]^{\frac{1}{\mathcal{R}-1}} \quad (1.70)$$

$$W(t) = \psi^{\frac{1}{\mathcal{R}-1}} \xi(t)^{-1} \mathbb{E}^\kappa \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_t^T \xi(s)^{\frac{\mathcal{R}}{\mathcal{R}-1}} ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \xi(T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \middle| \mathcal{F}_t \right] \quad (1.71)$$

$$V_\kappa(x, \lambda) = \frac{\psi^{\frac{\mathcal{R}}{\mathcal{R}-1}}}{\mathcal{R}} \mathbb{E}^\kappa \left[A^{\frac{1}{1-\mathcal{R}}} \int_0^T \xi(s)^{\frac{\mathcal{R}}{\mathcal{R}-1}} ds + (1-A)^{\frac{1}{1-\mathcal{R}}} \xi(T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right] - \frac{A(T-1)+1}{\mathcal{R}} \quad (1.72)$$

where

$$\psi = \left(\frac{\mathbb{E}^\kappa \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T \xi(s)^{\frac{\mathcal{R}}{\mathcal{R}-1}} ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \xi(T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right]}{x} \right)^{1-\mathcal{R}}$$

is the unique Lagrange multiplier for which budget constraint (2.61) is satisfied as an equality. Since

$$\xi(t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} := \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} = f(t)g(t) \quad (1.73)$$

where

$$\begin{aligned} f(t) &= \exp \left\{ -\frac{\mathcal{R}}{\mathcal{R}-1} \int_0^t \left(r(s) - \frac{|\theta_{\kappa+\lambda}(s)|^2}{2(\mathcal{R}-1)} \right) ds \right\} \\ g(t) &= \exp \left\{ -\frac{1}{2} \left(\frac{\mathcal{R}}{\mathcal{R}-1} \right)^2 \int_0^t |\theta_{\kappa+\lambda}(s)|^2 ds - \frac{\mathcal{R}}{\mathcal{R}-1} \int_0^t \theta_{\kappa+\lambda}(s)' \cdot dw_\kappa(s) \right\} \end{aligned}$$

we find it useful to assume that $\theta_{\kappa+\lambda}$ satisfies a Novikov condition, implying that that $g(t)$ is a P^κ -martingale. We may thus define the probability measure $Q_\theta(A) = \mathbb{E} [g(T) \mathbf{1}_{[A]}]$, $A \in \mathcal{F}$, under which the process

$$w_\theta(t) = w_\kappa(t) + \int_0^t \frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda}(s) dt$$

is a standard Brownian motion. Let $\mathbb{E}_\theta[\cdot]$ denote expectation with respect to this measure. Then according to conditional Bayes-rule and the optimal consumption and terminal wealth level (1.70):

$$\begin{aligned} W(t) &= (\psi/A)^{\frac{1}{\mathcal{R}-1}} \xi(\kappa + \lambda, t)^{\frac{1}{\mathcal{R}-1}} \mathbb{E}^\kappa \left[\int_t^T \frac{f(s)g(s)}{f(t)g(t)} + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \frac{f(T)g(T)}{f(t)g(t)} \middle| \mathcal{F}_t \right] \\ &= (\psi/A)^{\frac{1}{\mathcal{R}-1}} \xi(\kappa + \lambda, t)^{\frac{1}{\mathcal{R}-1}} \mathbb{E}_\theta \left[\int_t^T \frac{f(s)}{f(t)} ds + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right] \end{aligned}$$

The latter formulation allows us to derive an expression for the stochastic marginal propensity to

consume and therefore to achieve a convenient factorization for the optimal consumption process:

$$c^*(t) = \frac{W(t)}{\mathbb{E}_\theta \left[\int_t^T \frac{f(s)}{f(t)} ds + \left(\frac{A}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right]} = W(t) \text{ mpc}(t)$$

Let us now introduce the Levy-martingale

$$\tilde{f}(t, T) = \mathbb{E}_\theta \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \middle| \mathcal{F}_t \right] \quad (1.74)$$

Then, using the optimally invested wealth process (1.71), we have:

$$\psi = \left(\frac{\tilde{f}(0, T)}{x} \right)^{1-\mathcal{R}}$$

To write down a representation for the optimal portfolio policies, we apply Ito's lemma to the P^κ -supermartingale (see (2.60))

$$W(t)\xi(\kappa + \lambda, t) = \psi^{\frac{1}{\mathcal{R}-1}} g(t) \left(\tilde{f}(t, T) - \int_0^t f(s) ds \right)$$

to obtain:

$$\begin{aligned} \xi(\kappa + \lambda, t)W(t) &= \frac{x}{\tilde{f}(0, T)} \int_0^t \left(\tilde{f}(s, T) - \int_0^s f(u) du \right) \frac{\mathcal{R}}{1-\mathcal{R}} g(s) \theta_{\kappa+\lambda}(s)' dw_\kappa(s) \\ &\quad - \frac{x}{\tilde{f}(0, T)} \int_0^t f(s) g(s) ds + \frac{x}{\tilde{f}(0, T)} \int_0^t g(s) \phi(\omega, s)' dw_\theta(s) \\ &\quad + \frac{x}{\tilde{f}(0, T)} \int_0^t \frac{\mathcal{R}}{1-\mathcal{R}} g(s) \theta_{\kappa+\lambda}(s)' \phi(\omega, s) ds = \\ &= \int_0^t W(s) \xi(\kappa + \lambda, s) \frac{\mathcal{R}}{1-\mathcal{R}} \theta_{\kappa+\lambda}(s)' dw_\kappa(s) \\ &\quad + \int_0^t W(s) \xi(\kappa + \lambda, s) \frac{\phi(\omega, s)'}{\left(\tilde{f}(s, T) - \int_0^s f(u) du \right)} \cdot \left[dw_\kappa(s) + \frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda}(s) ds \right] \\ &\quad + \int_0^t \frac{\mathcal{R}}{1-\mathcal{R}} W(s) \xi(\kappa + \lambda, s) \theta_{\kappa+\lambda}(s)' \frac{\phi(\omega, s)}{\left(\tilde{f}(s, T) - \int_0^s f(u) du \right)} ds \\ &\quad \underbrace{- \frac{x}{\int_0^T \tilde{f}(0, s) ds} \int_0^t f(s) g(s) ds}_{- \int_0^t \xi(\kappa + \lambda, s) c^*(s) ds} = \end{aligned}$$

$$\begin{aligned}
&= - \int_0^t \xi(\kappa + \lambda, s) c^*(s) ds + \frac{x}{\tilde{f}(0, T)} \int_0^t W(s) \xi(\kappa + \lambda, s) \frac{\mathcal{R}}{1 - \mathcal{R}} \theta_{\kappa + \lambda}(s)' \cdot dw_{\kappa}(s) \\
&\quad + \frac{x}{\tilde{f}(0, T)} \int_0^t W(s) \xi(\kappa + \lambda, s) \frac{\phi(\omega, s)'}{\left(\tilde{f}(s, T) - \int_0^s f(u) du\right)} \cdot dw_{\kappa}(s)
\end{aligned}$$

In the last chain of equalities we have used the predictable integrand process $\phi(\omega, t)$ appearing in the stochastic integral representation of the Levy martingale $\tilde{f}(t, T)$:²⁹

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t \phi(\omega, s) \cdot dw_{\theta}(s) \quad (1.75)$$

$$\begin{aligned}
&= \mathbb{E}_{\theta} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \right] \\
&\quad + \int_0^t \mathbb{E}_{\theta} \left[\mathcal{D}_s \left(\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \right) \middle| \mathcal{F}_s \right] \cdot dw_{\theta}(s) \quad (1.76)
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}^{\theta} \left[\int_0^T f(s) ds \right] \\
&\quad + \int_0^t \mathbb{E}^{\theta} \left[\int_s^T \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \mathcal{D}_s f(u) du + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \mathcal{D}_s f(T) \middle| \mathcal{F}_s \right] \cdot dw_{\theta}(s) \quad (1.77)
\end{aligned}$$

where \mathcal{D} denotes the Malliavian derivative operator and the last equality follows from the Clark-Ocone formula.³⁰ We have thus obtained an alternative representation for the P^{κ} -local martingale (2.60):

$$\begin{aligned}
\mathbb{E}^{\kappa} \left[\int_0^T \xi(\kappa + \lambda, s) c(s) ds \middle| \mathcal{F}_t \right] &= \xi(\kappa + \lambda, t) W(t) + \int_0^t \xi(\kappa + \lambda, s) c(s) ds \\
&= x + \int_0^t W(s) \xi(\kappa + \lambda, s) \left[\frac{\mathcal{R}}{1 - \mathcal{R}} \theta_{\kappa + \lambda}(s)' + \frac{\phi(\omega, s)}{\left(\tilde{f}(s, T) - \int_0^s f(u) du\right)} \right] dw_{\kappa}(u)
\end{aligned}$$

The latter expression leads us to the optimal portfolio proportions by direct comparison with the RHS of (2.60):

$$\pi^*(t) = \frac{1}{1 - \mathcal{R}} \left[\frac{\|\sigma(t)\|^{-2} [\mu(t) - r(t) + \kappa_1(t)]}{(\bar{\lambda}(t) - r(t) \bar{1}_k + \bar{\kappa}(t))} \right] + \left[\frac{\sigma(t) \|\sigma(t)\|^{-2}}{\vartheta(t)} \right] \frac{\phi(\omega, t)}{\left(\tilde{f}(t, T) - \int_0^t f(s) ds\right)} \quad (1.78)$$

²⁹If the coefficient $b(t, x)$ and $\delta(t, x)$ are deterministic functions on $[0, T] \times \mathbb{R}$, then we conclude from the Markov property of the underlying state variable $X(t)$ that the expectation $\left(\tilde{f}(t, T) - \int_0^t f(s) ds\right)$ is a deterministic function of time and $X(t)$. Moreover,

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t d(s, X(s)) \delta(s, X(s)) dw_{\kappa + \lambda}(s)$$

for some deterministic function $d(t, x)$, from results on the representation of additive functional of diffusion processes due to Cinlar, Jacod, Protter and Sharpe (1980)

³⁰See Nualart(1995), Section , for details.

Summarizing, we can thus conclude that the program (1.67) reduces to:

$$\begin{aligned}
J(x) &= \inf_{\kappa \in K} \inf_{\lambda} V_{\kappa}(x, \lambda) \\
&= \inf_{\kappa \in K} \inf_{\lambda} \left(\frac{\psi^{\frac{\mathcal{R}}{\mathcal{R}-1}}}{\mathcal{R}} \mathbb{E}_{\theta} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \right] - \frac{A(T-1)+1}{\mathcal{R}} \right) \\
&= \inf_{\kappa \in K} \inf_{\lambda} \frac{x^{\mathcal{R}}}{\mathcal{R}} \left(\mathbb{E}_{\theta} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} f(T) \right] \right)^{1-\mathcal{R}} - \frac{A(T-1)+1}{\mathcal{R}}
\end{aligned} \tag{1.79}$$

This is equivalent to solving the following program:³¹

$$\begin{aligned}
V(t) &= \inf_{\kappa \in K} \inf_{\lambda} \mathbb{E}_{\theta} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_t^T \frac{f(s)}{f(t)} ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right] \\
\text{s.t. } dY(t) &= \left[\Lambda(\omega, t) + \frac{\bar{\kappa}(t)}{1-\mathcal{R}} - \frac{\mathcal{R}}{\mathcal{R}-1} (\bar{\lambda}(t) - r\bar{I}_k) \right] dt + \Xi(\omega, t) dw_{\kappa+\lambda}(t)
\end{aligned} \tag{1.80}$$

Conditions to solve the incomplete market problem without relying on Markovian assumptions for the opportunity set of the economy are best understood in the context of the martingale approach to stochastic control as reviewed - for instance - in Davis (1979) or Elliot (1982). It is easily verified that our regularity assumptions on the model primitives meet the requirements of the last reference for the problem to be well posed. In particular, the conditional expected utility to be minimized over shadow state-price processes and admissible model selections belongs to $L^1(\Omega, \mathcal{F}_t, P)$,³² which is a complete lattice. Therefore, the infimum exists and is \mathcal{F}_t -measurable. What is more, since the domain K is compact and locally convex, and the function $f(t)$ is convex, bounded and lower-semicontinuous, this infimum is actually attained.

The following propositions characterize the solution of sequential dynamic optimization problem (1.80). They apply ideas from the above mentioned approach to identify the min-max martingale measure (He and Pearson (1991)) and the optimal Girsanov kernel, respectively.

Proposition 6 *Let*

$$\chi_K(\kappa) = \begin{cases} 0 & \text{if } \kappa \in K \\ \infty & \text{if } \kappa \notin K \end{cases}$$

denote the indicator function (in the sense of convex analysis) of the set K and define the strictly

³¹Notice that expression in (1.79) suggests that program (1.80) characterizes the equilibrium if $0 \leq \mathcal{R} \leq 1$. For $\mathcal{R} < 0$ the equivalent control problem would involve a (strictly concave) maximization problem, rather than minimization. Since the optimality conditions of the latter case coincide with those prevailing in the case $0 < \mathcal{R} \leq 1$, we retain for brevity the inf notation for the rest of this Appendix.

³²as noted in Elliott (1982), being the measures P^{κ} equivalent, the null sets up to which $\mathbb{E}^{\kappa}[\cdot]$ is defined are independent of the control κ .

convex function let $g(Y, \lambda, \kappa, t) = \frac{\partial}{\partial t} \log f(t)$ as well as the Hamiltonian $\mathcal{H}(\kappa, \epsilon, t)$

$$\begin{aligned}\mathcal{H}(\kappa, \epsilon, t) &= -\sup_{\lambda \in \mathbb{R}} \left[-\left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda}(t) \right)' \cdot \epsilon(t) - V(t) g(Y, \lambda, \kappa, t) \right] \\ &= \inf_{\lambda \in \mathbb{R}} \left[V(t) g(Y, \lambda, \kappa, t) + \left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda}(t) \right)' \cdot \epsilon(t) \right] \\ &= \inf_{\lambda \in \mathbb{R}} H(\lambda, \kappa, \epsilon, t)\end{aligned}\tag{1.81}$$

then, for any admissible Girsanov kernel $\kappa \in K$

i) The process

$$\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^t f(s) ds + f(t) V(t)$$

is a $Q^{\kappa+\lambda}$ -submartingale. It is $Q^{\kappa+\lambda}$ -martingale if and only if λ achieves the infimum in (1.80)

ii) With $\tilde{\epsilon} = \frac{\mathcal{R}}{\mathcal{R}-1} \epsilon$, let $\lambda^*(Y, \kappa, \tilde{\epsilon}, t)$ denote the unique measurable function achieving the minimum in $H(\kappa, \epsilon, t)$, then the min-max measure for the incomplete-market consumption-investment problem is identified by the dynamic Kuhn-Tucker multiplier $\lambda^*(Y, \kappa, \tilde{a}_f, t)$, where

$$a_f(\kappa, t) = \frac{a(\kappa, t)}{f(t)}$$

and $a(\kappa, t)$ is the predictable integrand process in the stochastic integral representation:

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t a(Y, s) \cdot dw_{\kappa+\lambda^*}(s)$$

Furthermore, the process $a(Y, t)$ is of the form

$$a(Y, t)' = \Xi(\omega, t)' \tilde{a}(Y, t)\tag{1.82}$$

for a k -dimensional stochastic process $\tilde{a}(Y, t)$.

Let the Kuhn-Tucker multiplier λ be given as $\lambda^*(\kappa)$, the optimal control identified in the innermost optimization, and define the Hamiltonian

$$\begin{aligned}\tilde{\mathcal{H}}(\epsilon, t) &= \inf_{\kappa \in \mathbb{R}} \left[V(t) g(Y, \lambda^*, \kappa, t) + \left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda^*}(t)' - \Sigma(t)^{-1} \kappa(t) \right)' \cdot \epsilon(t) + \chi_K(\kappa) \right] \\ &= \inf_{\kappa \in K} \left[V(t) g(Y, \lambda^*, \kappa, t) + \left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda^*}(t)' - \Sigma(t)^{-1} \kappa(t) \right)' \cdot \epsilon(t) \right] \\ &= \inf_{\kappa \in K} \tilde{H}(\lambda^*, \kappa, \epsilon, t)\end{aligned}$$

then

iii) The process

$$\left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_0^t f(s)ds + f(t)V(t)$$

is a $Q^{\kappa+\lambda^*}$ -submartingale. It is $Q^{\kappa+\lambda^*}$ -martingale if and only if κ achieves the infimum in (1.80)

iv) With $\tilde{\epsilon} = \frac{\mathcal{R}}{\mathcal{R}-1} \epsilon$, let $\kappa^*(Y, \tilde{\epsilon}, t)$ denote the unique measurable function achieving the minimum in $\tilde{H}(Y, t)$, then the optimal Girsanov kernel that identifies the preference ordering representation of the agent is $\kappa^*(Y, \tilde{a}_f, t)$, where

$$a_f(Y, t) = \frac{a(Y, t)}{f(t)}$$

and $a(Y, t)$, the predictable integrand process in the stochastic integral representation:

$$\begin{aligned} \tilde{f}(t, T) &= \tilde{f}(0, T) + \int_0^t a(Y, s) \cdot dw_{\kappa^*+\lambda^*}(s) \\ &= \tilde{f}(0, T) + \int_0^t \tilde{a}(Y, s)' \Xi(\omega, s) dw_{\kappa^*+\lambda^*}(s) \end{aligned}$$

is the same process identified in ii).

Proof. i) We denote by $\mathcal{V}(t)$ the value function of the innermost optimization in (1.80) The process $\left[\left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_0^t f(s)ds + f(t)\mathcal{V}(t)\right]$ can be shown to admit a Right-Continuous-Left-Limits version following arguments of El Karoui and Quenez (1995). For any $\kappa \in K$, $t \leq s \leq T$ and a given process ν satisfying the regularity conditions mentioned at the beginning of this Appendix, consider the set $\mathcal{D}^{(t,s)}$ of controls λ which coincide with ν on $[t, s]$. Since $\mathcal{D}^{(t,s)}$ is a subset of the admissible Kuhn-Tucker multipliers, we have

$$\begin{aligned} \mathcal{V}(t) &\leq \inf_{\lambda \in \mathcal{D}^{(t,s)}} \frac{\tilde{f}(t, T) - \left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_0^t f(s)ds}{f(t)} \\ &= \inf_{\lambda \in \mathcal{D}^{(t,s)}} \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_t^s \frac{f(u)}{f(t)} du + \right. \\ &\quad \left. \frac{f(s)}{f(t)} \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_s^T \frac{f(u)}{f(s)} du + \left(\frac{1}{1-A}\right)^{\frac{1}{\kappa-1}} \frac{f(T)}{f(s)} \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_t \right] \end{aligned}$$

From Elliott (1982), Lemmas 16.11, 14 we know that by virtue of the ε -lattice property of the set of random variables $\left\{ \left(\tilde{f}(t, T) - \left(\frac{1}{A}\right)^{\frac{1}{\kappa-1}} \int_0^t f(s)ds\right)/f(t) \right\}_\lambda$ we can commute the orders of the infimum

and expectation in the last expression to obtain:

$$\begin{aligned}
\mathcal{V}(t) &\leq \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^s \frac{f(u)}{f(t)} du + \right. \\
&\quad \left. \frac{f(s)}{f(t)} \inf_{\lambda \in \mathcal{D}^{(t,s)}} \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_s^T \frac{f(u)}{f(s)} du + \left(\frac{1}{1-A} \right)^{\frac{1}{\kappa-1}} \frac{f(T)}{f(s)} \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^s \frac{f(u)}{f(t)} du + \frac{f(s)}{f(t)} \mathcal{V}(s) \middle| \mathcal{F}_t \right]
\end{aligned}$$

Then

$$\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^t f(u) du + f(t) \mathcal{V}(t) \leq \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^s f(u) du + f(s) \mathcal{V}(s) \middle| \mathcal{F}_t \right]$$

and we have obtained the submartingale property of $f(t) \mathcal{V}(t) + \left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^t f(u) du$ for any admissible λ and $\kappa \in K$.

Consider now an admissible Kuhn-Tucker multiplier λ^* and suppose that $f(t) \mathcal{V}(t) + \left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^t f(u) du$ evaluated at λ^* is a $Q^{\kappa+\lambda^*}$ -martingale; then the following chain of equalities holds

$$\begin{aligned}
\mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^T f(u) du + \left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} f(T) \middle| \mathcal{F}_t \right] &= \\
\mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^T f(u) du + f(T) \mathcal{V}(T) \middle| \mathcal{F}_t \right] &= \\
\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_0^t f(u) du + f(t) \mathcal{V}(t) &
\end{aligned}$$

therefore

$$\begin{aligned}
\mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^T \frac{f(u)}{f(t)} du + \left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right] &= \mathcal{V}(t) = \\
\inf_{\lambda} \mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^T \frac{f(u)}{f(t)} du + \left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \frac{f(T)}{f(t)} \middle| \mathcal{F}_t \right] &=
\end{aligned}$$

and λ^* is optimal. To show the converse, assume that λ^* is optimal; for any $t \leq s \leq T$ we may write:

$$\begin{aligned}
\mathcal{V}(t) &= \mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^s \frac{f(u)}{f(t)} du + \right. \\
&\quad \left. \frac{f(s)}{f(t)} \mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_s^T \frac{f(u)}{f(s)} du + \left(\frac{1}{1-A} \right)^{\frac{1}{\kappa-1}} \frac{f(T)}{f(s)} \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\kappa+\lambda^*} \left[\left(\frac{1}{A} \right)^{\frac{1}{\kappa-1}} \int_t^s \frac{f(u)}{f(t)} du + \frac{f(s)}{f(t)} \mathcal{V}(s) \middle| \mathcal{F}_t \right]
\end{aligned}$$

Then

$$\left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)du + f(t)\mathcal{V}(t) = \mathbb{E}^{\kappa+\lambda} \left[\left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^s f(u)du + f(s)\mathcal{V}(s) \middle| \mathcal{F}_t \right]$$

and the martingale property of $f(t)\mathcal{V}(t) + \left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)du$ follows.

ii) For any $\kappa \in K$, and any admissible process $\lambda(t)$, due to the assumptions on the stochastic coefficients of the opportunity set, the submartingale $\left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)du + f(t)\mathcal{V}(t)$ is of class D, therefore admits a unique Doob-Meyer decomposition which can be represented in the form:

$$L(\lambda, \kappa, t) = \left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)du + f(t)\mathcal{V}(t) = \mathcal{V}(0) + \int_0^t B(\lambda, \kappa, s)ds + M(\kappa, t) \quad (1.83)$$

for a nondecreasing process of bounded variation $\int_0^t B(\lambda, \kappa, s)ds$, with nonnegative density $B(\cdot)$ ³³ and a uniformly integrable $Q^{\kappa+\lambda}$ -martingale $M(\kappa, t)$. Recalling familiar results of stochastic analysis we may represent the latter process as the stochastic integral $\int_0^t a(\kappa, s) \cdot dw_{\kappa+\lambda}(s)$. For any pair of admissible controls (λ, ν) we obviously have :

$$\frac{L(\lambda, \kappa, t) - \left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)|_\lambda du}{f(t)|_\lambda} = \mathcal{V}(t) = \frac{L(\nu, \kappa, t) - \left(\frac{1}{A}\right)^{\frac{1}{\bar{\kappa}-1}} \int_0^t f(u)|_\nu du}{f(t)|_\nu} \quad (1.84)$$

Applying Ito's lemma and Girsanov Theorem to the last and second-to-last members, and recalling the decomposition (1.83) we may write

$$\begin{aligned} & \int_0^t \frac{1}{f(s)|_\lambda} \left[B(\lambda, \kappa, s) + \frac{\mathcal{R}}{\mathcal{R}-1} a(\kappa, s) \cdot (\theta_{\kappa+\nu}(t) - \theta_{\kappa+\lambda}(t)) \right] ds + \int_0^t \frac{1}{f(s)|_\lambda} a(\kappa, s) \cdot dw_{\kappa+\nu}(s) - \\ & \int_0^t \mathcal{V}(s)g(Y, \lambda, \kappa, s)ds = \int_0^t \frac{1}{f(s)|_\nu} B(\nu, \kappa, s)ds - \int_0^t \frac{1}{f(s)|_\nu} a(\kappa, s) \cdot dw_{\kappa+\nu}(s) - \int_0^t \mathcal{V}(s)g(Y, \nu, \kappa, s)ds \end{aligned}$$

Certainly, the uniqueness of the representation of the special semimartingale $\mathcal{V}(\cdot)$ mandates that bounded variation and local martingale parts be equivalent in the two representations obtained; therefore we conclude by direct comparison that the predictable integrand

$$a_f(\kappa, t) = \frac{a(\kappa, t)}{f(t)} \quad \forall \kappa \in K$$

is independent of the control and that, for any $t \in (0, T)$:

$$\frac{B(\nu, \kappa, t)}{f(t)|_\nu} = \frac{B(\lambda, \kappa, t)}{f(t)|_\lambda} + H(\nu, \kappa, \widetilde{a}_f, t) - H(\lambda, \kappa, \widetilde{a}_f, t)$$

Suppose now that the Kuhn-Tucker multiplier λ is optimal: the process $L(\lambda, \kappa, t)$ is then a $Q^{\kappa+\lambda}$ -martingale by virtue of the previous proposition and this implies $B(\lambda, \kappa, t) = 0$. For any admissible

³³W.l.o.g. in the present context, we have assumed this process absolutely continuous with respect to the Lebesgue measure. In particular, being $\int_0^t B(\cdot, s)ds$ nondecreasing, $B(\cdot) \geq 0$

ν the non negative process $\frac{B(\nu, \kappa, t)}{f(s)|_\nu}$ reduces to:

$$\frac{B(\nu, \kappa, t)}{f(s)|_\lambda} = H(\nu, \kappa, \widetilde{a}_f, t) - H(\lambda, \kappa, \widetilde{a}_f, t) \geq 0$$

hence, if κ is optimal:

$$\lambda(Y, \kappa, \widetilde{a}, t) = \arg \inf_{\nu} H(\nu, \kappa, \widetilde{a}, t)$$

The rest of the proof, in particular the standard argument ensuring that the minimizer of $H(\cdot)$ can be selected in a progressively measurable way, follows easily from Thm 16.35 in Elliott (1982).

As of the claim $a(Y, t) = \Xi(\omega, t)' \widetilde{a}(Y, t)$, notice that coefficients of the opportunity set (1.11) are deterministic function of the (possibly non Markovian) state variable Y . Though not apparent, let us conjecture that the optimal controls κ_1 and $\bar{\lambda}$ inherit this property, so that $f(t)$ is a function of time and the state Y . This conjecture will be verified in the sequel. Applying Clark-Ocone's formula (more details will be found in the proof of Proposition 6) we realize that $a(Y, t)$ is implicitly defined by the following integral equation

$$a(Y, t) = \mathbb{E}^{\kappa^* + \lambda^*} \left[\int_t^T \left(\frac{1}{A} \right)^{\frac{1}{\bar{\kappa}-1}} \mathcal{D}_t f(s) ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\bar{\kappa}-1}} \mathcal{D}_t f(T) \middle| \mathcal{F}_t \right] \quad (1.85)$$

where $\mathcal{D}_t \cdot$ is the Malliavin derivative operator:

$$\mathcal{D}_t f(s) = \nabla f(s) X(s) X(t)^{-1} \Xi(\omega, t) \mathbf{1}_{(s \geq t)}$$

and X is the first variation process associated with the vector process Y (see Nualart (1995) for details). Plugging the last expression into (1.85) we realize that

$$a(Y, t)' = \Xi(\omega, t)' \left(\mathbb{E}^{\kappa^* + \lambda^*} \left[\int_t^T \left(\frac{1}{A} \right)^{\frac{1}{\bar{\kappa}-1}} \nabla f(s) X(s) X(t)^{-1} ds + \left(\frac{1}{1-A} \right)^{\frac{1}{\bar{\kappa}-1}} \nabla f(T) X(T) X(t)^{-1} \middle| \mathcal{F}_t \right] \right)'$$

from which the claim follows.

iii) The proof is similar to *i)*. The expectations involved are performed with respect to the measure $Q^{\kappa + \lambda(\kappa)^*}$, where $\kappa \in K$ and $\lambda(\kappa)^*$ is the Kuhn-Tucker multiplier selected at the previous step.

iv) The proof exploits the arguments used in *ii)*. In particular, the sub-martingale property of the process $L(\lambda(\kappa)^*, \kappa, t)$, now defined with the value function $V(t)$ in place of $\mathcal{V}(t)$. If we let $(\kappa, v) \in K \times K$ denote two admissible Girsanov Kernels, then applying Ito's lemma and Girsanov

theorem to the analog of (1.84) we may write

$$\begin{aligned} \int_0^t \frac{1}{f(s)|_\kappa} & \left[B(\lambda(\kappa)^*, \kappa, s) + a(\kappa, s) \cdot \left[\left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{v+\lambda(v)^*}(t) - v(t) \right) - \left(\frac{\mathcal{R}}{\mathcal{R}-1} \theta_{\kappa+\lambda(\kappa)^*}(t) - \kappa(t) \right) \right] \right] ds \\ & + \int_0^t \frac{1}{f(s)|_\kappa} a(\kappa, s) \cdot dw_{v+\lambda(v)^*}(s) - \int_0^t \mathcal{V}(s) g(Y, \lambda(\kappa)^*, \kappa, s) ds = \\ & \int_0^t \frac{1}{f(s)|_v} B(\lambda(v)^*, v, s) ds - \int_0^t \frac{1}{f(s)|_v} a(\kappa, s) \cdot dw_{v+\lambda(v)^*}(s) - \int_0^t \mathcal{V}(s) g(Y, \lambda(v)^*, v, s) ds \end{aligned}$$

Mimicking the arguments outlined in *ii)* leads to the claim in *iv)* and completes the proof. ■

Notice that from this proposition it is apparent that $a(\kappa, t) = \phi(\omega, t)$, that is, the dual process $a(t)$ is coincident with the progressively measurable integrand $\phi(\omega, t)$ appearing in the stochastic integral representation of the Levy martingale $\tilde{f}(t, T)$ evaluated at the optimal Kuhn-Tucker multiplier. Since

$$\frac{1}{V(t) f(t)} = \frac{1}{\tilde{f}(t, T) - \int_0^t f(s) ds}$$

the first order condition for (1.81)

$$\frac{1}{1-\mathcal{R}} [\bar{\lambda}(t) - r\bar{1}_k + \bar{\kappa}(t)] + \frac{\vartheta(Y)a(t)'}{f(t)V(t)} = 0\bar{1}_\kappa \quad (1.86)$$

mandates directly $\pi_i = 0$, $i = 2, \dots, k+1$. From (1.86) we obtain³⁴

$$\theta_{\kappa+\lambda^*}(t) = \theta_0(\kappa_1) - (\mathcal{R}-1) \frac{a_f(t)'}{V(t)}$$

and substituting we obtain³⁵

$$\begin{aligned} \tilde{H}(\lambda(\kappa)^*, \kappa, \epsilon, t) &= V(t) \left(-\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} \theta_0(\kappa_1)' \theta_0(\kappa_1) \right) + \frac{\mathcal{R}}{\mathcal{R}-1} \theta_0(\kappa_1) \cdot a_f(t) \\ &\quad - \frac{\mathcal{R}}{2} \frac{a_f(t)' a_f(t)}{V(t)} - \bar{\kappa}(t)' \vartheta(Y) a_f(t)' - \kappa_1(t) (\sigma(t) \sigma(t)')^{-1} \sigma(t) a_f(t)' \end{aligned} \quad (1.87)$$

in Proposition 2 we conclude that $a(Y, t)' = \Xi(\omega, t)' \tilde{a}(Y, t)$, therefore the expression above reduces to

$$\tilde{H}(\lambda(\kappa)^*, \kappa, \epsilon, t) = V(t) \left(-\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} \theta_0(\kappa_1)' \theta_0(\kappa_1) \right) - \frac{\mathcal{R}}{2} \frac{\tilde{a}_f(t)' \tilde{a}_f(t)}{V(t)} - \bar{\kappa}(t)' \tilde{a}_f(t)'$$

³⁴We remind that $\theta_0(\kappa_1) = \sigma'(\sigma\sigma')^{-1}(\mu - r + \kappa_1)$, which is orthogonal to the second summand in the expression for $\theta_{\kappa+\lambda^*}$.

³⁵Notice that we have substituted the dual variable $\epsilon(t)$ with its 'optimal' value $a_f(t)$

The unique admissible minimizer of this expression is³⁶

$$\kappa_1(t) = \kappa_1^* = \begin{cases} -(\mu - r) & \text{if } -\sqrt{2h_1(Y)} < -\frac{\mu-r}{\sqrt{\sigma\sigma'}} < \sqrt{2h_1(Y)} \\ -\text{sgn}(\mu - r)\sqrt{2(\sigma\sigma')h_1(Y)} & \text{otherwise} \end{cases}$$

$$\bar{\kappa}(t) = -\sqrt{2\bar{h}(Y)} \frac{\tilde{a}_f(t)}{\sqrt{\tilde{a}_f(t)' \tilde{a}_f(t)}}$$

which fulfills the conjecture, made in the proof of Proposition 6, that the optimal control κ_1 is the deterministic function of the state Y . The Hamiltonian $\tilde{\mathcal{H}}(a_f, t)$ is easily seen to be

$$\tilde{\mathcal{H}}(a_f, t) = V(t) \left(-\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} \tilde{\theta}_0(\kappa_1^*)' \tilde{\theta}_0(\kappa_1^*) \right) - \frac{\mathcal{R}}{2} \frac{a_f(t)' a_f(t)}{V(t)} + \sqrt{2\bar{h}(Y)} \sqrt{a_f(t)' a_f(t)}$$

(1.88)

where

$$\tilde{\theta}_0(\kappa_1^*) = \begin{cases} 0 & \text{if } -\sqrt{2h_1(Y)} < -\frac{\mu-r}{\sqrt{\sigma\sigma'}} < \sqrt{2h_1(Y)} \\ (\sigma(t)\sigma(t'))^{-1}(\mu - r - \text{sgn}(\sigma\sigma')\sqrt{2(\sigma\sigma')h_1(Y)})^2 & \text{otherwise} \end{cases}$$

This ends the proof of Proposition 1.

Proof of Corollary 7

We remind that $w_\kappa(t) = w(t) - \int_0^t \Sigma^{-1}(s)\kappa(s)ds$ is a standard brownian motion under the measure P^κ . Let an adapted, square integrable solution $(p(t), b(t))$ of the following backward stochastic differential equation exists.

$$dp(t) = \left[p(t) \left(\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2} b(t)b(t)' + b(t)' \Xi(t) \Sigma(t)^{-1} \kappa(t) \right) - \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \right] dt + p(t)b(t)' \Xi(t) dw_\kappa(t)$$

$$p(T) = \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}}$$

³⁶Notice that the control κ_1 influences the component $\theta_0(\kappa_1)' \theta_0(\kappa_1)$ alone and does not control the state variables, therefore the optimality conditions mandate point-wise minimization of the ‘running cost’ with respect to this component.

$\kappa \in K$. Then taking into account the partitioned form of $\Sigma(t)^{-1}$ and applying Ito's lemma we have

$$\begin{aligned} d \left(p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right) &= -\frac{\mathcal{R}}{\mathcal{R}-1} p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} [r(t)dt + \theta_{\kappa+\lambda}(t)dw_{\kappa}(t)] \\ &+ \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \|\theta_{\kappa+\lambda}(t)\|^2 dt + p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} b(t)dw_{\kappa}(t) \\ &+ \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \left[p(t) \left(\frac{\mathcal{R}}{\mathcal{R}-1} r(t) - \frac{\mathcal{R}}{\mathcal{R}-1} b(t)'(\bar{\lambda}(t) - r(t)\bar{1}_k + \bar{\kappa}(t)) \right. \right. \\ &\quad \left. \left. + \frac{\mathcal{R}}{2} b(t)b(t)' + b(t)'\bar{\kappa}(t) \right) - \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \right] dt \end{aligned}$$

or

$$\begin{aligned} d \left(p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right) &= \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \left[(\bar{\lambda}(t) - r(t)\bar{1}_k + \bar{\kappa}(t))' (\bar{\lambda}(t) - r(t)\bar{1}_k + \bar{\kappa}(t)) + \right. \\ &\quad \left. (\mathcal{R}-1)^2 b(t)b(t)' + 2(\mathcal{R}-1)b(t)' (\bar{\lambda}(t) - r(t)\bar{1}_k + \bar{\kappa}(t)) \right] dt + \\ &\xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \left[p(t) \left(b(t)'\bar{\kappa}(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} (\sigma(t)\sigma(t'))^{-1} \sigma(t)'(\mu - r + \kappa_1)^2 \right) - \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \right] dt + [\dots]dw_{\kappa}(t) \end{aligned}$$

Integrating and taking expectations with respect to Q_{κ} we obtain

$$\begin{aligned} \mathbb{E}^{\kappa} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} dt + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \xi(\kappa + \lambda, T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right] &= p(0) + \\ \mathbb{E}^{\kappa} \left[\int_0^T \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \|\bar{\lambda}(t) - (r(t)\bar{1}_k - \bar{\kappa}(t) - (\mathcal{R}-1)b(t))\|^2 dt \right] &+ \\ \mathbb{E}^{\kappa} \left[\int_0^T p(t) \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \left(b(t)'\bar{\kappa}(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} (\sigma(t)\sigma(t'))^{-1} \sigma(t)'(\mu - r + \kappa_1)^2 \right) dt \right] \end{aligned}$$

therefore

$$\begin{aligned} \inf_{\kappa \in K} \inf_{\lambda} \mathbb{E}^{\kappa} \left[\left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} \int_0^T \xi(\kappa + \lambda, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} dt + \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}} \xi(\kappa + \lambda, T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right] &= \\ \inf_{\kappa \in K} \mathbb{E}^{\kappa} \left[\int_0^T p(t) \xi(\kappa + \lambda^*, t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \left(b(t)'\bar{\kappa}(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} (\sigma(t)\sigma(t'))^{-1} \sigma(t)'(\mu - r + \kappa_1)^2 \right) dt \right] \end{aligned}$$

where $\lambda^*(t) = [0 \quad (r(t)\bar{1}_k - \bar{\kappa}(t) - (\mathcal{R}-1)b(t))']$ and substitution in $\theta_{\kappa+\lambda}(t)$ leads to the expression reported in the corollary. It follows from the Kuhn-Tucker conditions that the minimization of the latter functional over admissible Girsanov kernels leads to the optimal control

$$\kappa_1^* = \begin{cases} -(\mu - r) & \text{if } -\sqrt{2h_1(Y)} < -\frac{\mu-r}{\sqrt{\sigma\sigma'}} < \sqrt{2h_1(Y)} \\ -\text{sgn}(\mu - r)\sqrt{2(\sigma\sigma')h_1(Y)} & \text{otherwise} \end{cases}$$

$$\bar{\kappa}^*(t) = -\sqrt{2\bar{h}(Y)} \frac{b(t)}{\sqrt{b(t)b(t)'}}$$

As of the characterization of the optimal portfolio policy in terms of the solution of the BSDE (1.13) the argument is similar to the proof of Proposition (1) and exploits the dynamic representation of the value function given by this BSDE.

Proof of Corollary 2

In order to obtain a characterization of the problem useful for computational purposes, let us focus our attention on the case in which (1.66) reduces to a strongly Markovian system, that is, all coefficients of the equations involved in the dynamics of the opportunity set are deterministic functions of time and the current state Y . In this case dynamic programming techniques imply that the value function V of the program (1.80) is a classical solution of the corresponding Hamilton-Jacobi-Bellman equation, the first order conditions of which are easily recovered from (1.87) and (1.86) above once we remind that in the markovian case the following equality holds

$$a_f(t) = \frac{\partial V(t)'}{\partial Y} \Xi(\omega, t)$$

therefore

$$\frac{1}{1-\mathcal{R}} [\bar{\lambda}(t) - r\bar{1}_k + \bar{\kappa}(t)] - \frac{\partial V(t)}{\partial Y} \frac{1}{V(t)} = 0\bar{1}_\kappa$$

or

$$\theta_{\kappa+\lambda^*}(t) = \theta_0(\kappa_1) + (\mathcal{R}-1) \frac{\partial V(t)}{\partial Y} \frac{1}{V(t)}$$

and

$$\tilde{H}(\lambda(\kappa)^*, \kappa, \Xi'V_Y, t) =$$

$$V(t) \left(-\frac{\mathcal{R}}{\mathcal{R}-1} r(t) + \frac{\mathcal{R}}{2(\mathcal{R}-1)^2} \theta_0(\kappa_1)' \theta_0(\kappa_1) \right) - \frac{\mathcal{R}}{2} \frac{\frac{\partial V(t)'}{\partial Y} \frac{\partial V(t)}{\partial Y}}{V(t)} - \bar{\kappa}(t)' \frac{\partial V(t)}{\partial Y}$$

This expression admits as unique admissible minimizer $\bar{\kappa}(t) = -\sqrt{2\bar{h}(Y)} \frac{\frac{\partial V(t)'}{\partial Y}}{\sqrt{\frac{\partial V(t)'}{\partial Y} \frac{\partial V(t)}{\partial Y}}}$ and κ_1 as in the proof of Proposition (6), therefore the value function of the consumption-investment problem under ambiguity is a classical solution of the semilinear partial differential equation

$$\frac{\partial V}{\partial t} + \tilde{\mathcal{H}}\left(\frac{\partial V(t)}{\partial Y}, t\right) + \Lambda(Y)' \frac{\partial V(t)}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 V(t)}{\partial Y \partial Y'} \right] + \left(\frac{1}{A} \right)^{\frac{1}{\mathcal{R}-1}} = 0$$

$V(T, Y) = \left(\frac{1}{1-A} \right)^{\frac{1}{\mathcal{R}-1}}$, where $\tilde{\mathcal{H}}(\cdot)$ is given in (1.88). Now let

$$G = V^{1-\mathcal{R}}$$

Then if V satisfies the latter equation one easily checks that $G(t, Y)$ solves the partial differential equation

$$\begin{aligned} \frac{\partial G}{\partial t} + \Lambda(Y)' \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi(Y) \Xi(Y)' \frac{\partial^2 G}{\partial Y \partial Y'} \right] + \sqrt{2\eta(Y)} \sqrt{\frac{\partial G'}{\partial Y} \frac{\partial G}{\partial Y}} \\ G \left[\mathcal{R} r - \frac{\mathcal{R}}{2(\mathcal{R} - 1)} (\sigma \sigma')^{-1} (\mu - r + \kappa_1^*)^2 \right] + (\mathcal{R} - 1) A^{\frac{1}{1-\mathcal{R}}} G^{\frac{\mathcal{R}}{\mathcal{R}-1}} = 0 \end{aligned} \quad (1.89)$$

with terminal condition $G(T, Y) = 1 - A$.

Logarithmic felicity function

By reasoning along the same lines of the proof of Proposition 1 one concludes that in this case the ambiguity-averse agent solves the program

$$\begin{aligned} \inf_{\kappa \in K} \inf_{\lambda} \mathbb{E}^{\kappa} \left[\int_t^T \int_t^s r(u) + \frac{1}{2} \theta_{\kappa+\lambda}(u)' \theta_{\kappa+\lambda}(u) du ds \mid \mathcal{F}_t \right] \\ \text{s.t.} \quad dY = [\Lambda(Y, t) + \bar{k}] dt + \Xi(Y, t) dw_{\kappa}(t) \end{aligned}$$

Quite clearly, a given probability measure P^{κ} on the space of sample paths of the state variable vector Y is invariant with respect to a particular choice of the dynamic Kuhn-Tucker multiplier $\lambda(t)$, therefore the constrained solution is recovered from the dual formulation by means of the simple point-wise minimization:

$$\lambda^*(t) = \arg \min_{\lambda} |\theta_{\kappa+\lambda}(t)|^2 = r(t) \bar{\Gamma}_k - \bar{\kappa}$$

from which $\theta_{\kappa+\lambda^*}(t) = \sigma(t)' (\sigma(t) \sigma(t)')^{-1} (\mu(t) - r(t) + \kappa_1(t))$, hence program (1.16). Since the control appearing in the running cost (κ_1) does not affect the dynamics of the state variable, point-wise minimization of the criterion implies the form reported in proposition 1 for the optimal controls, the HBJ equation solved by the value function and the content of Corollary (3).

Proof of Proposition 2

Let us first assume that an equilibrium exists. We first prove the expression for the equilibrium interest rate (1.24), the excess return (1.25), relation (1.26) and the volatility of the stock dynamics. By construction

$$\xi(\kappa^{(1)} + \lambda^1, t) = \frac{\mathcal{U}'(\varepsilon(t), \psi\gamma(t))}{\mathcal{U}'(\varepsilon(0), \psi)}$$

therefore, according to Ito's lemma applied to the left hand side of the expression above we have

$$\frac{d\mathcal{U}'(\varepsilon(t), \psi\gamma(t))}{\mathcal{U}'(\varepsilon(t), \psi\gamma(t))} = -r(t) dt - \left[\theta_0(\kappa_1^{(1)}) + \Xi' \left(\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1 \right) \right] \cdot dw_{\kappa^{(1)}}$$

On the other hand, one may apply Ito's lemma to the right hand side of the equality above and obtain the alternative representation

$$\begin{aligned}
d\mathcal{U}'(\varepsilon(t), \psi\gamma(t)) = & \left\{ \mathcal{U}'_{\varepsilon}\varepsilon \left(\mu_{\varepsilon} + \sigma_{\varepsilon} \cdot \mathbf{k}_1^{(1)} \right) + \mathcal{U}'_{\gamma}\gamma \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)'(\bar{\lambda}^1 - r\bar{\mathbf{I}}_k + \bar{\kappa}^1) + \right. \right. \\
& (\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1)'(\bar{\lambda}^1 - r\bar{\mathbf{I}}_k + \bar{\kappa}^1) + (\mathbf{k}_1^{(1)} - \mathbf{k}_1^{(2)})'\theta_0(\kappa_1^{(1)}) \left. \right] + \frac{1}{2}\mathcal{U}'_{\varepsilon\varepsilon}\varepsilon^2\sigma'_{\varepsilon}\sigma_{\varepsilon} + \\
& \frac{1}{2}\mathcal{U}'_{\gamma\gamma}\gamma^2 \left[(\mathbf{k}_1^{(1)} - \mathbf{k}_1^{(2)})'(\mathbf{k}_1^{(1)} - \mathbf{k}_1^{(2)}) + (\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1)'(\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1) \right] + \\
& \mathcal{U}'_{\gamma\varepsilon}\gamma\varepsilon\sigma_{\varepsilon}(\mathbf{k}_1^{(1)} - \mathbf{k}_1^{(2)}) \left. \right\} dt + \mathcal{U}'_{\varepsilon}\sigma_{\varepsilon}\varepsilon dw_{\kappa^{(1)}} + \\
& \mathcal{U}'_{\gamma}\gamma \left[(\mathbf{k}_1^{(1)} - \mathbf{k}_1^{(2)})' + \Xi'((\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1)) \right] dw_{\kappa^{(1)}}
\end{aligned}$$

Notice that the dynamics of the state variable need to be represented under the first agent's probability measure $Q^{\kappa^{(1)}}$. Furthermore, as it will become clear in the sequel, for the rows of the volatility matrix Ξ to span the kernel of the the stock's volatility σ in equilibrium, the same relation must hold for the the dividend process' volatility σ_{ε} , therefore $\sigma_{\varepsilon}\Xi' = 0\bar{\mathbf{I}}_k$. From the uniqueness of the special semi-martingale representation of the process $\mathcal{U}'(\varepsilon(t), \psi\gamma(t))$ we conclude that the drift and diffusion components of the two Ito's representation must be indistinguishable: from direct comparison of the drift components we obtain the the expression for the interest rate, whereas from direct comparison of the diffusion components we obtain the relation:

$$-\mathcal{U}'\left(\theta_0(\kappa_1^{(1)}) + \Xi'(\bar{\lambda}^1 - r\bar{\mathbf{I}}_{\kappa} + \bar{\kappa}^1)\right) = \mathcal{U}'_{\varepsilon}\sigma_{\varepsilon}\varepsilon + \mathcal{U}'_{\gamma}\gamma \left((\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)}) + \Xi'(\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1) \right)$$

Projecting this equality on $\text{Span}(\sigma)$ and $\text{Kernel}(\sigma)$ we obtain (1.25) and the equality

$$-\Xi' \left[\mathcal{U}'(\bar{\lambda}^1 - r\bar{\mathbf{I}}_{\kappa} + \bar{\kappa}^1) + \mathcal{U}'_{\gamma}\gamma \left(\bar{\lambda}^2 + \bar{\kappa}^2 - \bar{\lambda}^1 - \bar{\kappa}^1 \right) \right] = \mathcal{U}'_{\varepsilon}(I_{k+1} - \sigma'(\sigma\sigma')^{-1}\sigma)\sigma'_{\varepsilon}$$

Pre-multiplying both members of this expression by Ξ and keeping in mind the properties of this matrix we obtain (1.26).

As of the volatility process $\sigma(Y)$ appearing in the equity dynamics, Malliavin calculus provides us with a semi explicit representation in terms of expectations with respect to the probability measure implied by the dynamics of the state variable Y (under $Q^{\kappa^{(1)}}$, since in the solution technique the state price density of the first agent has been taken as 'equilibrium' state price density). This technique has been already exploited in the literature³⁷: one needs basically to apply Clark-Ocone formula in order to obtain the predictable integrand process $\alpha(t)$ in the stochastic integral representation of the martingale³⁸ (remind that $\mathcal{U}'(\varepsilon, \gamma, t) = \xi(\kappa^{(1)} + \lambda^1, t)$)

$$S(t)\mathcal{U}'(t) + \int_0^t \mathcal{U}'(s)\varepsilon(s)ds = \mathbb{E}^{\kappa^{(1)}} \left[\int_0^T \mathcal{U}'(s)\varepsilon(s)ds \middle| \mathcal{F}_t \right] = S(0) + \int_0^t \alpha(s) \cdot dw_{\kappa^{(1)}}(s) \quad (1.90)$$

³⁷see, for instance, Detemple and Serrat (2003)

³⁸we omit functional arguments once again in order to simplify notation.

that is

$$\begin{aligned} \alpha(t) = \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_{\varepsilon}(s) \varepsilon(s) \mathcal{D}_t \varepsilon(s) ds \middle| \mathcal{F}_t \right] - \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_{\gamma}(s) \varepsilon(s) \mathcal{D}_t \gamma(s) ds \middle| \mathcal{F}_t \right] + \\ \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'(s) \mathcal{D}_t \varepsilon(s) ds \middle| \mathcal{F}_t \right] \end{aligned} \quad (1.91)$$

where \mathcal{D}_t denotes the Malliavin derivative operator. But

$$\begin{aligned} \mathcal{D}_t \varepsilon(s) &= \mathbf{1}_{(s \geq t)} \varepsilon(s) \left(\sigma_{\varepsilon}(t)' + \int_t^s \mathcal{D}_t(\mu_{\varepsilon}(u) + \sigma_{\varepsilon} \mathbf{k}_1^{(1)}) du - \int_t^s \sigma_{\varepsilon}(u)' \mathcal{D}_t \sigma_{\varepsilon}(u) du + \int_t^s (\mathcal{D}_t \sigma_{\varepsilon}(u))' dw_{\kappa^{(1)}} \right) \\ \mathcal{D}_t \gamma(s) &= \mathbf{1}_{(s \geq t)} \gamma(s) \left(L(t)' + \int_t^s \mathcal{D}_t(\mu_{\varepsilon}(u) + \mathbf{k}_1^{(1)'}) \cdot \mathbf{k}_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)'(\bar{\lambda}^1 - r \bar{\mathbf{1}}_k + \bar{\kappa}^1) + \right. \\ &\quad \left. L(u) \Sigma^{-1}(u)(\kappa^{(1)} - \kappa^{(2)}) du - \int_t^s L(u)' \mathcal{D}_t L(u) du + \int_t^s (\mathcal{D}_t L(u))' dw_{\kappa^{(1)}} \right) \end{aligned} \quad (1.92)$$

where

$$L(t) = \left[(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)}) + \Xi(t)' (\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1) \right]$$

and we have made use of the representation of the process γ under the measure $Q^{\kappa^{(1)}}$. Now let the process $Y(t)$ be strongly markovian, that is, the coefficients $\Lambda(Y, t)$ and $\Xi(Y, t)$ be deterministic functions of the current state and time. Then we may define the first variation process X associated with the vector process Y as the solution of the following stochastic differential equation³⁹

$$dX = \frac{\partial \Lambda(Y, t)}{\partial Y} X(t) dt + \sum_{i=1}^{k+1} \frac{\partial \Xi^i(Y, t)}{\partial Y} X(t) dw_{\kappa_i^{(1)}}(t) \quad X(0) = I_k$$

and check (see Nualart (1995) for details) that for any $\phi \in C^1(\mathbb{R}^k)$ we have

$$\mathcal{D}_t \phi(Y(s)) = \nabla \phi(Y(s)) X(s) X(t)^{-1} \Xi(Y, t) \mathbf{1}_{(s \geq t)} \quad (1.93)$$

According to the last expressions, one may rewrite (1.91) in the following way

$$\begin{aligned} \alpha(t) = \sigma_{\varepsilon}(t)' \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_{\varepsilon}(s) \varepsilon(s)^2 ds \middle| \mathcal{F}_t \right] - L(t) \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'_{\gamma}(s) \varepsilon(s) \gamma(s) ds \middle| \mathcal{F}_t \right] + \\ \sigma_{\varepsilon}(t)' \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \mathcal{U}'(s) \varepsilon(s) ds \middle| \mathcal{F}_t \right] + \Xi(Y, t)' (\dots) \end{aligned} \quad (1.94)$$

and we easily identify the last term on the right hand side as $S(t) \mathcal{U}'(t) \sigma_{\varepsilon}$. Expression (1.94) needs to be compared to the volatility process of the first member of (1.90) given by Ito's lemma. Once

³⁹which is easily represented as the usual stochastic exponential

we project the latter on $\text{Span}(\sigma)$ we obtain:

$$\alpha(t) = S(t) \left[\mathcal{U}'_\varepsilon(t) \sigma'_\varepsilon(t) + \mathcal{U}'_\gamma(t) (k_1^{(2)} - k_1^{(1)}) \gamma(t) \right] + S(t) \mathcal{U}'(t) \sigma(t)' \quad (1.95)$$

We may then perform a similar projection for (1.94) and compare the result with (1.95) we obtain the system (1.27), whose solution gives the equilibrium volatility process.

Following the definition at the beginning of Section 1.4, in order to prove that the stated policies constitute an equilibrium we must show that: (i) markets are cleared by the proposed demands under the price system (S, r) when selected beliefs are $Q^{\kappa^{(i)*}}$, $i = 1, 2$; (ii) the latter are optimally chosen; (iii) consumption policies are optimal and financed by the given trading strategies.

Since in this general equilibrium framework we have confined ourselves either to the case of utility from intertemporal consumption or to the case of utility from terminal wealth, in what follows we will assume either $A = 0$ or $A = 1$.

i) The consumption policies (1.22) clear the good market as a consequence of expression (1.18), which follows from the state-dependent representative agent's marginal utility's being the inverse of the aggregate demand function (for the consumption good) with respect to the first agent's state price density (multiplied by ψ_1). Consider now the optimally invested wealth process of agent 2; since $\xi(\kappa^{(2)*} + \lambda^{2*}, t) = \mathcal{U}'(\varepsilon, \gamma, t) \gamma(t)$, we have:

$$\begin{aligned} W_2^{c_2, \pi_2}(t) &= \frac{1}{\gamma(t) \mathcal{U}'(\varepsilon, \gamma, t)} \mathbb{E}^{\kappa^{(2)}} \left[A \int_t^T \mathcal{U}'(\varepsilon, \gamma, s) \gamma(s) c_2^*(s) + (1 - A) \mathcal{U}'(\varepsilon, \gamma, T) \gamma(T) W_2^*(T) ds \middle| \mathcal{F}_t \right] \\ &= \frac{1}{\beta(t)} \mathbb{E}^{0_2} \left[A \int_t^T \beta(s) c_2^*(s) + (1 - A) \beta(T) W_2^*(T) \middle| \mathcal{F}_t \right] \end{aligned}$$

where we have denoted by W_2^* the optimally invested terminal wealth level and $\mathbb{E}^{0_2}[\cdot]$ denotes expectation with respect to the min-max martingale measure $Q^{0_2}(\cdot)$, defined by the relation

$$\mathbb{E}^{\kappa^{(2)}} \left[Z_0(\kappa^{(2)*} + \lambda^{2*}, T) \mathbf{1}_B \right] = Q^{0_2}(B) \quad B \in \mathcal{F}$$

The min-max martingale measure is preference dependent, therefore the selection made by agent 2 will differ in general from the selection made by agent 1, whose min-max measure Q^{0_1} is similarly defined.

Since first agent's shadow market corresponding to the state-price density $\xi(\kappa^{(1)} + \lambda^{1*}, t)$ (see proof of proposition 1) is dynamically complete, agent's 2 optimal consumption-invested wealth plan $(c_2^*(t), W_2^*(t))$ is attained by means of a self-financing, martingale generating trading strategy in the market driven by agent's 1 selected belief. Therefore:

$$x + \int_0^T \beta(s) \pi_2^*(s) \sigma(s) \cdot dw_{0_1}(s) = A \int_0^T \beta(s) c_2^*(s) + (1 - A) \beta(T) W_2^*(T)$$

Taking conditional expectations with respect to Q^{0_1} on both sides and applying the conditional

Bayes rule we obtain

$$x + \int_0^t \beta(s) \pi_2^*(s) \sigma \cdot dw_{0_1}(s) = \mathbb{E}^{0_1} \left[A \int_0^T \beta(s) c_2^*(s) + (1-A) \beta(T) W_2^*(T) \middle| \mathcal{F}_t \right] =$$

$$\frac{1}{\mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(s) c_2^*(s) + (1-A) \mathcal{U}'(T) W_2^*(T) \middle| \mathcal{F}_t \right] = W_2^{c_2, \pi_2}(t) \mathcal{U}'(t) + \int_0^t \mathcal{U}'(s) c_2^*(s) ds$$

and we realize that $W_2^{c_2, \pi_2}(t)$ admits the alternative representation:

$$\frac{1}{\mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_t^T \mathcal{U}'(s) c_2^*(s) ds + (1-A) \mathcal{U}'(T) W_2^*(T) \middle| \mathcal{F}_t \right]$$

It then follows from the good market clearing condition that:

$$W_1^{c_1, \pi_1}(t) + W_2^{c_2, \pi_2}(t) = \frac{1}{\mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_t^T \mathcal{U}'(s) (c_1^*(s) + c_2^*(s)) ds + (1-A) \mathcal{U}'(T) (W_1^*(s) + W_2^*(s)) \middle| \mathcal{F}_t \right] =$$

$$\frac{1}{\mathcal{U}'(t)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_t^T \mathcal{U}'(s) \varepsilon(s) ds + (1-A) \mathcal{U}'(T) \varepsilon(T) \middle| \mathcal{F}_t \right] = S(t) \quad (1.96)$$

and

$$\beta(t) (W_1^{c_1, \pi_1}(t) + W_2^{c_2, \pi_2}(t)) + A \int_0^t \beta(s) (c_1^*(s) + c_2^*(s)) ds =$$

$$S(0)(\eta_1 + \eta_2) + \int_0^t \beta(s) [\pi_1^*(s) W_1^{c_1, \pi_1}(t) + \pi_2^*(s) W_2^{c_2, \pi_1}(t)] \sigma(s) dw_{0_1}(s)$$

By virtue of (1.96), the consumption good market clearing and by direct comparison of the last expression with the cum-dividend stock dynamics (1.2) - namely, matching diffusion terms - we conclude that the stock market clearing condition holds for the posited consumption-investment policies and selected beliefs.

ii) This is the content of Proposition (3), where the equilibrium control problem that mandates optimality of the Girsanov kernels κ^i , $i = 1, 2$ (and of the Kuhn-Tucker multipliers λ^i) is illustrated. The equilibrium quantities reported in this proposition hold for any admissible likelihood.

iii) Let the controls κ^{i*} and λ^{i*} be optimal according to the criteria illustrated in Proposition 3. Furthermore, let the Kuhn-Tucker multipliers λ^i satisfy the equilibrium relation (1.26). By construction, we have $\mathcal{U}'(\varepsilon, \gamma, t) = \xi(\kappa^{(1)*} + \lambda^{1*}, t)$ therefore the definition of the shadow state price density $\xi(\cdot)$ implies that⁴⁰:

$$d\mathcal{U}'(t) = -\mathcal{U}'(t) r(t) dt - \mathcal{U}'(t) \theta_{\kappa^{(1)*} + \lambda^{1*}}(t) \cdot dw_{\kappa^{(1)}}(t)$$

⁴⁰As often the case in this Appendix, we drop most functional arguments in what follows when no confusion may arise. We will also use the notation $\mathcal{U}'(\psi, t)$ in order to emphasize the dependence of this inverse function on the ratio of the lagrange multipliers for the budget constraints, $\psi = \psi_2/\psi_1$

Let ψ^* be the positive ratio of the static lagrange multipliers ψ_2 and ψ_1 such that agent's 1 budget constraint is satisfied:

$$\frac{1}{\mathcal{U}'(\psi^*, 0)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T (\mathcal{U}'(\psi^*, t))^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} dt + (1-A) (\mathcal{U}'(\psi^*, T))^{\frac{\mathcal{R}_2}{\mathcal{R}_2-1}} \right] = \eta_1 \frac{1}{\mathcal{U}'(\psi^*, 0)} \mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, s) \varepsilon(s) ds + (1-A) \mathcal{U}'(\psi^*, T) \varepsilon(T) \right] \quad A = 0, 1 \quad (1.97)$$

or⁴¹

$$\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, t) v_1(\mathcal{U}'(\psi^*, t), t) dt + (1-A) \mathcal{U}'(\psi^*, T) v_1(\mathcal{U}'(\psi^*, T)', T) \right] = \eta_1 \mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, s) \varepsilon(s) ds + (1-A) \mathcal{U}'(\psi^*, T) \varepsilon(T) \right]$$

The individual multipliers are then recovered by means of the relation $\mathcal{U}'(\psi^*, 0) = \psi_1$, and it may easily be checked that agent's 2 budget constraint is satisfied as a consequence of market clearing. To see that a solution to (1.97) exists, rewrite the equation in the form

$$\frac{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, t) v_1(\mathcal{U}'(\psi^*, t), t) dt + (1-A) \mathcal{U}'(\psi^*, T) v_1(\mathcal{U}'(\psi^*, T)', T) \right]}{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, s) \varepsilon(s) ds + (1-A) \mathcal{U}'(\psi^*, T) \varepsilon(T) \right]} = \eta_1 < 1$$

Since, by definition of the inverse function $\mathcal{U}'(\cdot)$, $\lim_{\psi \rightarrow \infty} \mathcal{U}'(\psi, t) = \frac{\partial u^1}{\partial c}(\varepsilon(t))$, then, by continuity of the functions involved we have $\lim_{\psi \rightarrow \infty} \mathcal{U}'(\psi, t) v_1(\mathcal{U}'(\psi, t)', t) = \frac{\partial u^1}{\partial c}(\varepsilon(t)) \varepsilon(t)$ and applying Lebesgue's dominated convergence theorem twice we conclude that

$$\lim_{\psi \rightarrow \infty} \frac{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, t) v_1(\mathcal{U}'(\psi^*, t), t) dt + (1-A) \mathcal{U}'(\psi^*, T) v_1(\mathcal{U}'(\psi^*, T)', T) \right]}{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, s) \varepsilon(s) ds + (1-A) \mathcal{U}'(\psi^*, T) \varepsilon(T) \right]} = 1 > \eta_1$$

Furthermore, in light of the fact that $\lim_{\psi \rightarrow 0} \mathcal{U}'(\psi, t) = \infty$, we have $\lim_{\psi \rightarrow 0} v_1(\mathcal{U}'(\psi, t), t) = 0$ and an application of De L' Hospital rule to the above ratio allows us to write

$$\lim_{\psi \rightarrow 0} \frac{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, t) v_1(\mathcal{U}'(\psi^*, t), t) dt + (1-A) \mathcal{U}'(\psi^*, T) v_1(\mathcal{U}'(\psi^*, T)', T) \right]}{\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi^*, s) \varepsilon(s) ds + (1-A) \mathcal{U}'(\psi^*, T) \varepsilon(T) \right]} = 0 < \eta_1$$

Therefore we conclude that by continuity there exists a ψ^* in the domain $(0, \infty)$ such that the equation (1.97) is satisfied. As of uniqueness of this solution, rewrite (1.97) as

$$\mathbb{E}^{\kappa^{(1)}} \left[A \int_0^T \mathcal{U}'(\psi, t) (v_1(\mathcal{U}'(\psi, t), t) - \eta_1 \varepsilon(t)) dt + (1-A) \mathcal{U}'(\psi, T) (v_1(\mathcal{U}'(\psi, T)', T) - \eta_1 \varepsilon(T)) \right] = 0$$

⁴¹Remind that $v_1(\cdot)$ denotes the inverse marginal felicity function of agent 1.

or

$$\mathbb{E} \left[A \int_0^T \left(\mathcal{U}'(\psi^*, t)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}} - \mathcal{U}'(\psi^*, t) \varepsilon(t) \right) dt + (1-A) \left(\mathcal{U}'(\psi^*, T)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}} - \mathcal{U}'(\psi^*, T) \varepsilon(T) \right) \right] = 0$$

and notice that state dependent marginal utility of the representative agent', $\mathcal{U}'(\psi, t)$ is strictly decreasing in the ratio ψ , therefore if we assume $\mathcal{R}_1 > 0$, the expression on the left hand side is monotonically decreasing; for this case uniqueness of the solution ψ^* can then be guaranteed.

Consider the optimal consumption-portfolio policy for agent-1 reported in the proposition, (c_1^*, π_1^*) and define the vector $\hat{\pi}_1^* = (\pi_1^* \quad \tilde{\pi}_1^*)'$ where

$$\tilde{\pi}_1^* = (\bar{\lambda}^{1*} - r \bar{\mathbf{I}}_k + \bar{\kappa}^{1*}) + \Xi(t)' \delta(t)$$

so that

$$\hat{\pi}_1^*(t) = \Sigma(Y)^{\prime-1} \theta_{\kappa(1)* + \lambda^{1*}}(Y) + \Sigma(Y)^{\prime-1} \frac{\delta(t)}{W_1^*(t) \mathcal{U}'(t)}$$

In terms of the Kuhn-Tucker multiplier and the Girsanov kernel to be determined in the next Proposition. With the above choice of the parameter ψ the following chain of equalities holds for the $Q^{\kappa(1)}$ -martingale:

$$\begin{aligned} \mathbb{E}^{\kappa(1)} \left[A \int_0^T \frac{\mathcal{U}'(s)}{\mathcal{U}'(0)} c_1^*(s) ds + (1-A) \frac{\mathcal{U}'(T)}{\mathcal{U}'(0)} W_1^*(T) \middle| \mathcal{F}_t \right] &= \mathbb{E}^{\kappa(1)} \left[A \int_0^T \frac{\mathcal{U}'(s)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}}{\mathcal{U}'(0)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}} ds + (1-A) \frac{\mathcal{U}'(T)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}}{\mathcal{U}'(0)^{\frac{\mathcal{R}_1}{\mathcal{R}_1-1}}} \middle| \mathcal{F}_t \right] \\ \mathcal{U}'(t) W_1^{c_1^*, \pi_1^*}(t) + A \int_0^t \mathcal{U}'(s) c_1^*(s) ds &= \eta_1 S(0) + \int_0^t \delta(s)' \cdot dw_{\kappa(1)}(s) = \\ \eta_1 S(0) + \int_0^t W_1^{c_1^*, \pi_1^*}(s) \mathcal{U}'(s) [\Sigma' \hat{\pi}_1^*(s) - \theta_{\kappa(1)* + \lambda^{1*}}(s)]' \cdot dw_{\kappa(1)}(s) &\quad (1.98) \end{aligned}$$

where the second equality follows from the stochastic integral representation of the first member and the third from the definition of $\hat{\pi}_1^*$. Applying once again Ito's lemma to (1.98) - namely, to $(\mathcal{U}'(t) W_1^*(t)) / \mathcal{U}'(t)$, where the first term is represented as in (1.98) - we conclude that

$$\begin{aligned} W_1^{c_1^*, \pi_1^*}(t) &= \eta_1 S(0) + \int_0^t \left[W_1^{c_1^*, \pi_1^*}(s) \left(r(s) + \hat{\pi}_1^{*'}(s) \Sigma \theta_{\kappa(1) + \lambda^1}(s) \right) - c(s) \right] ds + \\ &\quad \int_0^t W_1^{c_1^*, \pi_1^*}(s) \hat{\pi}_1^{*'}(s) \Sigma \cdot dw_{\kappa(1)} \end{aligned}$$

or, recalling the form of $\theta_{\kappa(1) + \lambda^1}$ and expliciting products

$$\begin{aligned} W_1^{c_1^*, \pi_1^*}(t) &= \eta_1 S(0) + \int_0^t \left[W_1^{c_1^*, \pi_1^*}(s) \left(r(s) + \pi_1^*(s) \sigma(s) \cdot \mathbf{k}_1^{(1)}(s) \right) - c(s) \right] ds + \\ &\quad \int_0^t W_1^{c_1^*, \pi_1^*}(s) \pi_1^*(s) (\mu(s) - r(s)) ds + \int_0^t W_1^{c_1^*, \pi_1^*}(s) \pi_1^*(s) \sigma(s) \cdot dw_{\kappa(1)} \end{aligned}$$

so that c_1^* is financed by π_1^* and is attainable, given that we clearly have $W_1^{c_1^*, \pi_1^*}(t) > 0$. The

symmetric result holds for agent-2 as an immediate consequence of the consumption good and stock market clearing conditions. Consider now the convex dual of the felicity function u_1 :

$$\begin{aligned}\tilde{u}_1(\mathcal{U}'(\varepsilon(t), \psi^* \gamma(t))) &= \sup_{c_1} [u_1(c_1(t)) - \mathcal{U}'(\varepsilon(t), \psi^* \gamma(t))c_1(t)] = \\ &u_1(v_1(\mathcal{U}'(\varepsilon(t), \psi^* \gamma(t)), t)) - \mathcal{U}'(\varepsilon(t), \psi^* \gamma(t))v_1(\mathcal{U}'(\varepsilon(t), \psi^* \gamma(t)), t)\end{aligned}$$

and the budget feasible consumption plan c_1 ; in light of the last expression we have $[u_1(v_1(\mathcal{U}'(\cdot))) - \mathcal{U}'(\cdot)v_1(\mathcal{U}'(\cdot))] \geq u_1(c_1) - \mathcal{U}'(\cdot)c_1$, so that:

$$\begin{aligned}U^1(c_1^*) - U^1(c_1) &= \mathbb{E}^{\kappa^{(1)}} \left[\int_0^T (u_1(c_1^*(s)) - u_1(c_1(s))) ds \right] \\ &\geq \mathcal{U}'(\varepsilon(0), \psi^*) \mathbb{E}^{\kappa^{(1)}} \left[\int_0^T \frac{\mathcal{U}'(\varepsilon(s), \psi^* \gamma(s))}{\mathcal{U}'(\varepsilon(0), \psi^*)} (c_1^*(s) - c_1(s)) ds \right] \geq 0\end{aligned}$$

where the last inequality follows from (1.97). Therefore c_1^* is an optimal consumption plan for agent-1; repeating the argument for the convex dual of $u_2(\cdot)$ evaluated at $\mathcal{U}'(\cdot)\gamma$ leads to the optimality of c_2^* .

Proof of Corollary 4

Apply Ito's lemma to both sides of the good market clearing condition:

$$\left(\psi_1 \xi(\kappa^{(1)} + \lambda^1, t) \right)^{\frac{1}{\mathcal{R}_1 - 1}} + \left(\psi_1 \xi(\kappa^{(1)} + \lambda^1, t) \gamma(t) \right)^{\frac{1}{\mathcal{R}_2 - 1}} = \varepsilon(t)$$

and notice that $R_a^i = -c_i^*/(\mathcal{R}_i - 1) = (\psi_i \xi_i)^{\frac{1}{\mathcal{R}_i - 1}}/(\mathcal{R}_i - 1)$. Comparing drift and diffusion components of the Ito representations thus obtained and applying the line of reasoning used in the proof of Proposition (2) one obtains (1.28), (1.29) and the expression for the equilibrium interest rate reported in the Corollary.

Proof of Proposition 3

Due to competitive behavior, the agents determine their controls in equilibrium taking as exogenously given the state price density $\mathcal{U}'(\varepsilon(t), \psi\gamma)$ determined by the good market clearing condition. In particular this implies that they do not take advantage of the equilibrium relations (1.25), (1.24) and (1.26), instead they consider the equilibrium weighting process as given and intermediate consumption the only stochastic component in the equilibrium state price density. Since, by construction, we have

$$\frac{\mathcal{U}'(\varepsilon, \gamma, t)}{\mathcal{U}'(\varepsilon, \gamma, 0)} = \xi(\kappa^{(1)} + \lambda^1, t) \quad \gamma(t) \frac{\mathcal{U}'(\varepsilon, \gamma, t)}{\mathcal{U}'(\varepsilon, \gamma, 0)} = \xi(\kappa^{(2)} + \lambda^2, t)$$

following the steps that led us to the characterization of the individual consumption problem in the partial equilibrium framework, we determine the general equilibrium values of the Kuhn-Tucker multipliers λ and κ as solutions of the programs (1.31) and (1.32), equilibrium counterparts of (1.80).

Notice that in this section the parameter A is either 0 or 1.

In general, due to the functional dependence of the inverse function $\mathcal{U}'(\varepsilon, \gamma, t)$, the control problem to be solved in equilibrium by the agent 2 is subject to the dynamics of the three state variables γ, ε, Y , whereas competitive behavior suggests that the problem solved by agent 1 is subject to the dynamics of ε and Y alone.

Proof of Proposition 4

If both agents have logarithmic felicity functions, inversion of the good market clearing condition is easily seen to yield

$$\mathcal{U}'(t, \varepsilon, \gamma) = \frac{1}{\varepsilon(t)} \left(1 + \frac{1}{\gamma(t)} \right)$$

The optimal equilibrium allocation (1.22), expressed in terms of the equilibrium weighting process $\gamma(t)$ is then immediately obtained from the partial equilibrium optimal demand. Competitive behavior of the agents suggests that in their dynamic optimization problem performed at equilibrium prices the weighting process γ appearing above is considered exogenously given at its equilibrium value: $\gamma = \widehat{\gamma}$. Then program (1.31) and program (1.32) reduce to, respectively

$$\inf_{\kappa^{(1)} \in K^1} \inf_{\lambda} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T -\log \left(\frac{1}{\varepsilon(s)} \left(1 + \frac{1}{\widehat{\gamma}(s)} \right) \right) \middle| \mathcal{F}_t \right]$$

and

$$\inf_{\kappa^{(2)} \in K^2} \inf_{\lambda} \mathbb{E}^{\kappa^{(2)}} \left[\int_t^T -\log \left(\frac{\gamma(s)}{\varepsilon(s)} \left(1 + \frac{1}{\widehat{\gamma}(s)} \right) \right) \middle| \mathcal{F}_t \right]$$

subject to

$$dY = (\Lambda(t) + \overline{\kappa^i})dt + \Xi(t)dw_{\kappa^{(i)}} \quad i = 1, 2$$

Notice that the solutions of the stochastic differential equation for $\varepsilon(t)$ and $\gamma(t)$ under the measure $Q^{\kappa^{(2)}}$ are as follows

$$\varepsilon(t) = \varepsilon(0) \exp \left(\int_0^t \left(\mu_\varepsilon + \sigma_\varepsilon \cdot \mathbf{k}_1^{(1)} - \frac{1}{2} \sigma_\varepsilon' \cdot \sigma'_\varepsilon \right) ds + \int_0^t \sigma_\varepsilon \cdot dw_{\kappa^{(1)}} \right) \quad (1.99)$$

$$\begin{aligned} \gamma(t) = & \frac{\psi_1}{\psi_2} \exp \left(\int_0^t \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + (\overline{\lambda^2} - \overline{\lambda^1})' (\overline{\lambda^1} - r\overline{\mathbf{I}}_k + \overline{\kappa^1}) \right] ds - \right. \\ & \frac{1}{2} \left| \left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right) + \Xi(t)' \left(\overline{\lambda^2} - \overline{\lambda^1} + \overline{\kappa^2} - \overline{\kappa^1} \right) \right|^2 + \\ & \left. \left[\left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right) + \Xi(t)' \left(\overline{\lambda^2} - \overline{\lambda^1} + \overline{\kappa^2} - \overline{\kappa^1} \right) \right] dw_{\kappa^{(2)}}(t) \right) \end{aligned} \quad (1.100)$$

therefore the value function of agent 2, possibly after a localization argument becomes

$$\inf_{\kappa^{(2)} \in K^2} \inf_{\bar{\lambda}} \mathbb{E}^{\kappa^{(2)}} \left[\int_t^T \int_t^s \left[\mu_\varepsilon(u) - \frac{1}{2} \sigma_\varepsilon(u) \sigma_\varepsilon(u)' + \sigma_\varepsilon(u) \cdot \mathbf{k}_1^{(2)} - C(\kappa, \lambda, u) \right] du - \log \left(1 + \frac{1}{\widehat{\gamma}(s)} \right) ds \middle| \mathcal{F}_t \right]$$

where

$$C(\kappa, \lambda, t) = \left[\mathbf{k}_1^{(1)'} \cdot \mathbf{k}_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)' (\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1) \right] - \frac{1}{2} \left| \left(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)} \right) + \Xi(t)' \left(\bar{\lambda}^2 - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1 \right) \right|^2$$

and we realize why the current stochastic controls problem are being solved subject to the dynamics of the state variable Y alone. Notice that the controls $\bar{\lambda}^1$, $\bar{\kappa}^1$, $\kappa_1^{(1)}$ are taken as given by agent 2. Quite clearly the Kuhn-Tucker multiplier $\bar{\lambda}^2$ does not affect the dynamics of the state variable, therefore a necessary condition for its optimality is $\bar{\lambda}^{2*} = \arg \inf_{\bar{\lambda}} [\text{drift}(\log \varepsilon(t)) - C(\kappa, \lambda, t)]$, or

$$\bar{\lambda}^2 = 2(\bar{\lambda}^1 + \bar{\kappa}^1) - \bar{\kappa}^2 - r \bar{\Gamma}_k$$

but substitution of this expression into the equilibrium condition (1.26) yields $\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1 = -\frac{u'}{u'} (\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1)$, or $\bar{\lambda}^{1*} = r \bar{\Gamma}_k - \bar{\kappa}^1$. It is immediate to see that the optimal Girsanov kernel of agent 2 is given, in terms of the still to be determined optimal Girsanov kernel of agent 1, by

$$\begin{aligned} \mathbf{k}_1^{(2)*} &= \begin{cases} \mathbf{k}_1^{(1)*} - \sigma'_\varepsilon & \text{if } \left(\mathbf{k}_1^{(1)*} - \sigma'_\varepsilon \right)' \left(\mathbf{k}_1^{(1)*} - \sigma'_\varepsilon \right) \leq 2h_2(Y) \\ \sqrt{2h_2(Y)} \frac{\mathbf{k}_1^{(1)} - \sigma'_\varepsilon}{\sqrt{(\mathbf{k}_1^{(1)} - \sigma'_\varepsilon)' (\mathbf{k}_1^{(1)} - \sigma'_\varepsilon)}} & \text{otherwise} \end{cases} \\ \bar{\kappa}^{2*} &= -\sqrt{2h_2(Y)} \frac{(a + (\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1))}{\sqrt{(a + (\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1))' (a + (\bar{\lambda}^1 - r \bar{\Gamma}_k + \bar{\kappa}^1))}} \end{aligned}$$

where a is the predictable integrand process in the stochastic integral representation of the Levy martingale

$$\mathbb{E}^{\kappa^{(2)}} \left[\int_0^T \int_0^s \left(\mu_\varepsilon(u) - \frac{1}{2} \sigma_\varepsilon(u) \sigma_\varepsilon(u)' + \sigma_\varepsilon(u) \mathbf{k}_1^{(2)*}(s) - C(\kappa^*, \lambda^*, u) \right) du ds \middle| \mathcal{F}_t \right] \quad (1.101)$$

with $\bar{\lambda}^1$ and $\bar{\kappa}^1$ still considered as given in spite of the equilibrium condition (1.26). The last characterization follows easily from the partial equilibrium counterpart of the optimization problem. The markovian analog of this expression is similarly obtained.

Consider now the equilibrium optimization problem of agent 1. In light of (1.99), and the fact that his optimal dynamic Kuhn-Tucker multiplier has been determined by means of the equilibrium

relation (1.26), his value function may be written, possibly after a localization argument

$$\inf_{\kappa^{(1)} \in K^1} \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T \int_t^s \left[\mu_\varepsilon(u) - \frac{1}{2} \sigma_\varepsilon(u) \sigma_\varepsilon(u)' + \sigma_\varepsilon(u) \mathbf{k}_1^{(1)}(u) \right] du - \log \left(1 + \frac{1}{\widehat{\gamma}(s)} \right) ds \middle| \mathcal{F}_t \right]$$

from which one immediately obtains

$$\begin{aligned} \mathbf{k}_1^{(1)*} &= -\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma'_\varepsilon \sigma_\varepsilon}} \\ \overline{\kappa^{1*}} &= -\sqrt{2\overline{h^1}(Y)} \frac{\widetilde{a}}{\sqrt{\widetilde{a}' \widetilde{a}}} \end{aligned}$$

where \widetilde{a} is the predictable integrand appearing in the stochastic integral representation of the Levy martingale

$$\mathbb{E}^{\kappa^{(1)}} \left[\int_0^T \int_0^s \mu_\varepsilon(u) - \frac{1}{2} \sigma_\varepsilon(u) \sigma_\varepsilon(u)' - \sqrt{2h_1(Y)} \sqrt{\sigma_\varepsilon(u) \cdot \sigma_\varepsilon(u)'} du ds \middle| \mathcal{F}_t \right] \quad (1.102)$$

Notice that in light of this form of the optimal Girsanov Kernel $\mathbf{k}_1^{(1)*}$, the expression (1.34) for the optimal Girsanov kernel $\mathbf{k}_1^{(2)*}$ follows.

The expression (1.36) for the equilibrium interest rate and (1.37) for the equilibrium excess return under the reference probability measure are obtained by substituting the above controls into (1.24) and (1.25) and computing derivatives of the inverse function $\mathcal{U}'(\varepsilon, \gamma, t)$. The expression for the equilibrium weighting process is obtained by substituting the optimal controls into (1.19).

As of the equilibrium volatility of the stock price process, we just need to remind that $\mathcal{U}'_\varepsilon = -(1/\varepsilon^2)(1 + \gamma^{-1})$, $\mathcal{U}'_\gamma = -(1/(\varepsilon\gamma^2))$ and substitute into the general formula (1.27). Notice that we have made use twice of the equality

$$S(t)\mathcal{U}'(t) = \mathbb{E}^{\kappa^{(1)}} \left[\int_t^T (1 + \gamma(s)^{-1}) ds \middle| \mathcal{F}_t \right]$$

obtained by simple substitution of the inverse function $\mathcal{U}'(t)$ into the general stock pricing formula. Finally, the equilibrium portfolio fraction (1.60) invested in the stock by the first agent is obtained by considering the investment policy of the logarithmic investor and reminding the equilibrium excess return with respect to the measure $Q^{\kappa^{(1)}}$ given in (1.37). The policy of the second agent, (1.61) then follows from the market clearing relation $\pi_1 W_1 + \pi_2 W_2 = S$, where the equilibrium stock price process and optimally invested wealth processes of the agents have been substituted.

Proof of Proposition 5

The proof of this proposition mimics the steps of the previous proof, but the analysis is more involved due to the agents' adopting a felicity function of power type. Notice that inversion of the

good market clearing condition

$$\left(\psi_1 \xi(\kappa^{(1)} + \lambda^1, T)\right)^{\frac{1}{\mathcal{R}-1}} + \left(\psi_1 \xi(\kappa^{(1)} + \lambda^1, T) \gamma(T)\right)^{\frac{1}{\mathcal{R}-1}} = \varepsilon(T)$$

yields

$$\mathcal{U}'(\varepsilon, \gamma, T) = \left(\frac{\varepsilon(T)}{1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}-1}$$

from which the optimal equilibrium allocation of terminal consumption (1.43) (terminal wealth) expressed in terms of the equilibrium weighting process $\gamma(t)$ is then immediately obtained from the partial equilibrium optimal demands. Since $\beta(t)^{-1} \mathcal{U}'(\varepsilon, \gamma, t)$ must be a martingale, then

$$\beta(t)^{-1} \mathcal{U}'(t) = \mathbb{E}^{\kappa^{(1)}} \left[\beta(T)^{-1} \left(\frac{\varepsilon(T)}{1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}-1} \middle| \mathcal{F}_t \right] = \beta(t)^{-1} \left(\frac{\varepsilon(t)}{1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}-1}$$

so that $\mathcal{U}'(t) = \left(\frac{\varepsilon(t)}{1 + \gamma(t)^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}-1}$. In light of considerations similar to those made in the proof of the logarithmic case and in light of the partial equilibrium analog of the current program, we realize that program (1.31) and program (1.32) reduce to, respectively

$$\inf_{\kappa^{(1)} \in K^1} \inf_{\lambda} \mathbb{E}^{\kappa^{(1)}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}} \varepsilon(T)^{\mathcal{R}} \right]$$

and

$$\inf_{\kappa^{(2)} \in K^2} \inf_{\lambda} \mathbb{E}^{\kappa^{(2)}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}} \varepsilon(T)^{\mathcal{R}} \gamma(T)^{\frac{\mathcal{R}}{\mathcal{R}-1}} \right]$$

In order to see that the two control problems are subject to the dynamics of the state variable Y alone (under a new probability measure, as we will explain shortly), consider the problem of the first agent and notice that in light of (1.99) and of the fact that Ξ is in the kernel of σ_ε , we may perform the same change of probability measure that we used in the partial equilibrium framework and equivalently write

$$\inf_{\lambda} \inf_{\kappa^1 \in K^1} \mathbb{E}^{\kappa^{(1)}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}} e^{\int_0^T \mathcal{R}(\mu_\varepsilon + \sigma_\varepsilon k_1^{(1)}) dt - \frac{1}{2} \sigma_\varepsilon \sigma'_\varepsilon (\mathcal{R} - \mathcal{R}^2) dt} \right]$$

s.t.

$$dY = [\Lambda(t) + \overline{\kappa^1}(t)] dt + \Xi(t) dw_{\overline{\kappa}^{(1)}}$$

from which one immediately obtains

$$\begin{aligned} \mathbf{k}_1^{(1)*} &= -\sqrt{2h_1(Y)} \frac{\sigma'_\varepsilon}{\sqrt{\sigma_\varepsilon \sigma'_\varepsilon}} \\ \overline{\kappa^1}^* &= -\sqrt{2\overline{h^1}(Y)} \frac{V_Y}{\sqrt{V'_Y V_Y}} \end{aligned}$$

where $V(Y)$ is a classical solution of the Hamilton Bellman Jacobi equation

$$\begin{aligned} V_t + \Lambda(Y)V_Y - \sqrt{2h_1(Y)}\sqrt{V'_Y V_Y} + \frac{1}{2}\text{trace}[V_{YY}\Xi\Xi'] + \\ V \left[\mathcal{R} \left(\mu_\varepsilon(Y) - \sqrt{2h_1(Y)}\sqrt{\sigma_\varepsilon(Y)\sigma'_\varepsilon(Y)} \right) - \frac{1}{2}\sigma_\varepsilon(Y)\sigma'_\varepsilon(Y)(\mathcal{R} - \mathcal{R}^2) \right] = 0 \end{aligned} \quad (1.103)$$

$$V(T) = \left(\frac{1}{1 + \widehat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}}.$$

As of the equilibrium problem solved by agent 2, we apply once again the same line of reasoning used in the partial equilibrium framework. Notice that the equilibrium optimal controls of agent 1 are taken as given. (1.100) and (1.99) imply that, under the measure $Q^{\kappa^{(2)}}$ ⁴²

$$\begin{aligned} \varepsilon(t)^{\mathcal{R}} \gamma(t)^{\frac{\mathcal{R}}{\mathcal{R}-1}} = \varepsilon(0)^{\mathcal{R}} \exp \left(\int_0^t \left[\mathcal{R} \left(\mu_\varepsilon + \sigma_\varepsilon \cdot \mathbf{k}_1^{(2)} - \frac{1}{2}\sigma_\varepsilon \cdot \sigma'_\varepsilon \right) + \right. \right. \\ \left. \left. \frac{\mathcal{R}}{\mathcal{R}-1} \left(\mathbf{k}_1^{(1)'} \mathbf{k}_1^{(1)} + (\overline{\lambda^2} - \overline{\lambda^1})'(\overline{\lambda^1} - r\overline{1}_k + \overline{\kappa^1}) - \frac{1}{2}L'L \right) + \frac{1}{2} \left| \mathcal{R}\sigma_\varepsilon + \frac{\mathcal{R}}{\mathcal{R}-1}L' \right|^2 - \frac{1}{2} \left| \mathcal{R}\sigma_\varepsilon + \frac{\mathcal{R}}{\mathcal{R}-1}L' \right|^2 \right] ds \right. \\ \left. \int_0^t \left(\mathcal{R}\sigma_\varepsilon + \frac{\mathcal{R}}{\mathcal{R}-1}L' \right) \cdot dw_{\kappa^{(2)}} \right) \end{aligned}$$

In light of this form we may perform an absolutely continuous change of probability measure and apply Girsanov theorem to eventually rewrite the equilibrium dynamic program of agent 2 in the equivalent form

$$\inf_{\kappa^{(2)} \in K^2} \inf_{\lambda} \mathbb{E}^{\overline{\kappa}^{(2)}} \left[\left(\frac{1}{1 + \widehat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}} e^{F(T)} \right] \quad (1.104)$$

subject to

$$dY = \left[\Lambda(t) + \overline{\kappa^2}(t) - \Xi(t) \left(\mathcal{R}\sigma_\varepsilon(t)' + \frac{\mathcal{R}}{\mathcal{R}-1}L(t) \right) \right] dt + \Xi(t)dw_{\overline{\kappa}^{(2)}}$$

that is⁴³

$$dY = \left[\Lambda(t) + \overline{\kappa^2}(t) - \frac{\mathcal{R}}{\mathcal{R}-1} \left(\overline{\lambda^2} - \overline{\lambda^1} + \overline{\kappa^2} - \overline{\kappa^1} \right) \right] dt + \Xi(t)dw_{\overline{\kappa}^{(2)}}$$

The new probability measure and the new standard brownian motion $(w_{\overline{\kappa}^{(2)}}(t))$ are identified as

⁴²remind that

$$L(t) = \left[(\mathbf{k}_1^{(2)} - \mathbf{k}_1^{(1)}) + \Xi(t)' \left(\overline{\lambda^2} - \overline{\lambda^1} + \overline{\kappa^2} - \overline{\kappa^1} \right) \right]$$

⁴³remind that Ξ is in the Kernel of σ_ε and \mathbf{k}_1 is orthogonal to Ξ by construction.

usual and

$$\begin{aligned} F(T) = \int_0^T & \left[\mathcal{R} \left(\mu_\varepsilon + \sigma_\varepsilon \cdot k_1^{(2)} - \frac{1}{2} \sigma_\varepsilon \cdot \sigma'_\varepsilon \right) + \right. \\ & \left. \frac{\mathcal{R}}{\mathcal{R}-1} \left(k_1^{(1)'} k_1^{(1)} + (\bar{\lambda}^2 - \bar{\lambda}^1)' (\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1) - \frac{1}{2} L' L \right) + \frac{1}{2} \left| \mathcal{R} \sigma_\varepsilon + \frac{\mathcal{R}}{\mathcal{R}-1} L' \right|^2 \right] ds \end{aligned}$$

If we let $J(Y)$ denote the value function (1.104), dynamic programming techniques and considerations similar to those made in the partial equilibrium set-up suggest that

$$\bar{\lambda}^{2*} - \bar{\lambda}^1 + \bar{\kappa}^2 - \bar{\kappa}^1 = -(\mathcal{R}-1) \left(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1 \right) + \frac{J_Y}{J} \quad (1.105)$$

where $J(Y) = G(Y)^\alpha$, $\alpha = \frac{(\mathcal{R}-1)^2}{\mathcal{R}+(\mathcal{R}-1)^2}$, and $G(Y)$ is the value function of the program

$$\inf_{(k_2^{(2)*}, \kappa^2)} \mathbb{E}^{\bar{\kappa}^{(2)}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\frac{\mathcal{R}}{\alpha}} e^{\int_0^T K(t) dt} \right] \quad (1.106)$$

$$\text{s.t.} \quad dY = \left[\Lambda(Y) + \bar{\kappa}^2(t) - \frac{\mathcal{R} - \mathcal{R}^2}{\mathcal{R} - 1} \left(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1 \right) \right] dt + \Xi(t) dw_{\bar{\kappa}^{(2)}}$$

with

$$\begin{aligned} K(t) = \frac{1}{\alpha} & \left[\mathcal{R} \left(\mu_\varepsilon + \sigma_\varepsilon \cdot k_1^{(2)} - \frac{1}{2} \sigma_\varepsilon \cdot \sigma'_\varepsilon \right) + \right. \\ & \frac{\mathcal{R}}{\mathcal{R}-1} \left(k_1^{(1)'} k_1^{(1)} + (\bar{\kappa}^1 - \bar{\kappa}^2 - (\mathcal{R}-1)(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1))' (\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1) - \frac{1}{2} L^{*'} L^* \right) + \\ & \left. \frac{1}{2} \left| \mathcal{R} \sigma_\varepsilon + \frac{\mathcal{R}}{\mathcal{R}-1} L^{*'} \right|^2 \right] \end{aligned}$$

We have used the notation

$$L^*(t) = (k_1^{(2)} - k_1^{(1)}) - \Xi(t)'(\mathcal{R}-1) \left(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1 \right)$$

By virtue of techniques similar to those largely exploited in the previous proofs one easily realizes that the optimal Girsanov kernel for agent 2, $(k_1^{(2)*}, \bar{\kappa}^{2*})$ optimal control of program (1.106), is given by (1.47), where the value function $G(Y)$ of this program solves the nonlinear HJB equation

$$G_t + \left(\Lambda(Y) + \bar{\kappa}^{2*} - \frac{\mathcal{R} - \mathcal{R}^2}{\mathcal{R} - 1} \left(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1 \right) \right)' G_Y + \frac{1}{2} \text{trace}[\Xi \Xi' G_{YY}] + G K(t)|_{(k_1^{(2)*}, \bar{\kappa}^{2*})} = 0 \quad (1.107)$$

$G(T) = \left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\mathcal{R}-1}}} \right)^{\frac{\mathcal{R}}{\alpha}}$, where $K(t)|_{(k_1^{(2)*}, \bar{\kappa}^{2*})}$ denotes $K(t)$ evaluated at the optimal control (1.47).

To see why (1.48) gives the equilibrium Kuhn-Tucker multiplier of the first agent, notice that $\bar{\lambda}^1$ appears in the HJB solved by $G(Y)$ in the form $(\bar{\lambda}^1 - r \bar{1}_k + \bar{\kappa}^1)$, a quantity considered as exogenously

given by agent 2. Therefore if we substitute (1.105) into (1.26) and keep in mind that G is a function of $(\bar{\lambda}^1 - r\bar{1}_k + \bar{\kappa}^1)$, we realize that (1.26) provides an equation to be solved for this quantity, from which (1.48) then follows.

The equilibrium interest rate, excess return, equity volatility process and stock price process are all obtained by means of the appropriate substitutions into the general formulas provided in Proposition (2).

Finally, consider the equilibrium portfolio policies as detailed in the Proposition and let us focus on the determination of π_1^* , since π_2^* has been obtained from the market clearing relation $\pi_2 = S(t) - \pi_1 W_1$ and the appropriate substitutions. We basically apply the line of reasoning used to recover the partial equilibrium policies, suitably adapted to the current needs. The martingale $\mathcal{U}'(t)W_1(t)$ admits a stochastic integral representation of the form

$$\mathcal{U}'(t)W_1(t) = \mathbb{E}^{\kappa^{(1)}} [\mathcal{U}'(T)W_1^*(T) | \mathcal{F}_t] = \mathbb{E}^{\kappa^{(1)}} \left[\left(\frac{\varepsilon(T)}{1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}}} \right)^{\mathcal{R}} \middle| \mathcal{F}_t \right] = \eta_1 S(0) + \int_0^t \delta(s) \cdot dw_{\kappa^1}(s)$$

which, once compared to the representation (2.60) for the same process (remind that $\mathcal{U}'(t) = \psi(\kappa^{(1)} + \lambda^1, t)$ and that $c_1 = 0$) yields the following optimal investment process

$$\pi_1(t) = \Sigma(Y)^{\prime-1} \theta_{\kappa^{(1)*} + \lambda^{1*}}(Y) + \Sigma(Y)^{\prime-1} \frac{\delta(t)}{W_1^*(t)\mathcal{U}'(t)} \quad (1.108)$$

where Σ^{-1} is the volatility matrix of the fictitiously completed opportunity set. In light of Clark-Ocone formula and the chain rule for Malliavin calculus one may write

$$\delta(t) = \mathcal{R} \mathbb{E}^{\kappa^{(1)}} \left[\frac{\varepsilon(T)^{\mathcal{R}-1}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}}} \mathcal{D}_t \varepsilon(T) \middle| \mathcal{F}_t \right] - \frac{\mathcal{R}}{\mathcal{R}-1} \mathbb{E}^{\kappa^{(1)}} \left[\frac{\gamma(T)^{\frac{2-\mathcal{R}}{\mathcal{R}-1}} \varepsilon(T)^{\mathcal{R}}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}+1}} \mathcal{D}_t \gamma(T) \middle| \mathcal{F}_t \right]$$

where the Malliavin derivative processes of γ and ε are as in (1.92). But then according to consideration made in the proof of Proposition 2 we have

$$\delta(t) = \underbrace{\mathcal{R} \sigma_{\varepsilon}(t)' \mathbb{E}^{\kappa^{(1)}} \left[\frac{\varepsilon(T)^{\mathcal{R}}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}}} \middle| \mathcal{F}_t \right]}_{W_1^*(t)\mathcal{U}'(t)} - L(t) \frac{\mathcal{R}}{\mathcal{R}-1} \mathbb{E}^{\kappa^{(1)}} \left[\frac{\gamma(T)^{\frac{1}{\mathcal{R}-1}} \varepsilon(T)^{\mathcal{R}}}{(1 + \gamma(T)^{\frac{1}{\mathcal{R}-1}})^{\mathcal{R}+1}} \middle| \mathcal{F}_t \right] + \Xi(t)'(\dots)$$

Recalling the properties of the matrix process Ξ , the form of the matrix Σ^{-1} , the equilibrium shadow market price of risk of agent 1 and the equilibrium excess return relative to $Q^{\kappa^{(1)}}$, we conclude that the optimal stock demand (1.108) becomes (1.64).

Proof of Corollary 6

Consider the equilibrium worst case model selection solved by agent 1 according to Proposition 3:

$$\inf_{k^1 \in K^1} \mathbb{E}^{\bar{\kappa}^{(1)}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\bar{\kappa}-1}}} \right)^{\bar{\mathcal{R}}} e^{\int_0^T \bar{\mathcal{R}} (\mu_\varepsilon + \sigma_\varepsilon k_1^{(1)}) dt - \int_0^T \frac{1}{2} \sigma_\varepsilon^2 (\bar{\mathcal{R}} - \bar{\mathcal{R}}^2) dt} e^{\int_0^T \bar{\mathcal{R}} \sigma_\varepsilon dw_{\kappa^{(1)}} - \int_0^T \frac{1}{2} \bar{\mathcal{R}}^2 \sigma_\varepsilon^2 dt} \right]$$

One easily checks that the second exponential inside the expectation operator is an exponential martingale that may serve as density process for an absolutely continuous change of probability measure and the conditional Bayes rule implies that we may equivalently write the program above as the following optimization problem

$$\inf_{k^1 \in K^1} \mathbb{E}^{\bar{\kappa}^{(1)*}} \left[\left(\frac{1}{1 + \hat{\gamma}^{\frac{1}{\bar{\kappa}-1}}} \right)^{\bar{\mathcal{R}}} e^{\int_0^T [\bar{\mathcal{R}} (\mu_\varepsilon + \sigma_\varepsilon k_1^{(1)}) - \frac{1}{2} \sigma_\varepsilon^2 (\bar{\mathcal{R}} - \bar{\mathcal{R}}^2)] dt} \right]$$

where the expectation is taken with respect to the new measure, under which the stochastic process $w_{\kappa_1^{(1)}}^*(t) = w_{\kappa_1^{(1)}}(t) - \bar{\mathcal{R}} \sigma_\varepsilon t$ is a standard brownian motion. But μ_ε and σ_ε , coefficients of the endowment process, are constants, therefore the problem reduces to the point-wise minimization of the exponential within the expectation operator, and quite clearly we have $k_1^{(1)*} = \text{sgn}(\sigma_\varepsilon) \sqrt{2\bar{h}_1}$.

A similar factorization holds for the control problem of agent 2, as one realizes by exploiting the exponential form of $\varepsilon(t)^{\bar{\mathcal{R}}}$ and $\gamma(t)^{\frac{\bar{\mathcal{R}}}{\bar{\kappa}-1}}$, and we are left once again with a deterministic minimization problem, the solution of which is reported in the Corollary. As of the optimal portfolio policy, it is (1.64), subject to the lognormal law that governs the state variables involved, $\varepsilon(t)$ and $\gamma(t)$.

Chapter 2

Ambiguity Aversion, Bond Pricing and the Non-Robustness of some Affine Term Structures

We develop a continuous time general equilibrium model for the term structure of interest rates where economic agents are ambiguity averse and consider the possibility of a misspecified dynamic model for the latent risk factors driving interest rates. Aversion to ambiguity is parameterized through a specific form of Knightian uncertainty. We find that even a moderate level of ‘aggregate ambiguity’ significantly affects the implied term structures in equilibrium and drives the prices of common derivative securities toward the patterns observed in fixed income markets. Indeed, equilibrium equity premia and interest rates are characterized by a different functional form and random factors otherwise unpriced in the ‘standard’ paradigm do receive a premium for ambiguity which displays a particularly rich structure in the multiple factors setting. Examples of the impact of ambiguity aversion on popular factor models of the term structure are given in detail, both in cases for which the ‘level of concern’ for ambiguity is time varying and in cases for which is time invariant. Furthermore, we analyze the ‘robustness’ property of some classes with respect to model uncertainty, that is, the functional differences between equilibrium quantities and their counterparts arising in an economy with standard Savage-type preferences.

2.1 Introduction

This paper studies the influence of ambiguity aversion on the term structure of interest rates in a continuous-time general equilibrium economy. Ambiguity is the uncertainty deriving from an unprecise knowledge of the probability law that governs future realizations of economic factors. Ambiguity aversion refers to a situation in which investors dislike ambiguity about the distribution of asset returns. Although its distinction from standard risk aversion had been early pointed out¹, Ellsberg (1961) paradox and the literature inspired by this contribution deemed it relevant from a behavioral (and economical) point of view. Several recent academic papers have relied on ambiguity aversion to successfully address stylized facts considered as ‘puzzles’ according to the standard Savage expected utility modelling approach. Among these contributions we recall Uppal and Wang (2003), Epstein and Miao (2003), for the home-bias ‘puzzle’ and underdiversification, Anderson, Hansen and Sargent (2000), Chen and Epstein (2002), Maenhout (2001) and Sbuelz and Trojani (2002) for the equity premium ‘puzzle’. Dow and Werlang (1992) and Trojani and Vanini (2004) generate endogenous limited stock market participation as a consequence of agents’ optimizing behavior in the absence of market frictions, whereas Liu, Pan and Wang (2003) are able to mimic the typical ‘smirk’ shape of options’ implied volatilities.

A key observation arising in some of the mentioned literature is that ambiguity aversion influences mostly equity premia rather than assets prices. According to this intuition, the equilibrium term structure of interest rates should inherit the richest implications from agents’ concern for ambiguity. Quite surprisingly though, interest rate models under ambiguity have been largely unexplored so far and the aim of this paper is precisely to tackle this issue in detail.

In order to characterize the effects of ambiguity aversion on the term structure and derivative prices we start from the well-established general equilibrium framework of Cox Ingersoll and Ross (1985). We depart from this classical setting by treating the underlying exogenous state dynamics as an approximate description of the true data generating process, i.e. we model the reference belief of our agent as in the affine multidimensional framework outlined, for instance, in Dai and Singleton (2000). A concern for ambiguity is then induced by a max-min expected utility representation² for the relevant preference orderings where the representative agent regards as suitable for decision making purposes a worst case probabilistic description of the economic environment out of a set of relevant scenarios. We follow Anderson et al. (AHS, 1998, 2000) in the way we select the worst case scenarios by constraining their discrepancy from the approximate reference belief for asset prices. This choice translates into yield curve levels and shapes which cannot be supported in equilibrium by the non ambiguity averse counterpart of the model.

Implicit in the above discussion is the choice of a ‘constrained’ max-min expected utility representation as a convenient framework for a continuous time representation of preferences under ambiguity aversion, along the path initiated by Gilboa and Schmeidler (1989). Our representation is of the Recursive Multiple Prior Utility type, thus admitting an axiomatic foundation and implying a set of relevant likelihood which is ‘rectangular’, in the terminology introduced by Epstein and

¹Knight (1921)

²Gilboa and Schmeidler (1989), Epstein and Wang (1994), Epstein and Schneider (2003), Anderson, Hansen and Sargent (2003)

Schneider (2003).

As an additional step, we provide insight into those forms of concern for ambiguity, or ‘pessimism’, which preserve the functional form of the transition density of the state variable: in accordance with our intuition about the degree of ‘pessimism’ in the economy as an indicator of confidence in the reference model, the maximal discrepancy allowed between reference model and possible scenarios will parameterize such different, time invariant or time varying, forms of concerns. We will show that appropriate time varying specifications of this maximal discrepancy provide agents with selections of worst case transition densities that are consistent with those under the reference probability. Closed form characterizations of the impact of ambiguity aversion in these frameworks will emphasize that such a form of ‘robustness’ delivers equilibrium interest rates and equity premia that cannot be mimicked by the non ambiguity averse counterpart of the model.

We show that an ambiguity premium is responsible for this different behavior of key equilibrium quantities yet at small levels of concerns for ambiguity. Otherwise unpriced factors in the standard model receive a premium for ambiguity which is of a particularly rich structure in the multiple factors setting. All these features induce in equilibrium term structure levels and shapes that are very different from those arising in the ‘standard’ model. For instance, in a simple one factor model with square root dynamics we observe that for realistic parameter choices the levels of the yields to maturity are substantially lowered and the curvature is affected especially at shorter horizons. Examples of interest rate derivative securities prices show that popular market indicators like Black implied volatilities point toward a direction in accordance with empirical evidence. Such effects are present both in the version of the model with time-invariant pessimism - non-robust, in the sense outlined above - and in the version with time-varying pessimism; the additional layer of technical tractability gained in the latter case allows us to discuss some key points by explicit methods. It is nonetheless apparent in both frameworks that the specific form of these ambiguity aversion induced term structure effects cannot be obtained by a suitable parametrization of their ‘classical’ counterparts.

After a preliminary characterization of the equilibrium equity premia and short rate in terms of indirect utility functions, we solve for equilibrium quantities given any model in the admissible neighborhood and then attack the worst case model selection problem. Section 2.2 presents the reference belief for our ambiguity averse agent, the investment opportunity set represented under this probability measure, defines the set of relevant possible mispecifications and introduces the max-min expected utility optimization problem that implies worst case optimal consumption and portfolio policies under model uncertainty. Section 2.3 contains preliminary computations on the functional form of equilibrium interest rates and risk premia in terms of indirect utility function and characterizes the equilibrium by martingale methods in a second step. Section 2.4 focuses on explicit computations of optimal policies, equilibrium quantities and contingent claim prices in the framework of both state-dependent and state independent maximal discrepancy with the reference model, that is, time varying and invariant pessimism, in the terminology used above. Section 2.5 contains concluding remarks. All proofs have been relegated to the Appendix A; Appendix B contains an additional example of term structure model along the lines of what we have described as ‘time invariant pessimism’, whereas Appendix C treats a solution approach for an economy whose

representative agents is characterized by a more general felicity function than the logarithmic one used to derive the main results.

2.2 Model Setting

Our reference model setting is inspired by the standard framework of Cox Ingersoll and Ross (1985). On an infinite time horizon, uncertainty is generated by a $(k + 1)$ –dimensional standard brownian motion $Z(t)$ supported by a filtered probability space (Ω, \mathcal{F}, P) .³

2.2.1 Reference Belief

Under the probability measure P , often termed ‘reference belief’, the basic constituents of the opportunity set available to agents are⁴:

- A locally risk-less bond in zero net supply, with return $r(t)$.
- 1 linear technology producing a physical good which can be either reinvested or consumed. Its output rate evolves as

$$\frac{dQ(t)}{Q(t)} = \alpha(Y)dt + \sigma(Y) dZ(t) \quad (2.1)$$

- k financial assets in zero net supply, satisfying the stochastic differential equation

$$dS(t) = I_S \beta(Y)dt + I_S \vartheta(Y) dZ(t) \quad (2.2)$$

where I_S denotes $\text{diag}[S_1, S_2, \dots, S_k]$

- k driving state variables

$$dY(t) = \Lambda(Y)dt + \Xi(Y) dZ(t) \quad (2.3)$$

The equilibrium to be characterized is supported by a single, representative agent maximizing the expected lifetime utility from intertemporal consumption, based on a time preference rate δ and a logarithmic⁵ felicity function: $U(c, t) = e^{-\rho t} \log(c)$, $c > 0$.

2.2.2 Model Misspecification

The representative agent is uncertain about the belief according to which the evolution of the opportunity set is described, and considers scenarios around the reference model generated by absolutely

³We may have started with a brownian motion $w(t)$ with instantaneous correlation matrix Ω ; then, if Υ is a $(k + 1) \times (k + 1)$ matrix such that $\Omega = \Upsilon \Upsilon'$ and $Z(t)$ is a $(k + 1)$ –dimensional standard brownian motion, the brownian motion $\Upsilon Z(t)$ is indistinguishable from $w(t)$. Hence, provided that all diffusion matrices to appear be the result of a post multiplication by the matrix Υ , this setting is encompassed by our specification.

⁴All coefficients to appear are assumed to be continuous and uniformly bounded functions of the state variables. Furthermore, we impose a uniform ellipticity conditions on the matrix function $\Xi \Xi'$

⁵In Appendix C we analyze a finite time-horizon economy populated by an ambiguity averse representative agent who maximizes a CRRA utility of terminal wealth and we characterize the equilibrium in detail. We regard the logarithmic case suitable to emphasize the main intuition concerning ambiguity aversion while preserving simplicity of the main treatment.

continuous local contaminations, as in AHS (1998, 2000). Contaminations are described by contaminating vectors h : in probabilistic terms, Girsanov kernels that affect the drift of the reference diffusion process for the state variables. Aversion to model uncertainty arises by assuming that the representative agent is concerned with the worst case scenario in a neighborhood of the reference belief defined by:

$$h' h \leq 2\eta(Y) \quad (2.4)$$

In what follows we restrict our treatment to the class of Markov Girsanov kernels $h(t) = h(t, Y)$, $t > 0$, for some function $h(\cdot, \cdot)$ and require that they satisfy

$$\mathbb{E} \left[\int_0^t |h(s, Y)| ds \right] < \infty$$

for every $t > 0$. We denote by \mathcal{H} the class of admissible mispecifications.

As pointed out in the literature on ambiguity aversion⁶ this choice for the set of probabilistic models regarded as relevant by the agents has a clear interpretation in terms of limiting the maximum ‘distance’ allowed from the reference model: it corresponds to assuming a bound on the instantaneous rate of growth of the relative entropy between the ‘contaminated’ belief and the initial one; in particular, since this specification constrains the instantaneous evolution of the relative entropy and not just its global continuation value, the model delivers time consistent preference orderings or, in the terminology of Epstein and Schneider (2002), a rectangular set of priors. Furthermore, we posit a state dependent entropy bound $\eta(Y)$ in order to allow a form of ‘pessimism’ which could be time varying and tightened to the state of the economy⁷

Since the probabilistic scenarios we consider are mutually absolutely continuous, $Z_h(t) = Z(t) + \int_0^t h(s) ds$ is a standard brownian motion under the model contamination⁸ P^h . In this model setting agents posit the data generating process to have the representation (??) under an admissible belief P^h ; since the latter implies that $Z(t)$ is a brownian motion with drift, concern for ambiguity assumes the form of a change of drift in the dynamics specified under the reference measure.

⁶Appropriate references

⁷This specification was suggested in Trojani and Sbuelz (2003). This reference contains a thorough discussion on the point.

⁸If the discontinuous part of an adapted process k is null P -a.s, as in our framework, the Dolean-Dade exponential $\mathcal{E}(\cdot)$ is defined as

$$\mathcal{E}(k) = \exp \left(-k - \frac{\langle k, k \rangle}{2} \right)$$

Then the probability measure P^h is a contamination of the reference belief in the sense that

$$P^h(\cdot) = \mathbb{E} \left[\mathcal{E} \left(- \int h dZ \right) \mathbf{1}(\cdot) \right]$$

2.2.3 Max-min Expected Utility

The representative agent trades continuously at equilibrium prices in order to finance his consumption process $c(t)$; if we denote by Σ the $(k+1) \times (k+1)$ diffusion matrix of the available opportunity set,

$$\Sigma(Y) = \begin{bmatrix} \sigma(Y) \\ \vartheta(Y) \end{bmatrix} \begin{matrix} 1 \times (k+1) \\ k \times (k+1) \end{matrix} \quad (2.5)$$

then feasibility of consumption plans under the reference belief is mandated by the usual dynamic budget constraint, coupled with the appropriate integrability conditions⁹:

$$\frac{dW(t)}{W(t)} = \left[\omega(t) (\alpha(Y) - r(Y)) + v(t) (\beta(Y) - r(Y)) + \left(r(Y) - \frac{c(t)}{W(t)} \right) \right] dt + \pi'(t) \Sigma(Y) \cdot [dZ(t) + h(t) dt] \quad (2.6)$$

$W(t) > 0$ for every $t > 0$. As a consequence of Ito's lemma¹⁰:

$$\xi_h(t)W(t) + \int_0^t \xi_h(s)c(s)ds = x + \int_0^t W(s)\xi_h(s) [\Sigma(Y)' \pi(s) - \theta_h(Y)]' \cdot dZ(s) \quad (2.7)$$

where $\pi = [\omega \quad v]'$ will hereafter denote an \mathbb{R}^{k+1} -valued vector whose components are portfolio proportion invested in the technology and the financial assets, respectively; with a slight abuse of notation, we have set:

$$\theta_h(Y) = \Sigma(Y)^{-1} \begin{pmatrix} \alpha(Y) - r(Y) \\ \beta(Y) - r(Y) \bar{1}_k \end{pmatrix} + h(t) \quad (2.8)$$

whereas the state-price density $\xi_h(t)$ is governed by the SDE

$$\frac{d\xi_h(t)}{\xi_h(t)} = -r(t)dt - \theta_h(Y)dZ(t) \quad (2.9)$$

We conclude that the the ambiguity averse representative investor solves the max-min expected utility program

$$J(x, y) = \sup_{c, \pi} \inf_{h \in \mathcal{H}} \mathbb{E} \left[\int_0^\infty e^{-\delta s} \log(c(s)) ds \right] \quad (2.10)$$

s.t. (2.6)

where $W(0) = x$, $Y(0) = y$.

⁹In particular, for a trading strategy $\pi = [\omega \quad v]'$ to be admissible, we require that

$$\int_0^t |\omega(s)(\alpha(Y) - r(Y))| + |v(s)(\beta(Y) - r(Y))| + |\pi(s)' \Sigma(s)h(s)| + |\pi(s)' \Sigma(s)|^2 ds < \infty$$

hold a.s. for every $t > 0$.

¹⁰This expression is particularly useful for the derivation of the equivalent static form of the budget constraint needed in the martingale approach to consumption-investment. Since this approach is quite redundant in the context of a representative investor with logarithmic felicity, we have decided to confine the details of this derivation to the Appendix, where martingale methodologies are implemented in the framework of a representative agent with power felicity.

In a Cox, Ingersoll and Ross economy financial securities are in zero net supply, therefore their expected returns are really to be regarded as shadow prices for the constraint to hold a null portfolio weight on those. In light of this consideration we have the following definition of equilibrium.

Definition. An *equilibrium* is a collection (c^*, h^*, r^*, β^*) of a consumption policy, a model misspecification and returns on financial assets such that the following optimality and market clearing conditions are satisfied:

- 1) The Equilibrium consumption policy and model ‘misspecification’ h^* are optimal according to the preference ordering representation

$$\inf_{h: (h', h \leq 2\eta(Y))} \mathbb{E} \left[\int_0^\infty e^{-\delta s} \log(c(s)) ds \right]$$

- 2) Optimal consumption is financed by a trading strategy according to which wealth is totally invested in the technology:

$$\pi = [\omega \quad \begin{smallmatrix} 1 \times k \\ v \end{smallmatrix}]' \equiv [1 \quad \begin{smallmatrix} 1 \times k \\ 0 \end{smallmatrix}]'$$

2.3 Equilibrium

We emphasize that $J(x, y)$ is the equilibrium solution of (2.10), that is, the solution of the consumption-investment problem of the ambiguity averse representative agent evaluated at the market-clearing values of the interest rate and returns on financial assets. Quite clearly this value function stems from a joint treatment of the model selection problem implied by the maxmin expected utility representation and of the good optimality conditions at equilibrium prices. We argue that additional insight may be gained if the optimal Girsanov kernel h^* is chosen after the equilibrium interest rate and risk premia prevailing under any admissible model have been characterized, and their functional relation with the Girsanov kernels h has been clarified. To this end, interchanging¹¹ the order of maximization and minimization in (2.10) we realize that the innermost program is a standard problem whereby the equilibrium conditions are easily handled by means of constrained portfolio choice methods¹² The following Proposition takes advantage of this intuition to characterize the worst case model selection problem involved in the determination of the optimal Girsanov kernel h^* .

Proposition 7 *The value function of the ambiguity averse representative investor is given by*

$$J(x, y) = -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + V(y) \quad (2.11)$$

¹¹See Appendix A for a formal justification of this step.

¹²From He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991) we know that, for any $h \in \mathcal{H}$

$$\sup_{c, \pi \equiv [1 \quad \begin{smallmatrix} 1 \times k \\ 0 \end{smallmatrix}]'} \mathbb{E} \left[\int_0^\infty e^{-\delta s} \log(c(s)) ds \right] = \inf_{(r, \beta) \in \mathbb{R}^{k+1}} \sup_c \mathbb{E} \left[\int_0^\infty e^{-\delta s} \log(c(s)) ds \right]$$

As in Cox Ingersoll and Ross (1985a) the current analysis exploits the tractability of a logarithmic felicity function for the representative agent, and the fact that the constraints (equilibrium conditions) always bind in this brownian filtration setting. Nevertheless Appendix ... treats the case of a representative agent characterized by CRRA utility of terminal wealth and exploits the representation above,

where

$$V(y) = \inf_{h \in \mathcal{H}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \int_0^t \alpha(Y) - \frac{1}{2} \sigma(Y) \sigma(Y)' + \sigma(Y) \cdot h(s) ds dt \right] \quad (2.12)$$

subject to¹³

$$dY(t) = [\Lambda(Y) + \Xi(Y) h(t)] dt + \Xi(Y) dZ(t)$$

The equilibrium Girsanov kernel which identifies the solution of the worst case model selection problem is given by

$$h^*(\nu, Y, t) = -\sqrt{2\eta(Y)} \frac{\Xi(Y)' V_Y + \sigma(Y)'}{\sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')}} \quad (2.13)$$

$$\begin{aligned} & V_Y' \Lambda(Y) + \frac{1}{2} \text{trace} [\Xi(Y)' V_{YY} \Xi(Y)] - \\ & \sqrt{2\eta(Y)} \sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')} + \alpha(Y) - \frac{1}{2} \sigma(Y) \sigma(Y)' - \delta V(0, Y) = 0 \end{aligned} \quad (2.14)$$

The equilibrium interest rate process, expected returns on financial assets and market price of risk prevailing in this economy follow then as an immediate Corollary of Proposition 7.

Corollary 7 *The interest rate process and the risk premia on financial assets arising in equilibrium are given by*

$$\begin{aligned} r(Y) &= \alpha(Y) - \sigma(Y) \sigma(Y)' - \sqrt{2\eta(Y)} \sigma(Y) \cdot \frac{\Xi(Y)' V_Y + \sigma(Y)'}{\sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')}} \\ \beta(Y) &= r(Y) \mathbf{1}_k + \vartheta(Y) \left(\sigma(Y)' + \sqrt{2\eta(Y)} \frac{\Xi(Y)' V_Y + \sigma(Y)'}{\sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')}} \right) \end{aligned}$$

where $V(Y)$ solves the HBJ equation (2.14). Furthermore, the following factor market price of risk also holds

$$\lambda(Y) = \sigma(Y)' + \sqrt{2\eta(Y)} \frac{\Xi(Y)' V_Y + \sigma(Y)'}{\sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')}}$$

It is clear from (2.12) that the model selection problem involved in the maxmin expected utility representation amounts to solving a control program where the ‘control’ h affects the state variables by shifting the probability measure over the space of their sample paths in an absolutely continuous fashion. The linearity of the ‘running cost’ in the Girsanov kernel highlights the first order risk aversion effect induced by ambiguity aversion.

Differently from the results obtained in the classical Cox Ingersoll and Ross framework ($\eta(Y) = 0$), an intertemporal hedging component determined by a concern for ambiguity is present in the equilibrium expression of the market price of risk. In this situation the log utility investors will indeed exercise portfolio demand for intertemporal hedging purely due to a concern for model misspecifica-

¹³We remind that $Z_h(t) = Z(t) + \int_0^t h(s) ds$ is a P^h -standard brownian motion.

tion, and the market price of risk will reflect this feature in equilibrium. Quite clearly the additional term implies a different functional form for the latter quantity; in particular, risk factors which are unpriced in the standard model may receive a risk premium for ambiguity.

In order to look at a clarifying example of the last statement we are going to look at an example which is an excerpt of the presentation given in the presentation of Gersenzee.

Contingent Claim Pricing

In light of the expression for the factor market price of risk established in Corollary 7, standard arbitrage arguments imply the following change of drift, ϕ_Y , for the dynamics of the state variables under the risk neutral reference measure Q :

$$\phi_Y = \Xi \left[\sigma' + \sqrt{2\eta(Y)} \left(\frac{\Xi' V_Y + \sigma'}{\sqrt{(\Xi' V_Y + \sigma')' (\Xi' V_Y + \sigma')}} \right) \right] \quad (2.15)$$

where $V(Y)$ solves the HBJ equation (2.14). The price of a European contingent claim with maturity T and paying off at a rate $\Psi(Y, t)$, $t \leq T$ is easily characterized. This is the content of next Proposition.

Proposition 8 *The price at time t , $F(Y, t)$ of a contingent claim with instantaneous pay-off $\Psi(Y, t)$, $t \leq T$ satisfies the partial differential equation:*

$$\frac{1}{2} \text{trace} \left(\Xi \Xi' \frac{\partial^2 F}{\partial Y \partial Y'} \right) + (\Lambda - \phi_Y)' \frac{\partial F}{\partial Y} - r F + \frac{\partial F}{\partial t} = -\Psi(Y, t) \quad (2.16)$$

with boundary condition:

$$F(Y, T) = \Psi(Y, T)$$

where r is the equilibrium short rate given in Corollary 7 and ϕ_Y is as in (2.15).

The fundamental pricing equation is altered only indirectly by a concern for ambiguity, via the modified equilibrium interest rate r and the corresponding change of drift ϕ_Y . Therefore, the Feymann-Kac theorem gives the usual probabilistic representation of the price

$$F(Y, t) = E_Q \left[\int_t^T e^{-\int_t^s r(u) du} \Psi(Y, s) ds + \Psi(Y(T), T) \exp \left(- \int_t^T r_u du \right) \middle| \mathcal{F}_t \right] \quad (2.17)$$

Unlike the standard (i.e. non ambiguity averse) case¹⁴ in order to compute the expectation involved in the pricing representation we need to determine the functional form of the value function $V(Y)$, solution of (2.14). In this respect, if a concern for ambiguity is present the equilibrium perspective

¹⁴Notice that this property that characterizes the non ambiguity averse economy is due to the myopic portfolio behavior of the representative agent endowed with logarithmic felicity. The non ambiguity averse representative agent with CRRA utility over terminal wealth would support an equilibrium in which the intertemporal hedging property affects risk premia. See Appendix ...

cannot be separated from the pricing perspective yet in the simple case of a logarithmic felicity function.

2.4 Explicit Solutions

In the last section we have pointed out how the task of characterizing the equilibrium in an ‘ambiguity averse’ Cox Ingersoll and Ross economy, with a logarithmic form of the felicity function, amounts to solving a dynamic program in which the form of the ‘running cost’, affine in the contaminating parameter, highlights the first order risk aversion effects peculiar to this framework. In this section we explore in details those cases for which a suitable choice of the technology’s return process and state variables’ dynamics leads to tractable quantities in the solution approaches outlined previously. We are mainly interested in clarifying whether ambiguity aversion, as summarized by the worst-case model selection feature of the agents, preserves the structure that the model would have had if concern for unprecise data generating processes had not been present: by ‘structure of the model’ we really mean the family the transition density of the state variables belongs to, both under the physical and under risk-neutral measure. We label ‘non-robust’ those specification for which such closeness property under ambiguity aversion is not present in equilibrium.

At this stage a relevant choice to be made consists in a suitable functional form for the time varying entropy bound $\eta(Y)$. We first analyze the simple, but not so tractable choice $\eta = \text{constant}$ and we give explicit solutions essentially for the complete market case, with the remarkable exception of the two-factor Longstaff and Schwartz (1992) model in which the factor driving the instantaneous variance of the technology is subject to unprecise probabilistic description: the contingent claim pricing problems arising in this frameworks will be confined to the context of coefficient choices pertaining the affine specification analyzed, for instance, in Duffie and Kan (1996) and Dai and Singleton (2000), but an alternative, practically relevant specification is contained in the Appendix. Once we relax this assumption on $\eta(Y)$, we gain additional layers of tractability through suitable choices of ‘state dependent pessimism’: interesting conclusions about important pricing problems are drawn in those cases as well.

We have opted for the logical distinction time-varying/invariant pessimism because in a sense it mimics the distinction concerning the robustness of models: no ambiguity averse framework with constant parameter η may induce ‘robustness’ of the corresponding model.

Where no confusion may arise, we will often drop functional arguments in the expressions to follow for ease of exposition.

2.4.1 A Two-Factor Gaussian Model

We first analyze the influence of a concern for ambiguity in a simple modelling framework where the expected return on the production technology is an affine function of the state variables and these in turn evolve as Ornstein-Uhlenbeck stochastic processes. In particular, we consider the following

two factor Gaussian dynamics:

$$\begin{aligned}
\frac{dQ(t)}{Q(t)} &= (g_0 + g_1 Y_1(t) + g_2 Y_2(t) + Lh(t))dt + L dZ(t) \\
dY_1(t) &= [m_1(\bar{Y}_1 - Y_1(t)) + n_1 h_1(t) + qh_2(t)] dt + n_1 dZ_1(t) + q dZ_2(t) \\
dY_2(t) &= [m_2(\bar{Y}_2 - Y_2(t)) + n_2 h_1(t)] dt + n_2 dZ_1(t)
\end{aligned} \tag{2.18}$$

where $L \equiv [L_1 \ L_2 \ L_3] \in \mathbb{R}^3$ and $Z \equiv [Z_1 \ Z_2 \ Z_3]'$ is a three dimensional standard brownian motion. Very different implications arise when different choices of the ‘aggregate concern’ for ambiguity are made, that is, when the entropy bounds assumes different functional forms. In this Gaussian setting, a time varying bound will be seen to deliver interesting effects while preserving analytical tractability.

Constant entropy bound

The class of admissible likelihoods \mathcal{H} is identified by the entropy bound

$$h_1^2 + h_2^2 + h_3^2 \leq 2\eta \tag{2.19}$$

In Appendix B we briefly show that the effect of ambiguity aversion on the short rate reduces to a constant term and the drift correction (2.15) to be applied under the risk neutral reference measure is just a constant bivariate vector. One easily ends up with the following Proposition:

Proposition 9 *Let the class of admissible likelihoods \mathcal{H} be determined by the entropy bound (2.19), then the price of a zero coupon bond with maturity T under the model dynamics (2.18) is given by*

$$P(t, T) = \exp(A(t, T) + B(t, T)y_1 + C(t, T)y_2) \tag{2.20}$$

where $Y_1(t) = y_1$, $Y(t) = y_2$,

$$\begin{aligned}
B(t, T) &= \frac{(e^{-(T-t)m_1} - 1) g_1}{m_1} \\
C(t, T) &= \frac{(e^{-(T-t)m_2} - 1) g_2}{m_2}
\end{aligned}$$

and $A(t, T)$ is reported in Appendix A.

The difference between this model and its non ambiguity-averse counterpart is limited to the coefficient $A(t, T)$: although this leads in general to nontrivial effects on the behavior of both the yield curve and the slope of the yield curve, its state independence prevents this impact from affecting important indicators such as, for instance, the volatility structure of instantaneous forward rates, whose humped structure has been acknowledged as an important prerequisite for a model to fit the observed humped structure of implied Black volatilities of derivatives’ prices. In this respect, a time varying specification for the instantaneous entropy bound may help the ambiguity averse version of the model to achieve more ambitious goals.

Time-varying entropy bound

Consider a class of admissible likelihoods \mathcal{H} identified by the following entropy bound

$$h_1^2 + h_2^2 + h_3^2 \leq 2\eta (Y_1 - \bar{Y}_1)^2 \quad (2.21)$$

Intuitively, such a form of the function $\eta(\cdot)$ penalizes large (in absolute value) deviations of the conditional expected return on the technology from its parameter of mean reversion by postulating a concern for ambiguity which increases (at increasing rates) with the magnitude of this distance. The impact of ambiguity aversion is indeed much more pronounced in this case:

Proposition 10 *The price of a zero coupon bond with maturity T is*

$$P(t, T) = \exp (A(t, T) + B(t, T)y_1 + C(t, T)y_2 + D(t, T)|y_1|)$$

where $Y_1(t) = y_1$, $Y_2(t) = y_2$.

The absolute volatility of implied instantaneous forward rates is

$$\sigma_f(t, T) = \left(\left[n_1 \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right) + n_2 \frac{\partial B(t, T)}{\partial T} \right]^2 + q^2 \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right)^2 \right)^{\frac{1}{2}}$$

The functions $A(t, T)$, $B(t, T)$, $C(t, T)$ and $D(t, T)$, as well as their partial derivatives involved in the last expression are reported in Appendix B.

The functional forms of the coefficients driving the yield to maturity function are indeed different from a model in which no ambiguity aversion is considered, and, ceteris paribus, this difference may well enhance the ability of the model to recover observed shapes of popular (derivative) markets indicators. An humped volatility structure of instantaneous forward rates, for instance, is regarded as a desirable property for models aimed at a good derivative pricing performance: Figure 2.2 shows a volatility curve generated when a ‘small’ ($\eta = 0.005$) concern for ambiguity is present, to be compared with its counterpart generated by the ‘classical’ version of this Gaussian model (Figure 2.1).

2.4.2 Constant degree of pessimism

A constant value for the instantaneous entropy bound η induces a poor level of analytical tractability for the model selection problem (2.12), an exception being the Gaussian specification just analyzed; in this case we have seen that both time-varying and time-invariant pessimism specifications do generate important effects, nevertheless preserve the functional forms of equilibrium short rate and risk premia prevailing in the non ambiguity averse case. However, this ‘robustness’ feature is rather the exception than the rule. We thereafter discuss the equilibrium and pricing implications of two

examples a one factor square root model and a two factor square root model, both of which imply a concern for ambiguity that is essentially one dimensional. In the two factor example, inspired by Longstaff and Schwartz (1992), this concern is confined to the volatility of technological returns.

A one factor affine model

In what follows we analyze a one factor complete market setting¹⁵.

$$\alpha(Y) = g_0 + g_1 Y; \quad \sigma(Y) = l\sqrt{Y}; \quad \Lambda(Y) = \frac{n^2}{4} + mY; \quad \Xi(Y) = n\sqrt{Y} \quad (2.22)$$

The solution of our ‘model selection’ problem is easily characterized in this context.

Proposition 11 *The Girsanov Kernel $h^* = -\sqrt{2\eta}$ is optimal for the specification (2.22). The Value function for this problem is:*

$$V(Y) = A + B Y + C \sqrt{Y}$$

with

$$\begin{aligned} A &= \frac{1}{\delta} \left[\frac{n^2 \left(g_1 - \frac{l^2}{2} \right)}{4(m - \delta)} - \frac{\frac{n}{2} - \delta}{\frac{m}{2} - \delta} \left(\frac{-n \left(g_1 - \frac{l^2}{2} \right) \sqrt{2\eta}}{m - \delta} - l\sqrt{2\eta} \right) \frac{C}{2} + g_0 \right] \\ B &= \frac{1}{m - \delta} \left(g_1 - \frac{l^2}{2} \right) \\ C &= \frac{1}{\frac{m}{2} - \delta} \left[\frac{-n \left(g_1 - \frac{l^2}{2} \right) \sqrt{2\eta}}{m - \delta} - l\sqrt{2\eta} \right] \end{aligned}$$

Let us now assume without loss of generality that $g_0 = 0$ and $l > 0$; the additional assumption (corresponding to a similar one adopted in CIR (1985b)) $Y \geq 2\eta l^2 (g_1 - l^2)^{-2}$, coupled with the nonnegativity of the state variable Y under the optimal belief P^{h^*} will ensure that the short rate process

$$r(t) = (g_1 - l^2) Y - l\sqrt{2\eta}\sqrt{Y} \quad (2.23)$$

stays positive P^{h^*} - a.s.. The inversion of equation (2.23) leads to a second order equation in \sqrt{Y} , one of whose solutions is negative due to our previous assumptions on coefficients. If we define a new short rate process $\tilde{r} \equiv 2\eta l^2 + 4(g_1 - l^2)r$ then the equilibrium market price of risk $\lambda(t) = \sigma(Y)' - h^*$ reduces to

$$\lambda(t) = l\sqrt{Y} + \sqrt{2\eta} = \frac{l^2 \sqrt{2\eta} + l\sqrt{\tilde{r}}}{2(g_1 - l^2)} + \sqrt{2\eta}$$

¹⁵It is without loss of generality that we have assumed $u_0 = 0$, $u = 1$ and $r_0 = 0$, $r = 1$. The result would be substantially unaffected by the general case, but for a few additional terms in the expressions to follow.

and it is easy to deduce the following risk neutral dynamics for $\tilde{r}(t)$:

$$d\tilde{r}(t) = \left(a + b\tilde{r} - c\sqrt{\tilde{r}}\right) dt + \left(d + n\sqrt{\tilde{r}}\right) dZ^*(t) \quad (2.24)$$

where $Z^*(t) = Z(t) + \int_0^t \lambda(s) ds$ and the coefficients are reported in Appendix A. Using the dynamics of the state variable and standard arbitrage arguments we may write the fundamental valuation equation for the discount bond $P(\tilde{r}, \tau)$ as

$$\frac{\left(d + n\sqrt{\tilde{r}}\right)^2}{2} P_{\tilde{r}\tilde{r}}(\tilde{r}, \tau) + \left(a + b\tilde{r} - c\sqrt{\tilde{r}}\right) P_{\tilde{r}}(\tilde{r}, \tau) - \tilde{r} P(\tilde{r}, \tau) - P_{\tau}(\tilde{r}, \tau) = 0$$

to be solved with the initial condition $P(\tilde{r}, 0) = 1$.

It should be emphasized that this formulation of the pricing problem cannot be mimicked by any parametrization of the Cox Ingersoll and Ross economy where preferences are not ambiguity averse, due to the specific form of the risk premium (i.e. proportional to \sqrt{r}) that insures absence of arbitrage opportunity. We summarize a characterization of the term structure of interest rates in the following proposition.

Proposition 12 *Setting $r(\tau) = \bar{r}$, the price of a pure discount bond with time to maturity τ is given by the function:*

$$P(\tau, \bar{r}) = A(\tau) e^{B(\tau) \frac{\eta l^2 + (g_1 - l^2)r + \frac{1}{2}\sqrt{2\eta}\sqrt{2\eta l^2 + 4(g_1 - l^2)r}}{(g_1 - l^2)^2} + C(\tau) \frac{l\sqrt{2\eta} + \sqrt{2\eta l^2 + 4(g_1 - l^2)r}}{2(g_1 - l^2)}} \quad (2.25)$$

where¹⁶

$$\begin{aligned} A(\tau) &= \exp\left(\int_0^\tau \frac{n^2 B(t)}{4} - n\sqrt{2\eta} C(t) + \frac{n^2 C(t)^2}{8} dt\right) \\ B(\tau) &= \frac{a(1 - e^{\alpha\tau})}{2\alpha - (\alpha + d)(1 - e^{\alpha\tau})} \\ C(\tau) &= \sqrt{2\eta} \left(\frac{2a\left(n - \frac{ln^2}{d}\right)(1 - e^{\alpha\frac{\tau}{2}})^2}{\alpha(2\alpha - (\alpha + d)(1 - e^{\alpha\tau}))} \right) + \frac{2l}{d} \left(1 - \left(\frac{1}{d - \alpha + e^{\alpha\tau}(d + \alpha)} \right)^{\frac{d}{\alpha(d^2 - a^2)}} \times \right. \\ &\quad \left. (2d)^{\frac{d}{\alpha(d^2 - a^2)}} e^{\frac{d}{2}\tau - \frac{an^2}{2(d - \alpha)}\tau} \right) \end{aligned}$$

with $\alpha = \sqrt{d^2 + an^2}$, $d = m - nl$ and $a = 2(g_1 - l^2)$

The phase-plane analysis¹⁷ shows that under the mild additional assumptions (indeed satisfied by any ‘reasonable’ parametrization) $d < 0$ and $\frac{2(d+\alpha)}{n} + l > 0$ we have $C(\tau) > 0$. In light of this result it is easy to obtain an estimate of the impact of ambiguity on the equilibrium yield curve, at least

¹⁶Notice that the integration involved in the functional form of the coefficient A can be carried out explicitly. We do not report the (lengthy) expression.

¹⁷We do not include the phase-plane analysis. It is available upon request.

for small values of the instantaneous entropy bound;

$$\left. \frac{\partial}{\partial \sqrt{2\eta}} \left(-\frac{\log P(t, t + \tau)}{\tau} \right) \right|_{\eta=0} = -C(\tau) < 0$$

which leads to the conclusion that the equilibrium yield curves obtained when a concern for ambiguity is present are dominated by their classical counterparts in this simple one factor specification.

Our ability to solve both the Hamilton-Jacobi-Bellman equation and the pricing equation depends on the restriction we have imposed on the instantaneous conditional mean of the state variable, namely:

$$\Gamma(Y) = m_0 + mY = \frac{n^2}{4} + mY$$

If we posit

$$d(\sqrt{Y}) = \frac{1}{2} \left(m\sqrt{Y} - nl\sqrt{2\eta} \right) dt + \frac{n}{2} dZ \quad (2.26)$$

then Ito's lemma yields exactly the risk neutral dynamics we have ended up with by means of ambiguity aversion:

$$dY = \left(\frac{n^2}{4} + mY - nl\sqrt{2\eta}\sqrt{Y} \right) dt + n\sqrt{Y}dZ^* \quad (2.27)$$

Therefore it is not surprising that the change of variable $X = \sqrt{Y}$ leads us to the quadratic class of term structure models studied in the literature so far.

In order to gain insight into the solution corresponding to a general parameter choice, we pursue a numerical approach, whose conclusions indeed suggests that the qualitative evidence the previous analytics were pointing to is by no means restricted to that particular specification.¹⁸ A typical sample path of the short rate process (2.23) appears to be an almost parallel downward shift of its counterpart corresponding to the same $\omega \in \Omega$ and generated by a model with no concern for ambiguity (2.3); ceteris paribus a slight nonzero ambiguity aversion parameter suffices to generate yields to maturity which are almost a hundred basis points lower for all maturities, the effect on the curvature being limited to the very short end of the curve and progressively fading away.

The two factor affine model

The following model setting basically consists in the ambiguity averse extension of the Longstaff and Schwartz (1992) two factor model. We assume that agents display ambiguity aversion only over the probabilistic description of the state variable that drives the volatility of the returns on the technology; since the latter satisfies a stochastic differential equation which is autonomous and driven by the single brownian motion Z_3 , we henceforth assume $h = [0 \quad 0 \quad h_3]'$. Therefore, under

¹⁸An analytical approach, based on perturbation analysis, is the object of a Note.

the reference measure P , the opportunity set of the economy is described by the system¹⁹:

$$\begin{aligned}\frac{dQ(t)}{Q(t)} &= \left(g_1 Y_1 + g_2 Y_2 + l\rho\sqrt{Y_2} h_3(t)\right) dt + l\sqrt{Y_2} \left(\sqrt{1-\rho^2} dZ_1(t) + \rho dZ_3(t)\right) \\ dY_1 &= (a + m_1 Y_1) dt + n_1 \sqrt{Y_1} dZ_2(t) \\ dY_2 &= \left(\frac{n_2^2}{4} + m_2 Y_2 + n_2 \sqrt{Y_2} h_3(t)\right) dt + n_2 \sqrt{Y_2} dZ_3(t)\end{aligned}\quad (2.28)$$

The (one-dimensional) maximization of (1.81) gives again $h_3(t) = -\sqrt{2\eta}$ and (??) becomes²⁰

$$r(t) = g_1 Y_1 + (g_2 - l^2) Y_2 - l\rho\sqrt{2\eta}\sqrt{Y_2} \quad (2.29)$$

The separability of both the equilibrium and the asset pricing problem into problems involving a single state variable leads easily to the following characterizations.

Proposition 13 *The Value function of the ‘model selection’ problem corresponding to the specification (2.28) is given by*

$$V(Y_1, Y_2) = A + B Y_2 + C \sqrt{Y_2} + E Y_1$$

where the constant coefficients involved are reported in Appendix A. Furthermore the price of a pure discount bond with time to maturity $\tau = T - t$ is given by

$$\begin{aligned}P(\tau, Y_1, Y_2) &= A(\tau) \exp \left(D(\tau) Y_1 + C(\tau) \sqrt{Y_2} + B(\tau) Y_2 \right) \\ A(\tau) &= e^{\int_0^\tau \left(\frac{n_2^2 B(s)}{4} - \frac{n_2}{2} \sqrt{2\eta} C(s) + \frac{n_2^2 C(s)^2}{8} + a D(s) \right) ds} \\ B(\tau) &= \frac{2(g_2 - l^2)(1 - e^{\varphi\tau})}{2\varphi - (\varphi + \varrho)(1 - e^{\varphi\tau})} \\ C(\tau) &= \sqrt{2\eta} \left(\frac{2(g_2 - l^2) \left(n_2 - \frac{l\rho n_2^2}{\varrho} \right) (1 - e^{\varphi\frac{\tau}{2}})^2}{\varphi(2\varphi - (\varphi + \varrho)(1 - e^{\varphi\tau}))} \right) + \\ &\quad \frac{2l\rho}{\varrho} \left(1 - \left(\frac{1}{\varrho - \varphi + e^{\varphi\tau}(\varrho + \varphi)} \right)^{\frac{\varrho}{\varphi((\varrho)^2 - 4(g_2 - l^2)^2)}} \right) \times \\ &\quad (2\varrho)^{\frac{\varrho}{\varphi((\varrho)^2 - 4(g_2 - l^2)^2)}} e^{\frac{\varrho}{2}\tau - \frac{2(g_2 - l^2)f^2}{2(\varrho - \varphi)}\tau} \\ D(\tau) &= \frac{2g_1(1 - e^{\phi\tau})}{2\phi - (\phi + m_1)(1 - e^{\phi\tau})}\end{aligned}$$

where $\phi = \sqrt{2g_1 n_1^2 - m_1^2}$, $\varphi = -\sqrt{2m_2 n_2 \rho \sigma - m_2^2 + n_2^2(2g_2 - (2 + g_1)l^2)}$ and $\varrho = n_2 \rho l - m_2$

Similarly to the previous case, the impact of ambiguity aversion on the equilibrium yield curve is driven by the coefficient $C(\tau)$ for ‘moderate amounts’ of concern. In this respect the phase-plane

¹⁹In L & S (1992) the brownian motions driving the return on the technology and the state variable Y_2 are assumed to be correlated. We have denoted by ρ this instantaneous correlation

²⁰We assume $\alpha > 0$, $\sigma > 0$, $\beta - \sigma^2 > 0$ and $(\beta - \sigma^2)f^2 > \alpha e^2$.

analysis reveals that different signs of the correlation parameter lead to opposite implications: in particular, given the reasonable additional assumption $\varrho < 0$, a negative instantaneous correlation between the production technology and its volatility factor gives rise to uniformly higher yields to maturities when ambiguity is considered, whereas $\rho > 0$ and the mild assumption $\frac{2(\varrho + \varphi)}{f} + \sigma\rho > 0$ insure uniformly lower yields to maturity prevailing in ambiguity averse equilibria.

Similarly to the one dimensional case, the solutions of both the HBJ equation and the bond pricing equation depend on the particular choice $b = \frac{f^2}{4}$. Numerical experiments performed for the general show that this two factor model is able to generate a wider set of scenarios compared to the one dimensional counterpart. Again, in accordance with economic intuition and previous analytical findings, a different sign of the coefficient of correlation between innovations of the technology and of its volatility implies an opposite impact of ambiguity aversion both on the yield curve and on its derivatives. Figure 2.8 shows that for two opposite values of the correlation coefficient ($\rho = 0.5$ and $\rho = -0.5$) the mutual relationship of typical sample paths of the short rate corresponding to different η s looks quite different; the yield to maturities generated by these two different specifications inherit the same feature.

2.4.3 Options on zero coupon bonds

The aim of this subsection is to address the problem of pricing a (say call) option on a zero coupon bond within the model settings outlined above. In light of the inability to handle analytically every possible choice of coefficients, we pursue once again the strategy of solving a single specification ($m_0 = \frac{n^2}{4}$) and rely on numerical analysis otherwise. As of the first purpose, we restrict ourself to the one factor affine specification, being the analysis of the two remaining cases similar in spirit.

Proposition 14 *Let $X = \sqrt{Y}$ and let $C(t, X, T, s, \bar{K})$ denote the price of a European call option with strike \bar{K} , maturity T on a zero coupon bond with maturity s . Let $\mathcal{M} \equiv \{x \in \mathbb{R} : (X \leq X_l) \cup (X \geq X_u)\}$, with*

$$X_l = \frac{-C(s-T) - \sqrt{C(s-T)^2 + 4B(s-T) \log(\frac{K}{A(s-T)})}}{2B(s-T)}$$

$$X_u = \frac{-C(s-T) + \sqrt{C(s-T)^2 + 4B(s-T) \log(\frac{K}{A(s-T)})}}{2B(s-T)}$$

and

$$M(t, T) = X(t)e^{-\int_t^T \frac{m-n^2B(T-u)}{2} du} - \int_t^T e^{-\int_t^s \frac{m-n^2B(T-u)}{2} du} \left(\frac{nl\sqrt{2\eta}}{2} + \frac{n^2C(T-s)}{4} \right) ds$$

$$S(t, T) = \frac{n^2}{4} \int_t^T e^{-2\int_t^s \frac{m-n^2B(T-u)}{2} du} ds$$

$$\widetilde{M}(t, T) = X(t)e^{-\int_t^T \frac{m-n^2B(s-u)}{2} du} - \int_t^T e^{-\int_t^\tau \frac{m-n^2B(s-u)}{2} du} \left(\frac{nl\sqrt{2\eta}}{2} + \frac{n^2C(s-\tau)}{4} \right) d\tau$$

$$\widetilde{S}(t, T) = \frac{n^2}{4} \int_t^T e^{-2\int_t^\tau \frac{m-n^2B(s-u)}{2} du} d\tau$$

Then we have

$$C(t, X, T, s, \bar{K}) = P(t, s) \left[1 - \left(\Phi \left(\frac{X_u - \widetilde{M}(t, T)}{\sqrt{\widetilde{S}(t, T)}} \right) - \Phi \left(\frac{X_l - \widetilde{M}(t, T)}{\sqrt{\widetilde{S}(t, T)}} \right) \right) \right] - \bar{K} P(t, T) \left[1 - \left(\Phi \left(\frac{X_u - M(t, T)}{\sqrt{S(t, T)}} \right) - \Phi \left(\frac{X_l - M(t, T)}{\sqrt{S(t, T)}} \right) \right) \right]$$

where $\Phi(\cdot)$ is the cumulative standard Normal distribution function.

Numerical experiments allow us to overcome the restrictive assumption $m_0 = \frac{n^2}{4}$: in Figure (2.5) prices of a typical call option on a zero coupon are plotted against a wide range of values of the ambiguity aversion parameter: as expected prices are monotonically increasing in the amount of ‘ambiguity’ present in the economy. In order to analyze these effect on widely accepted market indicators, we compared the shapes of Black’s implied volatility curves for a call option on a zero coupon bond evaluated for different levels of the parameter η . We remind that Black’s implied volatility for such a contract is defined as solution (v) of the following equation²¹

$$\mathbb{E} \left[\frac{\xi_{h^*}(s)}{\xi_{h^*}(t)} (P(T, s) - \bar{K})^+ \middle| \mathcal{F}_t \right] = P(t, T) \left[\frac{P(t, s)}{P(t, T)} \Phi(d_1(v)) - \bar{K} \Phi(d_2(v)) \right]$$

where T is the maturity of the option, s is the maturity of the underlying and

$$d_1(v) = \frac{\log \left(\frac{P(t, s)}{P(t, T)} / \bar{K} \right) + v^2(T - t)/2}{v\sqrt{T - t}}$$

$$d_2(v) = \frac{\log \left(\frac{P(t, s)}{P(t, T)} / \bar{K} \right) - v^2(T - t)/2}{v\sqrt{T - t}}$$

Figure 2.6 shows that for the parametrization we have chosen higher degrees of model uncertainty increase the slope of the (almost linear) implied volatility curve. Evidence seems to point towards an inversion of this tendency for values of η above a certain point²²

These results should be compared to the richer pattern of Black implied caplet volatilities generated by the two factor model. We remind that Black’s implied caplet volatilities are defined as solutions of the following equation:

$$\mathbb{E} \left[\frac{\xi_{h^*}(s)}{\xi_{h^*}(t)} \tau \left[\frac{1}{\tau} \left(\frac{1}{P(T, s)} - 1 \right) - \bar{K} \right]^+ \middle| \mathcal{F}_t \right] = P(t, s) \tau \left[\frac{1}{\tau} \left(\frac{P(t, T)}{P(T, s)} - 1 \right) \Phi(d_1(v)) - \bar{K} \Phi(d_2(v)) \right]$$

²¹We remind that in Black’s model of bonds option pricing, bonds are assumed to follow a lognormal martingale under the forward risk neutral measure corresponding to the bond maturing at expiry. Similarly, in Black’s model of caplet pricing simple forward rates are lognormal martingales under this forward measure.

²²we will have more to say about this in the sequel.

where Φ is the cumulative standard normal distribution function, T is the maturity of the caplet, $\tau = s - T$ its tenor and

$$\begin{aligned} d_1(v) &= \frac{\log\left(\frac{1}{\tau}\left(\frac{P(t,T)}{P(T,s)} - 1\right)/\bar{K}\right) + v^2(T-t)/2}{v\sqrt{T-t}} \\ d_1(v) &= \frac{\log\left(\frac{1}{\tau}\left(\frac{P(t,T)}{P(T,s)} - 1\right)/\bar{K}\right) - v^2(T-t)/2}{v\sqrt{T-t}} \end{aligned}$$

Figure 2.9 anticipates that a suitable parametrization (namely, a negative coefficient of correlation between technological shocks and shocks on the volatility driving factor) leads to a negative sensitivity of (zero) call option prices with respect to model uncertainty. As a consequence, this two factor specification is able to reproduce the typical ‘smirk’ pattern of implied caplet volatilities observed on the market. Interestingly enough, the inability of the classical specification to generate the observed implied volatility pattern, evidenced in Figure 2.9, is progressively corrected by higher values of the ambiguity aversion parameter η .

As already pointed out Figure 2.5 shows that the one factor model generates prices European call options on zeros that are increasing in the parameter η , as expected after the previous discussion on yields to maturity; according to this consideration, the pattern highlighted in Figure 2.7 for Black implied caplet volatilities shouldn’t be surprising; moreover, we notice that the typical ‘smirk’ shape generated by the model implies that an higher concern for model uncertainty induces an increased leptokurticity and negative skewness on the risk neutral transition density of forward rates. A similar property arising in a framework of time varying pessimism for the short rate will be discussed in the sequel by analytic methods.

2.4.4 State dependent pessimism: robustness to ambiguity of some classes

So far we have been analyzing in detail the complete-markets, one factor specification of our ambiguity averse setting nested in the affine class and indeed we have been exploiting the analytical tractability arising in this case. In the attempt to extend our treatment to more general specifications, and in order to gain an additional layer of tractability from the involved problem arising in the general case, we will be imposing a state dependent entropy constraint, or, in a separate development, we will performing an initial change of reference measure under which the solution can be fully characterized. Time varying-pessimism is motivated by the intuition that the level of concern for ambiguity which affects decision makers may be linked to the state of the economy.

We discuss the contingent claim pricing problem in the context of single-factor affine dynamics. Appendix A contains details about the derivation of equilibrium model misspecifications and about multidimensional extensions.

The models that we are about to discuss allow for a fairly easy derivation of the optimal Girsanov kernels h^* ; as pointed out previously, these equilibrium solutions constitute the starting point in the study of the impact of ambiguity aversion on the contingent claim valuation problem. The specifications we analyze lead us to familiar pricing methodologies which allow for a direct parametric assess-

ment of this influence. We give details for three single factor specifications, the multi-dimensional extensions treated in Appendix A being qualitatively similar:

Affine coefficients

We consider a one factor affine specification with state dependent entropy constraint of the form $h' \cdot h \leq \frac{2\eta}{Y} Y$:

$$\begin{aligned} dQ &= g_1 Y dt + l \rho dZ_1^h + l \sqrt{1 - \rho^2} dZ_2^h \\ dY &= m_1 (\bar{Y} - Y) dt + n \sqrt{Y} dZ_1^h \end{aligned}$$

Let $m_0 = m_1 \bar{Y}$, then this specifications admits as optimal Girsanov kernels the processes

$$\begin{aligned} h_1^* &= \sqrt{\frac{2\eta}{\bar{Y}}} Y \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} \\ h_2^* &= \sqrt{\frac{2\eta}{\bar{Y}}} Y \frac{l \sqrt{1 - \rho^2}}{k} \end{aligned} \quad (2.30)$$

with k solution of (2.51) in Appendix A. Therefore the equilibrium market price of risk arising in this ambiguity averse preference setting is proportional to the square root of the state variable Y , whereas the equilibrium short rate is an affine function of the latter:

$$\begin{aligned} \theta_{h^*} &= \sigma'(Y) + h^* = \sqrt{Y} \left[l \rho + \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} \right. \\ &\quad \left. l \sqrt{1 - \rho^2} \left(1 + \frac{\sqrt{\frac{2\eta}{\bar{Y}}}}{k} \right) \right] \\ r^* &= \left(g_1 - l^2 - l \rho \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} - \right. \\ &\quad \left. \sqrt{\frac{2\eta}{\bar{Y}}} \frac{l^2 (1 - \rho^2)}{k} \right) Y \end{aligned} \quad (2.31)$$

In light of the familiar functional forms of these expressions it is not surprising therefore that closed form solution can be obtained for several contingent claim pricing problems.

Geometric Ornstein-Uhlenbeck dynamics for the state variables

If both drift and diffusion components of the technological returns are linear in the state variable ($g_1 Y$ and $l Y$) respectively, and the latter evolves as a geometric Ornstein-Uhlenbeck

$$dY = (m_0 + m_1 Y) dt + n Y dw$$

where $w(t)$ has instantaneous correlation ρ with the brownian component of the technology. In this case a similar analysis holds, provided we impose the following state dependent instantaneous

entropy constraint:

$$h' \cdot h \leq 2\eta Y^2$$

The optimal Girsanov kernels are entropy constraint are

$$\begin{aligned} h_1^* &= Y\sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} \\ h_2^* &= Y\sqrt{2\eta} \frac{l\sqrt{1-\rho^2}}{\lambda b(\lambda, k) - k} \end{aligned}$$

where

$$b(\lambda, k) = \frac{\left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1-\rho^2) + \lambda^2(-1+2\rho^2)) l^2} \right)}{\lambda^2 - n^2}$$

and (λ, k) solves the system of equations (2.52) in Appendix A. It turns out that the equilibrium market price of risk is affine in the state variable whereas the equilibrium short rate is a quadratic function of the latter:

$$\begin{aligned} \theta_{h^*} &= \sigma'(Y) + h^* = Y \begin{bmatrix} l\rho + \sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} \\ l\sqrt{1-\rho^2} + \sqrt{2\eta} \frac{l\sqrt{1-\rho^2}}{\lambda b(\lambda, k) - k} \end{bmatrix} \\ r^* &= g_1 Y - \left(\frac{l^2}{2} + l\rho\sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} + \sqrt{2\eta} \frac{l^2(1-\rho^2)}{\lambda b(\lambda, k) - k} \right) Y^2 \end{aligned}$$

The risk-neutral dynamics of the state variable do not belong to the Geometric Ornstein-Uhlenbeck case anymore, since a term appearing in the drift is proportional to the square of Y , therefore the pricing problem can be treated along the lines involved in the analysis of (2.57)

An additional example

In the previous examples we postulated suitable forms for the state dependent entropy constraints in order to gain analytical tractability of the dynamic belief selection problem. The aim of the current case is to attack the problem under the hypothesis of constant instantaneous entropy constraint:

$$h' \cdot h \leq 2\eta$$

To this end let us consider the affine specification treated in (2.4.4) and a preliminary change of reference measure

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \exp \left(-\kappa \cdot Z(t) - \frac{\kappa' \cdot \kappa}{2} t \right)$$

where $\kappa = (\kappa_1, \kappa_2)$ is a bivariate vector of constants. Girsanov theorem then easily shows that if we solve our max-min expected utility program relative to this new reference model, we obtain as

optimal misspecifications

$$\begin{aligned} h_1^* &= \sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} \\ h_2^* &= \sqrt{2\eta} \frac{l\sqrt{1-\rho^2}}{\lambda b(\lambda, k) - k} \end{aligned}$$

where $b(\lambda, k)$ is still

$$b(\lambda, k) = \frac{\lambda k - \rho l + \sqrt{k^2 - 2\lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2\rho^2)) l^2}}{\lambda^2 - 1}$$

but (λ, k) now solves the system of equations (2.54) in Appendix A. The equilibrium market price of risk and the equilibrium short rate are given by:

$$\begin{aligned} \theta_{h^*} &= \sigma'(Y) + h^* = \sqrt{Y} \begin{bmatrix} l\rho \\ l\sqrt{1-\rho^2} \end{bmatrix} + \begin{bmatrix} \sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} \\ \sqrt{2\eta} \frac{l\sqrt{1-\rho^2}}{\lambda b(\lambda, k) - k} \end{bmatrix} \\ r^* &= (g_1 - l^2) Y - \left[l\rho \left(\kappa_1 + \sqrt{2\eta} \frac{l\rho - nb(\lambda, k)}{\lambda b(\lambda, k) - k} \right) + \sqrt{2\eta} l\sqrt{1-\rho^2} \left(\kappa_2 + \frac{l\sqrt{1-\rho^2}}{\lambda b(\lambda, k) - k} \right) \right] \sqrt{Y} \end{aligned}$$

The equilibrium risk-neutral dynamics of the state variable are then qualitatively similar to its dynamics under the changed reference measure Q , since a term appearing in the drift is proportional to the square root of Y , therefore the conclusions reached for the specification (2.22) hold true.

The effects of ambiguity aversion on the term structure and options on zeros

The pricing frameworks outlined, together with a suitable generalization of the results concerning the one factor-complete markets case, allow to investigate the effects of ambiguity aversion on yield to maturities, first, and options on zeros next. As apparent from (2.31), model (??) with state-dependent entropy constraint delivers closed form solutions for both quantities of interest and the functional dependence of those from the ambiguity aversion parameter can be studied directly. As of the extension of (??), given that the pricing framework is qualitatively not dissimilar to the complete market example, but indeed quantitatively richer in predictions, closed-form solutions and explicit functional dependencies are not available but in a special case and numerical methods (or perturbation analysis) have to be called for. In what follows we concentrate on the affine framework (??) and work out solutions for the ‘ambiguous’ versions of the one factor square-root dynamics and the two factor Longstaff & Schwartz (1992) model when time-varying pessimism of the form (2.55) is involved.

CIR economy with state dependent entropy constraint

Let us remind that in this case the investment opportunity set evolves under the reference measure according to

$$\begin{aligned}\frac{dQ(t)}{Q(t)} &= g_1 Y dt + \sqrt{Y} \left[l \rho (dZ_1 + h_1 dt) + l \sqrt{1 - \rho^2} (dZ_2 + h_2 dt) \right] \\ dY(t) &= m_1 (\bar{Y} - Y) + n \sqrt{Y} (dZ_1 + h_1 dt)\end{aligned}$$

From the optimal Girsanov kernel selected (namely (2.30)) and the form assumed by the market price of risk (2.31) we deduce the risk neutral dynamics of the state variables once the optimal belief has been selected, hence the pricing equation

$$\begin{aligned}V_t + \left[m_1 (\bar{Y} - Y) - nY \left(l\rho + \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} \right) \right] V_Y + \frac{n^2 Y}{2} V_{YY} - \\ V \left(g_1 - l^2 - l\rho \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} - \sqrt{\frac{2\eta}{\bar{Y}}} \frac{l^2(1 - \rho^2)}{k} \right) Y = 0\end{aligned}$$

where k solves (2.51), the boundary conditions being $V(T, Y) = 1$ and $V(T, Y) = (P(T, s) - K)^+$ for the zero coupon bond and the call on a zero maturing in s , respectively.

Standard separation of variables allows us to obtain the following expression for the yield to maturity of a zero maturing in T :

$$y(t, T) = -\frac{\log P(t, T)}{T - t} = \frac{B(T - t) Y - A(T - t)}{T - t}$$

with

$$\begin{aligned}B(T - t) &= \frac{2b(k)}{n\bar{\theta}(k) + m_1 - \coth \left[\frac{(T-t) \sqrt{2n^2 b(k) + (n\bar{\theta}(k) + m_1)^2}}{2} \right]} \sqrt{2n^2 b(k) + (n\bar{\theta}(k) + m_1)^2} \\ A(T - t) &= \int_0^{T-t} m_1 \bar{Y} B(u) du\end{aligned}$$

where

$$\begin{aligned}b(k) &= g_1 - l^2 - l\rho \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} - \sqrt{\frac{2\eta}{\bar{Y}}} \frac{l^2(1 - \rho^2)}{k} \\ \bar{\theta}(k) &= l\rho + \sqrt{\frac{2\eta}{\bar{Y}}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k}\end{aligned}$$

and the hyperbolic cotangent may be substituted with the more familiar expression:

$$\coth x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

We remark that the expression for $A(T - t)$ is explicitly integrable and the (lengthy) result is reported

in the Appendix. If we consider the positive root of the quadratic equation (2.51) and make the assumption $(1 - 2n) > 0$ ²³, then the short rate is always lower in the ambiguity averse economy. Furthermore, in order to insure that the latter stays non negative, we need to impose $b(k) > 0$.

We are now able to describe the effect of ambiguity aversion on the term structure by computing the sensitivity of the yield to maturity to the ambiguity aversion parameter η . Let us first compare the result to the yield prevailing in an economy without model uncertainty ($\eta = 0$). It is easy to see that the ambiguity averse preference ordering delivers lower yields to maturity compared to those prevailing in the classical setting for all parameter sets satisfying the conditions depicted above.

The partial derivative of the yield to maturity with respect to η and the partial derivative of the slope of the yield curve with respect to η when the latter is null, describe the direct impact of a small perturbation of the reference model (when concern for such a case is present) both on the level and on the curvature of the yield curve.²⁴ These derivatives, though available in closed form, are involved enough not to allow an analytical investigation of their sign; this is due essentially to the nonlinear dependence of $b(k)$ and $\theta(k)$ on k and on the latter's (in turn) nonlinear dependence on η . Nevertheless the evidence reported in Figure (2.11), (2.12), (2.13), Figures (2.15) and (2.16) especially, and several exercises that have not been reported, show that for all parameter sets of interest the sensitivity of the yield on the ambiguity aversion parameter has proven negative, and decreasing up to an intermediate time to maturity, after which the the partial derivative reverts its tendency while still keeping its negative sign; whereas the sensitivity of the slope of the yield curve on the same parameter is negative for the short end of the yield curve and negative but fading away at increasing maturities. It should be noted that the correlation coefficient ρ determines a substantially different impact of model uncertainty: the closer ρ to -1 the less severe the shift of the yield curve.

In this respect, it may be instructive to notice that for reasonable parameter sets the sensitivity of the mean reversion parameter $b(k)\bar{Y}m_1$ with respect to the instantaneous correlation is null in the non ambiguity averse model and negative, decreasing with ρ otherwise (see Figure 2.14).

The global evidence seems to points toward the presence of nontrivial effects of model uncertainty on the term structure even when the reference model chosen displays the 'robustness' property with respect to ambiguity. Furthermore, as emphasized below, an alternative parametrization of the reference model may not reproduce identical effects.

In light of this evidence it is tempting to conjecture that $y(t, T)$ is indeed decreasing in η whenever the restrictions outlined above are satisfied.

The valuation of options on zero coupon bonds is similarly simplified in the current framework by the fact that the risk-neutral dynamics of the state variable retain their transition density prevailing under the objective measure. The formulation of the problem in terms of short rate dynamics is most convenient to our present purposes. Since under the optimal risk-neutral belief we observe:

$$dr(t) = [m_1\bar{r} - (m_1 + n\bar{\theta}(k))r(t)]dt + n\sqrt{b(k)}\sqrt{r(t)}dZ^*(t)$$

²³since n is an instantaneous volatility, the assumption seems mild to us

²⁴Since the entropy bound appears in the form $\sqrt{2\eta}$ in all relevant expressions, we compute derivatives with respect to this quantity in order to avoid singularities at $\eta = 0$

with $\bar{r} = b(k)\bar{Y}$

$$\begin{aligned} a(t, T) &= \left[\frac{4(m_1 + n\bar{\theta}(k))}{n^2 b(k)(1 - \exp((m_1 + n\bar{\theta}(k))(T - t)))} \right] \\ v &= \frac{4m_1 \bar{r}}{n^2 b(k)} = \frac{4m_1 \bar{Y}}{n^2} \\ d(t, T) &= a(T - t)r(t) \exp((m_1 + n\bar{\theta}(k))(T - t)) \\ c &= \sqrt{(m_1 + n\bar{\theta}(k))^2 + 2n^2 b(k)} \end{aligned}$$

then the risk neutral transition density of the short rate is

$$p^*(r(T)|r(t)) = a(t, T) p_{\chi^2}(a(t, T)r(T); v, d(t, T)|r(t))$$

with $p_{\chi^2}^2$ the non-central chi square pdf with v degrees of freedom and non centrality parameter $d(t, T)$. The effect of model uncertainty on the likelihood of the key state variable is apparent both through the the scale parameter $a(t, T)$ and the non centrality parameter $d(t, T)$: direct study of the behavior of the distribution is again possible and it suffices to observe Figure (2.17) to observe that for reasonable parameter sets increasing concern for model uncertainty induces densities which are more leptokurtic and negatively skewed.

It should be noted that for different admissible probability measures on our filtered probability space, the implied risk neutral density functions of the short rate are mutually singular (this can be deduced from the (nonlinear) dependence of the diffusion component on η); therefore the most natural formulation of the problem from an economic perspective, highlights how this ambiguity averse framework cannot be mimicked by any choice of the market price of risk within a (classic) partial equilibrium model that specifies square-root dynamics. Direct computation of the relevant risk-neutral expectation leads to the price of a call option expiring in T on a zero coupon bond maturing in $s > T$:

$$\begin{aligned} C(t, X, T, s, \bar{K}) &= P(t, s) \chi^2 \left(2r^*[\gamma + \psi + \bar{B}(s - T)]; v, \frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi + \bar{B}(s - T)} \right) - \\ &\quad \bar{K} P(t, T) \chi^2 \left(2r^*[\gamma + \psi]; v, \frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi} \right) \end{aligned}$$

with $\gamma = 2c/[n^2 b(k)(\exp(c(T - t)) - 1)]$, $\psi = ((m_1 + n\bar{\theta}(k)) + c)/n^2 b(k)$, $r^* = \log(\bar{A}(s - T)/\bar{K})/\bar{B}(s - T)$ and $\bar{A}(s - T)$, $\bar{B}(s - T)$ are the coefficient appearing in the zero coupon bond price after the change of variable $Y \rightarrow r$ has been performed in the pricing problem. An increasing ambiguity aversion parameter η displays a double effect:

- an increasing effect on the degree of moneyness of the option, as it can be argued by the previous discussion about yields to maturity, therefore a higher ‘moneyness’ rate r^* and a wider range of integration $[0, r^*]$; call options are always more expensive as a consequence of this effect;

- an effect on the likelihood of the option to expire in the money which is a priori indeterminate: the increased negative skew we have observed above may be counterbalanced by the pronounced leptokurticity of the distribution of the short rate at maturity. Figure (2.18) shows a situation in which the initial tendency towards higher option prices is contrasted by the fattening of the right tail of the distribution (of the short rate) once ambiguity aversion is further increased.

The conclusions raised in the latter point and suggested by the behavior depicted in Figure (2.18) are confirmed by an inspection of Black's implied volatilities²⁵ corresponding to those option prices (Figure 2.19). For moderate levels of ambiguity in the economy the model enhances the typical 'smirk' shape that we observe on market data by increasing the steepness of the curve for out of the money profiles; this effect is more and more pronounced for ambiguity aversion parameters increasing up to a certain threshold, after which additional layers of ambiguity do flatten out the profile of the curve.

We achieve more clarity about this point by writing down a decomposition of the sensitivity of the option price with respect to the ambiguity aversion parameter η . We exploit the smoothness of Regularized Hypergeometric functions to perform the following differentiations:

$$\frac{\partial C(t, X, T, s, \bar{K})}{\partial \eta} = \tilde{C}_1 + \tilde{C}_2 \quad (2.32)$$

where

$$\tilde{C}_1 = \frac{\partial P(t, s)}{\partial \eta} \chi_1^2 + P(t, s) p_{\chi_1^2} \frac{\partial (2r^*[\gamma + \psi + \bar{B}(s - T)])}{\partial \eta} - \bar{K} \frac{\partial P(t, T)}{\partial \eta} \chi_2^2 + P(t, T) p_{\chi_2^2} \frac{\partial (2r^*[\gamma + \psi])}{\partial \eta}$$

$$\begin{aligned} \tilde{C}_2 = \frac{1}{2} P(t, s) \left(\chi_1^2 - \widetilde{\chi}_1^2 \right) \frac{\partial}{\partial \eta} \left(\frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi + \bar{B}(s - T)} \right) - \\ \frac{1}{2} \bar{K} P(t, T) \left(\chi_2^2 - \widetilde{\chi}_2^2 \right) \frac{\partial}{\partial \eta} \left(\frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi} \right) \end{aligned}$$

$$\begin{aligned} \chi_1^2 &= \chi^2 \left(2r^*[\gamma + \psi + \bar{B}(s - T)]; v, \frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi + \bar{B}(s - T)} \right) \\ \chi_2^2 &= \chi^2 \left(2r^*[\gamma + \psi]; v, \frac{2\gamma^2 r(t) \exp(c(T - t))}{\gamma + \psi} \right) \end{aligned}$$

$p_{\chi_1^2}, p_{\chi_2^2}$ are the corresponding densities at $2r^*[\gamma + \psi + \bar{B}(s - T)]$ and $2r^*[\gamma + \psi]$, respectively, whereas $\widetilde{\chi}_1^2$ and $\widetilde{\chi}_2^2$ are identical to χ_1^2 and χ_2^2 but for one more degree of freedom.

The term \tilde{C}_1 characterizes the effect on the moneyness of the option that we have discussed previously and it can easily be seen to be positive, as a consequence of the positive derivative of zeros' prices with respect to η and of its being increasing with time to maturity. The second term, \tilde{C}_2 , quantifies the effect due to the distortion of the risk-neutral transition density of the state variable arising when

²⁵See the the corresponding exercise concerning time invariant pessimism for a reminder on Black's implied volatility.

higher concern for ambiguity is present, and it has been expressively characterized as a modification of the option price where the initial prices have been weighted by the sensitivities to η of non centrality parameters and the quantiles have been replaced by the difference with their analogous displaying one more degree of freedom. If we interpret the parameter η as a proxy of the degree of ambiguity concern present in the economy, and if we advocate a pragmatic point of view similar to those admitting the Greeks in the Black-Scholes economy, then we may argue that η , though constant, could be subject to an unprecise description and may therefore influence our pricing methodology: reasoning along these lines, (2.32) may be thought to provide a (first order) hedging methodology, whose portfolio weights in zero coupon bonds with maturity s and T , $P(t, s)$, $P(t, T)$, and a locally risk-less asset, are explicitly provided by the expression.

Two factor Longstaff and Schwartz (1992) model with time-varying pessimism

Our aim is to extend the analysis of this model to a framework in which the whole dynamics of the technological return, in excess of the factor driving its instantaneous volatility, are subject to an unprecise probabilistic description. In other words, we seek a Girsanov kernel of the form $h = [h_1 \ 0 \ h_3]'$ and represent the opportunity set under the reference belief P by the system of SDEs:

$$\begin{aligned} \frac{dQ(t)}{Q(t)} &= \left[\alpha Y_1 + b Y_2 + \sigma \sqrt{Y_2} \left(\rho h_3(t) + \sqrt{1 - \rho^2} h_1(t) \right) \right] dt + \sigma \sqrt{Y_2} \left(\sqrt{1 - \rho^2} dZ_1(t) + \rho dZ_3(t) \right) \\ dY_1 &= a(\bar{Y}_1 - Y_1) dt + e \sqrt{Y_1} dZ_2(t) \\ dY_2 &= \left(c(\bar{Y}_2 - Y_2) + f \sqrt{Y_2} h_3(t) \right) dt + f \sqrt{Y_2} dZ_3(t) \end{aligned}$$

In accordance with (2.55) we posit an instantaneous entropy constraint of the form:

$$h_1^2 + h_3^2 \leq 2 \frac{\eta}{Y_2} Y_2$$

The HBJ equation arising in the infinite time horizon likelihood selection process is then a special case of (2.56). In particular, the value function $V(Y_1, Y_2)$ satisfies

$$\begin{aligned} a(\bar{Y}_1 - Y_1)V_{Y_1} + \frac{e^2 Y_1}{2} V_{Y_1 Y_1} - Y_2 \sqrt{\frac{2\eta}{Y_2}} \sqrt{(\sigma \rho - f V_{Y_2})^2 + \sigma^2(1 - \rho^2)} + c(\bar{Y}_2 - Y_2)V_{Y_2} + \\ \frac{f^2 Y_2}{2} V_{Y_2 Y_2} - Y_2 \left(b - \frac{\sigma^2}{2} \right) - \alpha Y_1 - \beta V = 0 \end{aligned}$$

After (2.56) we realize that the solution of this problem is additively separable: $V(Y_1, Y_2) = V^1(Y_1) + V^2(Y_2)$, where $V^1(Y_1)$ and $V^2(Y_2)$ satisfy the autonomous ODEs:

$$\begin{cases} a(\bar{Y}_1 - Y_1)V_{Y_1}^1 + \frac{e^2 Y_1}{2} V_{Y_1 Y_1}^1 - \alpha Y_1 - \beta V^1 = 0 \\ Y_2 \sqrt{\frac{2\eta}{Y_2}} \sqrt{(\sigma \rho - f V_{Y_2}^2)^2 + \sigma^2(1 - \rho^2)} + c(\bar{Y}_2 - Y_2)V_{Y_2}^2 + \frac{f^2 Y_2}{2} V_{Y_2 Y_2}^2 - Y_2 \left(b - \frac{\sigma^2}{2} \right) - \beta V^2(Y_2) = 0 \end{cases}$$

The solution of the first equation admits the familiar representation in terms of hypergeometric functions; what concerns us more in light of the form of the Girsanov kernels is the fact that the solution of the second equation may easily be recovered along lines borrowed from the one-dimensional case, obtaining

$$V^2(Y_2) = Y_2 \frac{\left(\rho \sigma - \sqrt{k^2 - (-1 + \rho^2) \sigma^2}\right)}{f} + C$$

with $c_0 = c\bar{Y}$,

$$C = \frac{\left(\rho \sigma - \sqrt{k^2 - (-1 + \rho^2) \sigma^2}\right) c_0}{f\beta}$$

and k solution of the quadratic equation

$$\frac{\sigma^2}{2} - \sqrt{\frac{2\eta}{\bar{Y}}} k - \frac{\beta}{f} \left(\rho \sigma - \sqrt{k^2 - (-1 + \rho^2) \sigma^2}\right) - b - \left(\rho \sigma - \sqrt{k^2 - (-1 + \rho^2) \sigma^2}\right) \frac{c}{f} = 0$$

Not dissimilarly from the one-dimensional case

$$\begin{aligned} h_1^* &= \sqrt{\frac{2\eta Y_2}{\bar{Y}_2}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} \\ h_2^* &= \sqrt{\frac{2\eta Y_2}{\bar{Y}_2}} \frac{l\sqrt{1 - \rho^2}}{k} \end{aligned}$$

Since the risk premia are suitable adaptations of (2.31), we deduce the pricing equation

$$\begin{aligned} V_t + \left[c(\bar{Y}_2 - Y_2) - fY_2 \left(\sigma\rho + \sqrt{\frac{2\eta}{\bar{Y}_2}} \frac{\sqrt{k^2 - (-1 + \rho^2) \sigma^2}}{k} \right) \right] V_{Y_2} + a(\bar{Y}_1 - Y_1)V_{Y_1}^1 + \frac{f^2 Y_2}{2} V_{Y_2 Y_2} + \\ \frac{e^2 Y_1}{2} V_{Y_1 Y_1}^1 - V \left[\left(b - \sigma^2 - \sigma\rho \sqrt{\frac{2\eta}{\bar{Y}_2}} \frac{\sqrt{k^2 - (-1 + \rho^2) \sigma^2}}{k} - \sqrt{\frac{2\eta}{\bar{Y}_2}} \frac{\sigma^2(1 - \rho^2)}{k} \right) Y_2 + \alpha Y_1 \right] = 0 \end{aligned}$$

If we carry on the inversion procedure outlined in Longstaff and Schwartz (1992) to restate the opportunity set in terms of dynamics of the short rate and of instantaneous variance of the short rate, then the influence of ambiguity aversion is clearly summarized by the fact that the system we obtain cannot be mimicked by any alternative form of the risk premium selected in a partial equilibrium framework, due to the direct impact of the instantaneous entropy constraint on diffusion components. To see this, let

$$\mu(\eta) = b - \sigma^2 - \sigma\rho \sqrt{\frac{2\eta}{\bar{Y}_2}} \frac{\sqrt{k^2 - (-1 + \rho^2) \sigma^2}}{k} - \sqrt{\frac{2\eta}{\bar{Y}_2}} \frac{\sigma^2(1 - \rho^2)}{k}$$

then the short rate and the instantaneous variance of its changes are, respectively

$$\begin{aligned} r &= \alpha Y_1 + \mu(\eta) Y_2 \\ v &= \alpha^2 e^2 Y_1 + \mu(\eta)^2 f^2 Y_2 \end{aligned} \quad (2.33)$$

Following Longstaff and Schwartz (1992), we may solve this system of equation for (Y_1, Y_2) , apply Ito's lemma to (r, v) and substitute to obtain the risk neutral dynamics of the newly defined state variables.

$$\begin{aligned} dr &= \left(\frac{f(v - e^2 r \alpha) \theta(\eta)}{e^2 \alpha - f^2 \mu(\eta)} + c \mu(\eta) \left(\bar{Y}_2 + \frac{v - e^2 r \alpha}{\mu(\eta)(e^2 \alpha - f^2 \mu(\eta))} \right) + a \alpha \left(\bar{Y}_1 + \frac{-v + f^2 r \mu}{\alpha(e^2 \alpha - f^2 \mu(\eta))} \right) \right) dt \\ &\quad + e \alpha \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha(e^2 \alpha - f^2 \mu(\eta))} \right)} dZ_{2h}^* + f \mu(\eta) \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta)(e^2 \alpha - f^2 \mu(\eta))} \right)} dZ_{3h}^* \\ dv &= \left(\frac{f^3(v - e^2 r \alpha) \theta(\eta) \mu(\eta)}{e^2 \alpha - f^2 \mu(\eta)} + c f^2 \mu(\eta)^2 \left(\bar{Y}_2 + \frac{v - e^2 r \alpha}{\mu(\eta)(e^2 \alpha - f^2 \mu(\eta))} \right) \right) dt + \\ &\quad a e^2 \alpha^2 \left(\bar{Y}_1 + \frac{-v + f^2 r \mu(\eta)}{\alpha(e^2 \alpha - f^2 \mu(\eta))} \right) dt + e^3 \alpha^2 \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha(e^2 \alpha - f^2 \mu(\eta))} \right)} dZ_{2h}^* + \\ &\quad f^3 \mu(\eta)^2 \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta)(e^2 \alpha - f^2 \mu(\eta))} \right)} dZ_{3h}^* \end{aligned}$$

where $\theta(\eta) = \sigma \rho + \sqrt{\frac{2\eta}{Y_2}} \frac{\sqrt{k^2 - (-1 + \rho^2) \sigma^2}}{k}$

The effect of model uncertainty is captured by the (nonlinear) dependence on the parameter η of the diffusion components of both the short rate and the instantaneous variance of short rate increments: the set of absolutely continuous probability measures considered as relevant by the agents, P^{h*} (and their risk-neutral counterparts), implies a set of mutually singular densities for the equilibrium economically relevant state variables.

Although for what concerns the present application we are mainly interested in zero-coupon bond option prices, we notice that, not surprisingly ²⁶, zero-coupon bond prices assume the following form:

$$P(r, v, \tau) = A_1(\tau)^{2 \frac{a \bar{Y}_1}{e^2}} A_2(\tau)^{2 \frac{c \bar{Y}_2}{f^2}} e^{\kappa \tau + B_1(\tau) r + B_2(\tau) v}$$

²⁶see Longstaff and Schwartz (1992)

where

$$\begin{aligned}
A_1(\tau) &= \frac{2\phi}{(a + \phi)(\exp(\phi\tau) - 1) + 2\phi} \\
A_2(\tau) &= \frac{2\psi}{(c + f\theta(\eta) + \psi)(\exp(\psi\tau) - 1) + 2\psi} \\
B_1(\tau) &= \frac{\alpha\phi(\exp(\psi\tau) - 1)A_2(\tau) - \mu(\eta)\psi(\exp(\psi\tau) - 1)A_1(\tau)}{\phi\psi(\mu(\eta) - \alpha)} \\
B_2(\tau) &= \frac{\psi(\exp(\phi\tau) - 1)A_1(\tau) - \phi(\exp(\psi\tau) - 1)A_2(\tau)}{\phi\psi(\mu(\eta) - \alpha)}
\end{aligned}$$

$$\phi = \sqrt{2\alpha + a^2} \quad \psi = \sqrt{2\mu(\eta) + (c + f\theta(\eta))^2} \quad \kappa = \frac{aY_1}{e^2}(a + \phi) + \frac{cY_2}{f^2}(\psi + c + f\theta(\eta))^2$$

Let us now analyze the influence of model uncertainty on the pricing of option on zero coupon bonds in the current framework. We briefly remind that if relevant expectations are taken with respect to the T -forward martingale measure²⁷ then we may compute the price of a call option expiring in T on zero coupon bond with maturity s as

$$\begin{aligned}
C(t, Y_1, Y_2, T, s, \bar{K}) &= P(r, v, T - t) \times \\
&\quad \mathbb{E}_T \left[(A_1(s - T)^2 \frac{aY_1}{e^2} A_2(s - T)^2 \frac{cY_2}{f^2} e^{\kappa(s-T) + B_1(s-T)r(T) + B_2(s-T)v(T)} - \bar{K})^+ \middle| \mathcal{F}_t \right] \\
&= P(r, v, T - t) \times \\
&\quad \mathbb{E}_T \left[(A_1(s - T)^2 \frac{aY_1}{e^2} A_2(s - T)^2 \frac{cY_2}{f^2} e^{\kappa(s-T) + \tilde{B}_1(s-T)Y_1(T) + \tilde{B}_2(s-T)Y_2(T)} - \bar{K})^+ \middle| \mathcal{F}_t \right]
\end{aligned}$$

with $\tilde{B}_1(\cdot) = \alpha B_1(\cdot) + \alpha^2 e^2 B_2(\cdot)$ and $\tilde{B}_2(\cdot) = \mu(\eta) B_1(\cdot) + \mu(\eta)^2 f^2 B_2(\cdot)$. The transition densities of the state variables under the T -forward measure are independent noncentral chi-squares; conditional on a sample path of the process Y_1 , the ‘moneyness region’ is one dimensional and determined

²⁷We recall that the restriction to \mathcal{F}_t of the Radon-Nikodým derivative of the forward martingale measure with respect to the risk-neutral measure (where the numeraire is the locally risk-less bond) is given by

$$\begin{aligned}
\frac{dP_h^T}{dP_h^*} \bigg|_{\mathcal{F}_t} &= \frac{P(r, v, T - t)}{\exp(\int_0^t r(u) du) P(r, v, T)} \\
&= \exp \left(- \int_0^t \left(B_1(T - s) e \alpha \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha (e^2 \alpha - f^2 \mu(\eta))} \right)} + B_2(T - s) e^3 \alpha^2 \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha (e^2 \alpha - f^2 \mu(\eta))} \right)} \right) dZ_{2h}^*(s) \right. \\
&\quad \left. - \int_0^t \left(B_1(T - s) f \mu(\eta) \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta) (e^2 \alpha - f^2 \mu(\eta))} \right)} + B_2(T - s) f^3 \mu(\eta)^2 \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta) (e^2 \alpha - f^2 \mu(\eta))} \right)} \right) dZ_{3h}^*(s) \right. \\
&\quad \left. - \frac{1}{2} \int_0^t \left(B_1(T - s) e \alpha \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha (e^2 \alpha - f^2 \mu(\eta))} \right)} + B_2(T - s) e^3 \alpha^2 \sqrt{-\left(\frac{-v + f^2 r \mu(\eta)}{\alpha (e^2 \alpha - f^2 \mu(\eta))} \right)} \right)^2 ds \right. \\
&\quad \left. - \frac{1}{2} \int_0^t \left(B_1(T - s) f \mu(\eta) \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta) (e^2 \alpha - f^2 \mu(\eta))} \right)} + B_2(T - s) f^3 \mu(\eta)^2 \sqrt{-\left(\frac{v - e^2 r \alpha}{\mu(\eta) (e^2 \alpha - f^2 \mu(\eta))} \right)} \right)^2 ds \right)
\end{aligned}$$

by the inequation

$$\widetilde{B}_2(s-T)Y_2 \leq \log \left(\frac{A_1(s-T)^{2\frac{a\overline{Y}_1}}{e^2} A_2(s-T)^{2\frac{c\overline{Y}_2}}{f^2}}{\overline{K}} \right) - \widetilde{B}_1(s-T)y_1 - \kappa(s-T)$$

where y_1 is a realization of Y_1 at expiry. Since conditionally on this realization the valuation problem is not dissimilar to the one dimensional problem considered in the previous section, the following chain of equalities holds, where $\mathcal{F}_t^{Y_1(T)}$ denotes the sub-sigma field generated by the random variable $Y_1(T)$:

$$\begin{aligned} C(t, r, v, T, s, \overline{K}) &= P(r, v, T-t) \times \\ \mathbb{E}_T \left[\mathbb{E}_T \left[(A_1(s-T)^{2\frac{a\overline{Y}_1}}{e^2} A_2(s-T)^{2\frac{c\overline{Y}_2}}{f^2} e^{\kappa(s-T) + \widetilde{B}_1(s-T)Y_1(T) + \widetilde{B}_2(s-T)Y_2(T)} - \overline{K}) + \left| \mathcal{F}_t^{Y_1(T)} \right| \mathcal{F}_t \right] \right] &= \\ P(r, v, s-t) \mathbb{E}_T \left[\chi^2 \left(2y_2^*(y_1) [\gamma + \psi + \widetilde{B}_2(s-T)]; \varphi, \frac{2\gamma^2 Y_2(t) \exp(h(T-t))}{\gamma + \psi + \widetilde{B}_2(s-T)} \right) \middle| \mathcal{F}_t \right] &- \\ \overline{K} P(r, v, T-t) \mathbb{E}_T \left[\chi^2 \left(2y_2^*(y_1) [\gamma + \psi]; \varphi, \frac{2\gamma^2 Y_2(t) \exp(h(T-t))}{\gamma + \psi} \right) \middle| \mathcal{F}_t \right] &= \end{aligned}$$

$$P(r, v, s-t) \Omega_1(r, v, t, T, s) - \overline{K} P(r, v, T-t) \Omega_2(r, v, t, T)$$

with

$$\Omega_1(r, v, t, T, s) = \int_0^{y_1^*} \chi^2 \left(2y_2^*(Y_1) [\gamma + \psi + \widetilde{B}_2(s-T)]; \varphi, \frac{2\gamma^2 Y_2(t) \exp(h(T-t))}{\gamma + \psi + \widetilde{B}_2(s-T)} \right) p_T^*(Y_1(T)|Y_1(t)) dY_1$$

$$\Omega_2(r, v, t, T) = \int_0^{y_1^*} \chi^2 \left(2y_2^*(Y_1) [\gamma + \psi]; \varphi, \frac{2\gamma^2 Y_2(t) \exp(h(T-t))}{\gamma + \psi} \right) p_T^*(Y_1(T)|Y_1(t)) dY_1$$

and (r, v) substituted in the last expression by means of (2.33). Furthermore

$$y_2^*(y_1) = \frac{\log \left(\frac{A_1(s-T)^{2\frac{a\overline{Y}_1}}{e^2} A_2(s-T)^{2\frac{c\overline{Y}_2}}{f^2}}{\overline{K}} \right) - \widetilde{B}_1(s-T)y_1 - \kappa(s-T)}{\widetilde{B}_2(s-T)}$$

y_1^* is such that $y_2(y_1^*) = 0$, that is

$$y_1^* = \frac{\log \left(\frac{A_1(s-T)^{2\frac{a\overline{Y}_1}}{e^2} A_2(s-T)^{2\frac{c\overline{Y}_2}}{f^2}}{\overline{K}} \right) - \kappa(s-T)}{\widetilde{B}_1(s-T)}$$

$$\varphi = \frac{4c\overline{Y}_2}{f^2} \quad h = \sqrt{(c + f\theta(\eta))^2 + 2f^2} \quad \gamma = 2h/[f^2(\exp(h(T-t)) - 1)] \quad \psi = ((c + f\theta(\eta)) + h)/f^2$$

η	0.001	0.002	0.005	0.02	0.07	0.09	0.1
C	0.0028548	0.00847366	0.0401189	0.0809638	0.093576	0.099781	0.10869

Table 2.1: Two factor model with time varying pessimism. An example of prices of call options (expiring in 3 years) on a zero coupon bond (with time to maturity 4 years) for different values of the parameter η . The parameter set is $\{\bar{X} = 0.07, \bar{Y} = 0.05, \sigma = 0.134, \rho = 0.2, e = 0.186, f = 0.186, \beta = 0.03, \alpha = 0.2, b = 0.1, c = 0.3, a = 0.4, Y_1(t) = 0.08, Y_2(t) = 0.09, strike = .9\}$

and the forward neutral transition density of the state variable Y_1 is

$$p_T^*(Y_1(T)|Y_1(t)) = q(t, T) p_{\chi^2}(q(t, T) Y_1(T); v, \delta(t, T) | Y_1(t))$$

with $p_{\chi^2}^2$ the non-central chi square pdf with v degrees of freedom and non centrality parameter $\delta(t, T)$; $v = 4a\bar{Y}_1/e^2$, $q(t, T) = 2[\gamma_{Y_1} + \psi_{Y_1} + \widetilde{B}_1(T - t)]$, $\delta(t, T) = \frac{4\gamma_{Y_1}^2 Y_1(t) \exp(h_{Y_1}(T-t))}{q(t, T)}$ and γ_{Y_1} , ψ_{Y_1} , h_{Y_1} are identical to their previously defined counterparts with the parameters of the process Y_1 substituted.

An inspection of the expression for the option price process reveals that the influence of model uncertainty is indeed harder to grasp analytically than in the one dimensional case due to the additional layer of dependence on the parameter η induced by the integration of the chi square cumulative density function. Nevertheless, in light of the fact that this integration is easy to perform numerically, we may rely on explicit computations in order to gain some intuition. As was the case for the complete-market case, the flexible parametric structure of this two factor model is able to give rise to a variety of qualitative predictions. In Table 2.1 we have reported series of call options prices which seems to be monotonically increasing with the ambiguity aversion parameter η ceteris paribus, however a different parameter set (in particular, a different sign of the correlation parameter ρ) may well have generated an opposite behavior.

2.5 Conclusions

We develop a continuous time general equilibrium model for the term structure of interest rates where economic agents are averse to model uncertainty. A concern for an ‘ambiguous’ probabilistic description of the environment on which agent base their decision making process has been largely shown to be both economically and behaviorally relevant in terms of predictions on key economic indicators; we contribute to this strand of the literature by studying clarifying the equilibrium influence of ambiguity aversion on a widely investigated topic like factor models of the term structure and address the consequences for simple, but relevant pricing problems. We have emphasized that a small concern for model uncertainty significantly affects the implied term structures in equilibrium, implying risk premia and interest rates with a different functional form than in standard models. Moreover, otherwise unpriced factors in the standard model receive a premium for model uncertainty which is of a particularly rich structure in the multiple factor setting. All of these features induce in equilibrium term structure levels and shapes that are very different from those generated by a set-up characterized by standard Von Neumann-Morgenstern preferences. Work in progress includes the

analysis of the impact of model uncertainty in a multi factor term structure model and the estimation of yield curve models where a concern for model uncertainty is explicitly taken into account.

Appendix A

Proof of Proposition 7

Notice that the regularity conditions required for an application of the Saddle Point Theorem for infinite dimensional spaces are met in our framework. See Sion (1958) and Ky-Fan (1953). Therefore the value function $J(x, y)$ may be alternatively characterized as

$$J(x, y) = \inf_{h \in \mathcal{H}} \sup_{c, \pi} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \log(c(t)) dt \right] \quad (2.34)$$

Let us first assume that the time horizon T is finite. According to the martingale formulation of the consumption-investment problem to solve in the first step, it is well known that optimality of c implies $c^*(t) = \exp(-\delta t)/\xi_h(t)\psi$, where the lagrange multiplier ψ is solution of $\mathbb{E} \left[\int_0^T \xi(s)c^*(s)ds \right] = x$, i.e $\psi = (1 - \exp(-\delta T))/\delta x$. This leads to

$$c^*(t) = \delta \left(\frac{x e^{-\delta t}}{\xi_h(t)(1 - e^{-\delta T})} \right) \quad (2.35)$$

Let

$$J_h^T(x, y) = \mathbb{E} \left[\int_0^T e^{-\delta t} \log(c^*(t)) dt \right]$$

By virtue of (2.9) and (2.35) one obtains

$$J_h(x, y) = \frac{e^{-T\delta}(1 - e^{T\delta} + T\delta)}{\delta} + \log \left(\frac{\delta x}{1 - e^{-\delta T}} \right) \left(\frac{1 - e^{-\delta T}}{\delta} \right) + \mathbb{E} \left[\int_0^T e^{-\delta t} \int_0^t r(s) + \frac{\theta_h(s)' \theta_h(s)}{2} ds dt \right]$$

In the infinite time horizon case it follows that

$$J_h(x, y) = \lim_{T \rightarrow \infty} J_h^T(x, y) = -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + \mathbb{E} \left[\int_0^\infty e^{-\delta t} \int_0^t r(s) + \frac{\theta_h(s)' \theta_h(s)}{2} ds dt \right]$$

See Karatzas & Shreve (1998) for the differences on the technical assumptions underlying the two frameworks.

As a consequence of our inversion of the order of optimizations leading to the value function (2.34), we may consider a given Girsanov kernel h satisfying (2.4) and the corresponding probability measure P^h ; within this model, the equilibrium interest rate process and excess return on financial assets are inferred from Cox Ingersoll and Ross (1985a) and (2.8)

$$r_h(Y) = \alpha(Y) - \sigma(Y)\sigma'(Y) + \sigma(Y) \cdot h(t) \quad (2.36)$$

$$\beta_h(Y) = \alpha(Y)\bar{1}_k - \sigma(Y)(\sigma'(Y) - h(t))\bar{1}_k + \vartheta(Y)(\sigma'(Y) - h(t)) \quad (2.37)$$

Accordingly, the following equilibrium market price of risk also holds under P^h

$$\lambda_h(Y) = \sigma'(Y) - h(t) \quad (2.38)$$

Let us notice that in Appendix ..., where the case of representative agent maximizing CRRA utility of terminal wealth is analyzed, these equilibrium quantities will be derived by suitable dynamic optimization methods.

We may then conclude that the value function $J(x, y)$ is given by the following program.

$$\begin{aligned} J(x, y) &= -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + \inf_{h \in \mathcal{H}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \int_0^t \left(r(s) + \frac{\theta_h(s)' \theta_h(s)}{2} \right) ds dt \right] \\ &= -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + \inf_{h \in \mathcal{H}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \int_0^t \left(\alpha(Y) - \frac{\sigma(Y)' \sigma(Y)}{2} + \sigma(Y) \cdot h(s) \right) ds dt \right] \\ &= -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + V(y) \end{aligned}$$

Dynamic programming mandates the following necessary condition for optimality of h :

$$\inf_{h \in \mathcal{H}} \left\{ V_Y' [\Lambda(Y) - \Xi(Y) h(t)] + \frac{1}{2} \text{trace} [\Xi(Y)' V_{YY} \Xi(Y)] + \alpha(Y) - \frac{1}{2} \sigma(Y) \sigma(Y)' - \sigma(Y) \cdot h(t) - \delta V \right\} = 0 \quad (2.39)$$

Due to the convexity of the functional appearing in curly brackets in the control h , the condition is also sufficient for the optimality of h ²⁸. The complementary slackness condition corresponding to the minimization (2.39) implies

$$h^*(Y) = -\frac{1}{\psi} [\Xi(Y)' V_Y + \sigma(Y)']$$

where

$$\psi = \frac{1}{\sqrt{2\eta(Y)}} \sqrt{(\Xi(Y)' \nu(t) + \sigma(Y)')' (\Xi(Y)' \nu(t) + \sigma(Y)')}$$

Therefore, the process

$$h^*(Y) = -\sqrt{2\eta(Y)} \frac{\Xi(Y)' \nu(t) + \sigma(Y)'}{\sqrt{(\Xi(Y)' \nu(t) + \sigma(Y)')' (\Xi(Y)' \nu(t) + \sigma(Y)')}} \quad (2.40)$$

constitutes an optimal feed-back control. We then conclude that the value function of our model selection problem solves the nonlinear second order Hamilton-Bellman-Jacobi PDE :

$$\begin{aligned} &V_Y' \Lambda(Y) + \frac{1}{2} \text{trace} [\Xi(Y)' V_{YY} \Xi(Y)] - \\ &\sqrt{2\eta(Y)} \sqrt{(\Xi(Y)' V_Y + \sigma(Y)')' (\Xi(Y)' V_Y + \sigma(Y)')} + \alpha(Y) - \frac{1}{2} \sigma(Y) \sigma(Y)' - \delta V(0, Y) = 0 \end{aligned}$$

²⁸Reference to Fleming and Soner (1993)

Proof of Corollary 7

The equilibrium interest rate, risk premia on financial assets and factor market price of risk follow by substituting (2.40) into the corresponding quantities prevailing under a generic admissible model P^h , i.e. (2.36), (2.37) and (2.38).

Appendix B

A Multivariate Gaussian Model

This Appendix derives the equilibrium Girsanov kernel and the term structure of interest rates for the two factor gaussian model analyzed in Subsection 2.4.1.

i) Constant entropy bound. Proof of Proposition 9

Let us adopt the notation

$$g = [g_1 \ g_2] \quad Y = [Y_1 \ Y_2]' \quad \bar{Y} = [\bar{Y}_1 \ \bar{Y}_2]' \quad M = \text{diag}[m_1, m_2] \quad N = \begin{bmatrix} n_1 & q \\ n_2 & 0 \end{bmatrix}$$

In light of the constant instantaneous entropy bound (2.19) the HBJ equation (2.14) reads

$$V_Y' M (\bar{Y} - Y) + \frac{1}{2} \text{trace} [V_{YY} N N'] - \sqrt{2\eta} \sqrt{(N' V_Y + L')' (N' V_Y + L')} + g_0 + g \cdot Y - \frac{1}{2} L L' - \delta V = 0$$

Standard separation of variables suggests that the latter has a classical solution of the form

$$V(Y) = B' \cdot Y + A$$

where

$$\begin{aligned} B &= (M' + \delta I_k)^{-1} g' \\ A &= \delta^{-1} \left(\bar{Y}' M' B + g_0 - \frac{L L'}{2} - \sqrt{2\eta} \sqrt{(N' B + L')' (N' B + L')} \right) \end{aligned}$$

By virtue of Proposition 7 the equilibrium Girsanov kernel is given by the following expression²⁹:

$$h^* = -\sqrt{2\eta} \frac{N' B + L'}{\sqrt{(N' B + L')' (N' B + L')}}$$

whereas we deduce from Corollary 7 the following equilibrium short rate:

$$r^* = g_0 + g \cdot Y - L L' - \sqrt{2\eta} L \frac{N' B + L'}{\sqrt{(N' B + L')' (N' B + L')}}$$

²⁹More explicitly

$$h^* = -\sqrt{2\eta} \left[\frac{\frac{\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1}{\sqrt{\left(\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1\right)^2 + \left(L_2 + \frac{q g_1}{m_1 + \delta}\right)^2 + L_3^2}}}{\frac{\frac{q g_1}{m_1 + \delta} + L_2}{\sqrt{\left(\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1\right)^2 + \left(L_2 + \frac{q g_1}{m_1 + \delta}\right)^2 + L_3^2}}} \right]$$

which is affine in Y . Similarly, from (2.15), the dynamics of the state variables under the risk neutral reference measure Q are affected by the change of drift ϕ with components:

$$\begin{aligned}\phi_1 &= n_1 \tilde{\phi}_1 + q \tilde{\phi}_2 \\ \phi_2 &= n_2 \tilde{\phi}_1\end{aligned}$$

where

$$\begin{aligned}\tilde{\phi}_1 &= L_1 + \sqrt{2\eta} \frac{\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1}{\sqrt{\left(\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1\right)^2 + \left(L_2 + \frac{q g_1}{m_1 + \delta}\right)^2 + L_3^2}} \\ \tilde{\phi}_2 &= L_2 + \sqrt{2\eta} \frac{\frac{q g_1}{m_1 + \delta} + L_2}{\sqrt{\left(\frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1\right)^2 + \left(L_2 + \frac{q g_1}{m_1 + \delta}\right)^2 + L_3^2}}\end{aligned}$$

Notice that, as a consequence of the affine structure of the model's being preserved under ambiguity aversion, the transition density of the equilibrium short rate is Gaussian, as well as the transition density of the integral over time of the short rate. Therefore (2.20) is obtained by performing the standard (lognormal) integration involved in the price process representation (2.17) corresponding to the case of a zero coupon bond. The coefficient $A(t, T)$ is given by the following (easy to compute) integral:

$$\begin{aligned}A(t, T) = \int_t^T & \left[g_0 + LL' + \sqrt{2\eta} L h^* - (m_1 \bar{Y}_1 - \phi_1) B(s, T) - (m_2 \bar{Y}_2 - \phi_2) C(s, T) - \right. \\ & \left. \frac{1}{2} (n_1^2 + q^2) B(s, T)^2 - \frac{1}{2} n_2^2 C(s, T)^2 - n_1 n_2 B(s, T) C(s, T) \right] ds\end{aligned}$$

ii) Time-varying entropy bound. Proof of Proposition 10

Let $\tilde{Y} = \bar{Y} - Y$. Taking into account the instantaneous entropy bound (2.21), the HBJ equation (2.14) is expressed as follows in terms of \tilde{Y}

$$\begin{aligned}-V_{\tilde{Y}}' M \tilde{Y} + \frac{1}{2} \text{trace} [V_{\tilde{Y}\tilde{Y}} N N'] - \sqrt{2\eta \tilde{Y}_1^2} \sqrt{(-N' V_{\tilde{Y}} + L')' (-N' V_{\tilde{Y}} + L')} + \\ g_0 + g \cdot (\bar{Y} - \tilde{Y}) - \frac{1}{2} LL' - \delta V = 0\end{aligned}$$

We argue that the value function arising in this case is takes the form

$$V(\tilde{Y}) = A + B\tilde{Y}_1 + C\tilde{Y}_2 + D|\tilde{Y}_1|$$

Separation of variables leads to the following expressions for the coefficients involved in the last equation³⁰:

$$\begin{aligned} A &= \frac{1}{\delta} \left(g_0 + g_1 \bar{Y}_1 + g_2 \bar{Y}_2 - \frac{LL'}{2} \right) \\ C &= \frac{g_2}{m_2 + \delta} \end{aligned}$$

and

$$\begin{aligned} B + D &= \frac{1}{m_1^2 - 2\eta(q^2 + n_1^2)} \{ 2q\eta L_2 - g_1 m_1 + 2\eta L_1 n_1 + 2C\eta n_1 n_2 + \\ &\quad \frac{1}{2} \left(4(2q\eta L_2 - g_1 m_1 + 2\eta n_1 (L_1 + C n_2))^2 - \right. \\ &\quad \left. 4(-m_1^2 + 2\eta(q^2 + n_1^2)) \left(-g_1^2 + 2\eta(L_2^2 + L_3^2 + (L_1 + C n_2)^2) \right) \right) \}^{\frac{1}{2}} \\ B - D &= \frac{1}{m_1^2 - 2\eta(q^2 + n_1^2)} \{ 2q\eta L_2 - g_1 m_1 + 2\eta L_1 n_1 + 2C\eta n_1 n_2 - \\ &\quad \frac{1}{2} \left(4(2q\eta L_2 - g_1 m_1 + 2\eta n_1 (L_1 + C n_2))^2 - \right. \\ &\quad \left. 4(-m_1^2 + 2\eta(q^2 + n_1^2)) \left(-g_1^2 + 2\eta(L_2^2 + L_3^2 + (L_1 + C n_2)^2) \right) \right) \}^{\frac{1}{2}} \end{aligned}$$

so that

$$\begin{aligned} B &= \frac{1}{m_1^2 - 2\eta(q^2 + n_1^2)} (2q\eta L_2 - g_1 m_1 + 2\eta L_1 n_1 + 2C\eta n_1 n_2) \\ D &= \frac{1}{2[m_1^2 - 2\eta(q^2 + n_1^2)]} \left(4(2q\eta L_2 - g_1 m_1 + 2\eta n_1 (L_1 + C n_2))^2 - \right. \\ &\quad \left. 4(-m_1^2 + 2\eta(q^2 + n_1^2)) \left(-g_1^2 + 2\eta(L_2^2 + L_3^2 + (L_1 + C n_2)^2) \right) \right)^{\frac{1}{2}} \end{aligned}$$

Once again from (2.15) we may conclude that the dynamics of the state variables under the risk neutral reference measure \mathbb{Q} are the affected by the change of drift ϕ with components:

$$\begin{aligned} \phi_1 &= n_1 \tilde{\phi}_1 + q \tilde{\phi}_2 \\ \phi_2 &= n_2 \tilde{\phi}_1 \end{aligned}$$

³⁰Quite clearly

$$h^* = \sqrt{2\eta \tilde{Y}_1^2} \left[\begin{array}{c} \frac{n_1 \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) + n_2 C + L_1}{\sqrt{\left(n_1 \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) + n_2 C + L_1 \right)^2 + \left(L_2 + q \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) \right)^2 + L_3^2}} \\ \frac{L_2 + q \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right)}{\sqrt{\left(n_1 \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) + n_2 C + L_1 \right)^2 + \left(L_2 + q \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) \right)^2 + L_3^2}} \\ \frac{L_3}{\sqrt{\left(n_1 \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) + n_2 C + L_1 \right)^2 + \left(L_2 + q \left(B + \frac{|\tilde{Y}_1|}{Y_1} D \right) \right)^2 + L_3^2}} \end{array} \right]$$

where

$$\begin{aligned}\tilde{\phi}_1 &= L_1 + \sqrt{2\eta \tilde{Y}_1^2} \frac{n_1 \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right) + n_2 C + L_1}{\sqrt{\left(n_1 \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right) + n_2 C + L_1 \right)^2 + \left(L_2 + q \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right) \right)^2 + L_3^2}} \\ \tilde{\phi}_2 &= L_2 + \sqrt{2\eta \tilde{Y}_1^2} \frac{L_2 + q \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right)}{\sqrt{\left(n_1 \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right) + n_2 C + L_1 \right)^2 + \left(L_2 + q \left(B + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D \right) \right)^2 + L_3^2}}\end{aligned}$$

in light of which the pricing equation (2.16) satisfied by the price of zero coupon bond with maturity T , $P(t, T)$, becomes, in terms of the state variables $\tilde{Y} = \bar{Y} - Y$:

$$P_t - P'_Y \left(M \tilde{Y} - \phi \right) + \frac{1}{2} \text{trace} \left[P_{\tilde{Y}\tilde{Y}} N N' \right] - \left[g_0 + g \left(\bar{Y} - \tilde{Y} \right) - LL' - Lh^* \right] P = 0$$

$P(T, T) = 1$. This equation may be solved by reasoning along the lines leading to the value function of the corresponding max-min expected utility problem. In particular, we argue that the following holds

$$P(t, T) = \exp \left(A(t, T) + B(t, T) \tilde{Y}_1 + C(t, T) \tilde{Y}_2 + D(t, T) |\tilde{Y}_1| \right)$$

Solving the ordinary differential equations arising from a standard separation of variables we obtain the following expressions for the coefficients involved in the last equation:

$$\begin{aligned}A(t, T) &= \int_t^T \left[\left(B(t, s) + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D(t, s) \right) (n_1 L_1 - q L_2) - C(t, s) n_2 L_1 + (g_0 + g \cdot \bar{Y} - LL') - \right. \\ &\quad \left. \frac{1}{2} (n_1^2 + q^2) \left(B(t, s) + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D(t, s) \right)^2 + \frac{1}{2} n_2^2 C(t, s)^2 + \right. \\ &\quad \left. n_1 n_2 \left(B(t, s) + \frac{|\tilde{Y}_1|}{\tilde{Y}_1} D(t, s) \right) C(t, s) \right] ds = 0 \\ C(t, T) &= \frac{(1 - e^{-(T-t)m_2}) g_2}{m_2}\end{aligned}$$

Furthermore

$$\begin{aligned}B(t, T) + D(t, T) &= \frac{1}{S_+ (S_+ - m_2) m_2} \left[\left(1 - e^{-S_+ (T-t)} \right) (S_+ - m_2) (L \tilde{h}^+ + g_1) m_2^2 + \right. \\ &\quad \left. \left(m_2 e^{-S_+ (T-t)} - (m + S_+ (e^{-m(T-t)} - 1)) g_2 \tilde{h}_1^+ n_2 \right) \right] \\ B(t, T) - D(t, T) &= \frac{1}{S_- (S_- - m_2) m_2} \left[\left(1 - e^{-S_- (T-t)} \right) (L \tilde{h}^- - g_1) m_2^2 - \right. \\ &\quad \left. \left(e^{(T-t)m_2 - 1} \right) S_- g_2 \tilde{h}_1^- n_2 + \left(1 - e^{-S_- (T-t)} \right) m_2 (L \tilde{h}^- S_- - S_- g_1 - g_2 \tilde{h}_1^- n_2) \right]\end{aligned}$$

where $\tilde{h}^+ = [\tilde{h}_1^+ \tilde{h}_2^+ \tilde{h}_3^+]'$ $= \frac{h^*}{\sqrt{Y_1^2}}$, with $\tilde{Y}_1 > 0$, $\tilde{h}^- = [\tilde{h}_1^- \tilde{h}_2^- \tilde{h}_3^-]'$ $= \frac{h^*}{\sqrt{Y_1^2}}$, with $\tilde{Y}_1 < 0$, and

$$\begin{aligned} S_- &= m_1 + n_1 \tilde{h}_1^- + n_2 \tilde{h}_2^- \\ S_+ &= m_1 - n_1 \tilde{h}_1^+ - n_2 \tilde{h}_2^+ \end{aligned}$$

The coefficients $B(t, T)$ and $D(t, T)$ are then trivially obtained:

$$\begin{aligned} B(t, T) &= \frac{1}{2} \left\{ \frac{1}{S_+(S_+ - m_2)m_2} \left[\left(1 - e^{-S_+(T-t)}\right) (S_+ - m_2)(L\tilde{h}^+ + g_1)m_2^2 + \right. \right. \\ &\quad \left. \left(m_2 e^{-S_+(T-t)} - (m + S_+(e^{-m(T-t)} - 1))g_2\tilde{h}_1^+ n_2 \right) \right] + \\ &\quad \frac{1}{S_-(S_- - m_2)m_2} \left[\left(1 - e^{-S_-(T-t)}\right) (L\tilde{h}^- - g_1)m_2^2 - \right. \\ &\quad \left. \left(e^{(T-t)m_2-1} \right) S_- g_2 \tilde{h}_1^- n_2 + \left(1 - e^{-S_-(T-t)}\right) m_2 (L\tilde{h}^- S_- - S_- g_1 - g_2 \tilde{h}_1^- n_2) \right] \right\} \\ D(t, T) &= \frac{1}{2} \left\{ \frac{1}{S_+(S_+ - m_2)m_2} \left[\left(1 - e^{-S_+(T-t)}\right) (S_+ - m_2)(L\tilde{h}^+ + g_1)m_2^2 + \right. \right. \\ &\quad \left. \left(m_2 e^{-S_+(T-t)} - (m + S_+(e^{-m(T-t)} - 1))g_2\tilde{h}_1^+ n_2 \right) \right] - \\ &\quad \frac{1}{S_-(S_- - m_2)m_2} \left[\left(1 - e^{-S_-(T-t)}\right) (L\tilde{h}^- - g_1)m_2^2 - \right. \\ &\quad \left. \left(e^{(T-t)m_2-1} \right) S_- g_2 \tilde{h}_1^- n_2 + \left(1 - e^{-S_-(T-t)}\right) m_2 (L\tilde{h}^- S_- - S_- g_1 - g_2 \tilde{h}_1^- n_2) \right] \right\} \end{aligned}$$

The expression for the volatility of the instantaneous forward rates implied by the model, $\sigma_f(t, T)$, is obtained by applying Ito's lemma for convex functions (Tanaka-Meyer formula, see for example Karatzas and Shreve (1991), Theorem 7.1) to the instantaneous forward rate $-\frac{\partial \log P(t, T)}{\partial T}$. According to this formula, we have

$$|\bar{Y}_1 - Y_1(T)| = |\bar{Y}_1 - Y_1(t)| + \int_t^T \text{sgn}(\bar{Y}_1 - Y_1(s)) dY_1(s) + \int_t^T \mathbf{1}_{Y_1(s)=\bar{Y}_1} (n_1^2 + q^2) ds$$

The diffusion component of the instantaneous forward rate is thus given by the vector

$$\begin{bmatrix} -n_1 \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right) - n_2 \frac{\partial B(t, T)}{\partial T} \\ -q \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right) \end{bmatrix}$$

Therefore

$$\begin{aligned} \frac{d \text{Var} \left(d \left(-\frac{\partial \log P(t, T)}{\partial T} \right) \right)}{dt} &= \left[n_1 \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right) + n_2 \frac{\partial B(t, T)}{\partial T} \right]^2 + \\ &\quad q^2 \left(\frac{\partial B(t, T)}{\partial T} + \text{sgn}(\bar{Y}_1 - Y_1(t)) \frac{\partial D(t, T)}{\partial T} \right)^2 \end{aligned}$$

from which the expression reported in the main text for the absolute volatility follows. In particular, the partial derivatives $\frac{\partial B(t, T)}{\partial T}$, $\frac{\partial C(t, T)}{\partial T}$ and $\frac{\partial D(t, T)}{\partial T}$ are easily computed from the expressions above.

Proof of Proposition 11

The HBJ equation (??) becomes in this case:

$$\frac{1}{2}n^2YV_{YY} + \left(\frac{n^2}{4} + mY - n\sqrt{2\eta}\sqrt{Y}\right)V_Y + \left(g_0 + g_1Y - \frac{1}{2}l^2Y - l\sqrt{2\eta}\sqrt{Y}\right) - \delta V(Y) = 0 \quad (2.41)$$

Standard separation of variables leads to the solution reported in the Proposition.

Proof of Proposition 12

In light of the dynamics specified for the state variables and the technological returns (??, ??), the equilibrium interest rate process is $r(t) = (g_1 - l^2)Y - l\sqrt{2\eta}\sqrt{Y}$ (see Proposition). According to Proposition ??, the price of a zero coupon bond with time to maturity $\tau = T - t$ solves the boundary value problem :

$$\begin{aligned} -P_\tau(\tau, Y) + \frac{1}{2}n^2YP_{YY}(\tau, Y) + \left(\frac{n^2}{4} + (m - nl)Y - n\sqrt{2\eta}\sqrt{Y}\right)P_Y(\tau, Y) \\ + \left[(g_1 - l^2)Y - l\sqrt{2\eta}\sqrt{Y}\right]P(\tau, Y) = 0 \end{aligned} \quad (2.42)$$

with the boundary condition $P(0, Y) = 1$. The ansatz $P(\tau, Y) = \exp\left(A(\tau) + B(\tau)Y + C(\tau)\sqrt{Y}\right)$ allows to invoke a standard separation of variables argument, according to which the coefficients $A(\tau)$, $B(\tau)$ and $C(\tau)$ are solutions of the following ordinary differential equations

$$\begin{aligned} \frac{dB(\tau)}{d\tau} &= \frac{n^2}{2}B(\tau)^2 + (m - nl)B(\tau) - (g_1 - l^2) & A(0) &= 0 \\ \frac{dC(\tau)}{d\tau} &= C(\tau)\left(\frac{m - nl}{2} - \frac{n^2}{2}B(\tau)\right) - \sqrt{2\eta}(nB(\tau) - l) & B(0) &= 0 \\ \frac{dA(\tau)}{d\tau} &= \frac{n^2}{8}C(\tau)^2 + \frac{n^2}{4}B(\tau) - \frac{n}{2}\sqrt{2\eta}C(\tau) & C(0) &= 0 \end{aligned}$$

Under the assumption $(m - nl)^2 + 2n^2(g_1 - l^2) > 0$ the solutions of these ODEs are those reported in the Proposition.

In order to clarify the sign of the coefficient $C(\tau)$, which is responsible for the the first order effect of the instantaneous entropy bound η , we need to implement a phase-plane analysis. Consider first the evolutionary equation of $B(\tau)$. The stationary points of this coefficient are, for each τ

$$B^u = \frac{-d + \alpha}{n^2} > 0 \quad B^d = \frac{-d - \alpha}{n^2} < 0$$

where $d = m - nl$ and $\alpha = \sqrt{d^2 + 2n^2(g_1 - l^2)}$. The statements about the signs of the stationary

points hold under the reasonable assumption $d > 0$. Since

$$\frac{dB(\tau)}{d\tau} < 0 \iff B(\tau) \in [B^d, B^u]$$

and $B(0) = 0$ we conclude that $B(\tau) < 0$.

Consider now the evolutionary equation of the coefficient $C(\tau)$. Quite clearly

$$\frac{dC(\tau)}{d\tau} \geq 0 \iff C(\tau) \geq \sqrt{2\eta} \frac{nB(\tau) - l}{\frac{d}{2} - \frac{n^2}{2}B(\tau)} := F(\tau)$$

Notice that $C(0) = 0 > -2\sqrt{2\eta}l/d = F(0)$, therefore $\frac{dC(\tau)}{d\tau} > 0$ in $\tau = 0$. Now

$$\frac{dF(\tau)}{d\tau} = \frac{2\sqrt{2\eta}n(d - ln)\frac{dB(\tau)}{d\tau}}{\left(\frac{d}{2} - \frac{n^2}{2}B(\tau)\right)^2} < 0$$

under the assumption $d - ln = m - 2ln > 0$, so that we may conclude by a simple limit argument that $0 < F(\tau) < C(\tau)$ for any τ and that $\frac{dC(\tau)}{d\tau} > 0$, so that $C(\tau) > 0$.

Alternative argument, which actually allows to drop the assumption $m - 2ln > 0$

By variation of constants we know that the solution of the ordinary differential equation satisfied by the coefficient $C(\tau)$ may be

$$C(\tau) = e^{\int_0^\tau \frac{d - n^2 B(s)}{2} ds} \int_0^\tau e^{\int_0^s \frac{d - n^2 B(u)}{2} du} \sqrt{2\eta} (l - nB(s)) ds$$

This suggests that

$$B(s) < \frac{l}{n} \implies C(\tau) > 0 \quad \forall s > 0$$

but $B(s) < 0$, $\forall s > 0$ therefore the first condition is always verified and $C(\tau) > 0$.

Proof of Proposition 13

Taking into account that $\langle Y_1, Y_2 \rangle(t) = 0$, the HBJ equation (??) reads:

$$\begin{aligned} & \frac{1}{2}e^2 Y_1 V_{Y_1 Y_1} + \frac{1}{2}f^2 Y_2 V_{Y_2 Y_2} + (a + cY_1) V_{Y_1} \\ & + \left(\frac{f^2}{4} + dY_2 - f\sqrt{Y_2}\sqrt{2\eta} \right) V_{Y_2} + \left[\alpha Y_1 + \left(\beta - \frac{\sigma^2}{2} \right) Y_2 - \sigma\rho\sqrt{2\eta}\sqrt{Y_2} \right] - \delta V = 0 \end{aligned} \quad (2.43)$$

This problem is clearly additively separable. The solution follows by a standard separation of variables argument:

$$V(Y_1, Y_2) = A + B Y_2 + C \sqrt{Y_2} + E Y_1$$

$$\begin{aligned}
A &= \frac{1}{\delta} \left(\frac{a\alpha}{-c+\delta} + \frac{f^2 \left(\beta - \frac{\sigma^2}{2} \right)}{4(d-\delta)} + \frac{f\eta(2f\beta + 2d\sigma - 2\delta\sigma - f\sigma^2)}{(d-2\delta)(d-\delta)} \right) \\
B &= \frac{1}{d-\delta} \left(\beta - \frac{\sigma^2}{2} \right) \\
C &= -\frac{\sqrt{2\eta}(2f\beta + 2d\sigma - 2\delta\sigma - f\sigma^2)}{(d-2\delta)(d-\delta)} \\
E &= \frac{\alpha}{\delta - c}
\end{aligned}$$

As of the initial value problem satisfied by the price of the zero coupon bond

$$\begin{aligned}
P_\tau + \frac{1}{2}e^2 Y_1 P_{Y_1 Y_1} + \frac{1}{2}f^2 Y_2 P_{Y_2 Y_2} + (a + cY_1) P_{Y_1} + \left(\frac{f^2}{4} + (d - \sigma\rho f)Y_2 - f\sqrt{Y_2}\sqrt{2\eta} \right) P_{Y_2} + \\
\left[\alpha Y_1 + (\beta - \sigma^2)Y_2 - \sigma\rho\sqrt{2\eta}\sqrt{Y_2} \right] P = 0 \quad (2.44)
\end{aligned}$$

$P(0) = 1$, the separability of this equation exploited for the solution of the corresponding HBJ equation, coupled with the solution of the one-dimensional case, suggests that an exponentially affine form the state variables $Y_1, Y_2, \sqrt{Y_2}$ achieves the desired separation of variables and yields a system of ordinary differential equations for the coefficients, solutions of which are those reported in the text.

Options on zeros in a one factor affine model with constant degree of pessimism

In order to characterize the solution of our zero-th order model, $m_0 = \frac{n^2}{4}$, it is convenient to apply the change of variable $X = \sqrt{Y}$. Let $C(t, X, T, s, \bar{K})$ denote the price in t of a call option with strike \bar{K} and expiration T on a zero coupon bond with maturity s . The standard valuation principle dictates that:

$$C(t, X, T, s, \bar{K}) = \mathbb{E} \left[\frac{\xi_{h^*}(T)}{\xi_{h^*}(t)} \left[P(T, s, X) - \bar{K} \right]^+ \middle| \mathcal{F}_t \right] = P(t, T, X) \mathbb{E}_T \left[(P(T, s, X) - \bar{K}) \mathbf{1}_{\mathcal{M}} \middle| \mathcal{F}_t \right]$$

where

$$\mathcal{M} = \{X \in \mathbb{R} : P(T, s, X) - \bar{K} \geq 0\} \equiv \left\{ X \in \mathbb{R} : B(s - T)X^2 + C(s - T)X \geq \log \left(\frac{\bar{K}}{A(s - T)} \right) \right\}$$

and $\mathbb{E}_T[\cdot]$ denotes expectation with respect to the forward neutral measure³¹. According to the (2.26), (2.27) and the change of numeraire toolkit, the dynamics of the state variable under this

³¹Jamshidian (1991), Geman and Rochet (1995)

measure are³²

$$dX(t) = \frac{1}{2} \left[(m - n^2 B(T-t)) X - nl\sqrt{2\eta} - \frac{n^2}{2} C(T-t) \right] dt + \frac{n}{2} dZ_T(t) \quad (2.45)$$

with $Z_T(t) = Z(t) + \int_0^t \frac{n}{2} (2B(T-t)X + C(T-t)) ds$ a standard brownian motion under the new measure. It is easy to infer from³³ (2.45) that the forward-neutral transition density of the state variable is equal to a Gaussian density function with mean and variance given respectively by $(T > t)$

$$M(t, T) = X(t) e^{-\int_t^T \frac{m-n^2 B(T-u)}{2} du} - \int_t^T e^{-\int_t^s \frac{m-n^2 B(T-u)}{2} du} \left(\frac{nl\sqrt{2\eta}}{2} + \frac{n^2 C(T-s)}{4} \right) d\mathbf{2}.46$$

$$S(t, T) = \frac{n^2}{4} \int_t^T e^{-2 \int_t^s \frac{m-n^2 B(T-u)}{2} du} du ds \quad (2.47)$$

Since the bond price process satisfies the following SDE

$$dP(t, T) = r(t)P(t, T)dt + P(t, T)(2B(T-t)X + C(T-t))\frac{n}{2} dZ(t)$$

Ito's lemma yields the following representation under the T-forward measure for the forward price with delivery in T of the zero coupon bond that expires in s :

$$dP_F(t, T, s) = d \left(\frac{P(t, s)}{P(t, T)} \right) = P_F(t, T, s) [2(B(s-t) - B(T-t))X + C(T-t) - C(s-t)] dZ_T(t) \quad (2.48)$$

with solution $(T > u > t)$

$$P_F(u, T, s) = P_F(t, T, s) \mathcal{E} \left(\int_t^u (2(B(s-\tau) - B(T-\tau))X + C(T-\tau) - C(s-\tau)) dZ_T(\tau) \right)$$

where \mathcal{E} is the usual stochastic exponential. The P_T -martingale³⁴ $P_F(\cdot)$ may therefore serve as density process for the equivalent change of probability measure³⁵

$$\frac{dP_s}{dP_T} = \mathcal{E} \left(\int_0^T (2(B(s-\tau) - B(T-\tau))X + C(T-\tau) - C(s-\tau)) dZ_T(\tau) \right)$$

³²notice that Ito's lemma applied to $P(t, T)$ gives

$$dP(t, T) = r(t)P(t, T)dt + P(t, T)(2B(T-t)X + C(T-t))\frac{n}{2} dZ(t)$$

³³The solution of this SDE reads

$$X(t) = e^{\int_0^t \frac{m-n^2 B(T-s)}{2} ds} \left[X(0) - \int_0^t e^{-\int_0^s \frac{m-n^2 B(T-u)}{2} du} \left(\frac{nl\sqrt{2\eta}}{2} + \frac{n^2 C(T-s)}{4} \right) ds + \int_0^t e^{-\int_0^s \frac{m-n^2 B(T-u)}{2} du} \frac{n}{2} dZ_T \right]$$

³⁴By direct computation (similar to those leading to the bond pricing formula) one easily checks that the Novikov condition

$$\mathbb{E}_T \left[\exp \left(\frac{1}{2} \int_0^t (2(B(s-\tau) - B(T-\tau))X + C(T-\tau) - C(s-\tau))^2 d\tau \right) \right] < \infty$$

is indeed satisfied.

³⁵This measure is the restriction to the sigma-field \mathcal{F}_T of the s -forward measure. See Musiela and Rutkowski (1997).

According to Girsanov theorem, the process

$$Z_s(t) = Z_T(t) - \int_0^t (2(B(s-\tau) - B(T-\tau))X + C(T-\tau) - C(s-\tau))d\tau$$

is a P_s - standard brownian motion, therefore under the new probability measure the transition density of the state variable is still Gaussian with mean and variance given by ($T > t$)

$$\begin{aligned}\widetilde{M}(t, T) &= X(t)e^{-\int_t^T \frac{m-n^2B(s-u)}{2}du} - \int_t^T e^{-\int_t^\tau \frac{m-n^2B(s-u)}{2}du} \left(\frac{nl\sqrt{2\eta}}{2} + \frac{n^2C(s-\tau)}{4} \right) d\tau \\ \widetilde{S}(t, T) &= \frac{n^2}{4} \int_t^T e^{-2\int_t^\tau \frac{m-n^2B(s-u)}{2}du} d\tau\end{aligned}\quad (2.50)$$

Finally

$$\begin{aligned}C(t, X, T, s, \bar{K}) &= P(t, T) \mathbb{E}_T [(P(T, s) - \bar{K}) \mathbf{1}_{\mathcal{M}} | \mathcal{F}_t] \\ &= P(t, T) \mathbb{E}_T [P_F(T, T, s) \mathbf{1}_{\mathcal{M}} | \mathcal{F}_t] - P(t, T) \bar{K} P_T((X(T) \leq X_l) \cup (X(T) \geq X_u) | \mathcal{F}_t) \\ &= P(t, s) \mathbb{E}_T \left[\left. \frac{dP_s}{dP_T} \right|_{\mathcal{F}_t} \mathbf{1}_{\mathcal{M}} \middle| \mathcal{F}_t \right] - P(t, T) \bar{K} P_T((X(T) \leq X_l) \cup (X(T) \geq X_u) | \mathcal{F}_t) \\ &= P(t, s) P_s((X(T) \leq X_l) \cup (X(T) \geq X_u) | \mathcal{F}_t) - \\ &\quad P(t, T) \bar{K} P_T((X(T) \leq X_l) \cup (X(T) \geq X_u) | \mathcal{F}_t) \\ &= P(t, s) \left[1 - \left(\Phi \left(\frac{X_u - \widetilde{M}(t, T)}{\sqrt{\widetilde{S}(t, T)}} \right) - \Phi \left(\frac{X_l - \widetilde{M}(t, T)}{\sqrt{\widetilde{S}(t, T)}} \right) \right) \right] - \\ &\quad \bar{K} P(t, T) \left[1 - \left(\Phi \left(\frac{X_u - M(t, T)}{\sqrt{S(t, T)}} \right) - \Phi \left(\frac{X_l - M(t, T)}{\sqrt{S(t, T)}} \right) \right) \right]\end{aligned}$$

The coefficient $A(T-t)$ of the yield to maturity under one factor (??) with state dependent pessimism

Here is the form of $A(T-t)$

$$\begin{aligned}
A(T-t) = & - \left(\bar{Y} m_1 \left(-2n\theta \operatorname{arctanh} \left[\frac{(n\theta + m_1)^2 \tanh\left(\frac{t\sqrt{2n^2 b(k) + (n\theta + m_1)^2}}{2}\right)}{\sqrt{\left((n\theta + m_1)^2 (2n^2 b(k) + (n\theta + m_1)^2)\right)}} \right] \sqrt{2n^2 b(k) + (n\theta + m_1)^2} + \right. \right. \\
& \left. \left(n t \theta - \log(2n^2 b(k) + (n\theta + m_1)^2) + \log(n^2 b(k) \left(1 + \cosh(t\sqrt{2n^2 b(k) + (n\theta + m_1)^2}) \right) + (n\theta + m_1)^2) \right) \times \right. \\
& \left. \sqrt{\left((n\theta + m_1)^2 (2n^2 b(k) + (n\theta + m_1)^2)\right)} + \right. \\
& \left. m_1 \left(-2 \operatorname{arctanh} \left[\frac{(n\theta + m_1)^2 \tanh\left(\frac{t\sqrt{2n^2 b(k) + (n\theta + m_1)^2}}{2}\right)}{\sqrt{\left((n\theta + m_1)^2 (2n^2 b(k) + (n\theta + m_1)^2)\right)}} \right] \sqrt{2n^2 b(k) + (n\theta + m_1)^2} + \right. \right. \\
& \left. \left. t \sqrt{\left((n\theta + m_1)^2 (2n^2 b(k) + (n\theta + m_1)^2)\right)} \right) \right) \left(n^2 \sqrt{\left((n\theta + m_1)^2 (2n^2 b(k) + (n\theta + m_1)^2)\right)} \right)^{-1}
\end{aligned}$$

Worked-out examples of time varying pessimism

One state variable

It suffices to consider the case of two Brownian motions with instantaneous correlation ρ , the general case being qualitatively identical.

W.l.o.g. , $\sqrt{Y} \left(l\rho dZ_1^h + l\sqrt{1-\rho^2} dZ_2^h \right)$ and $n\sqrt{Y} dZ_1^h$ are, respectively, the diffusion components of the technology and the state variable, whereas $m_1(\bar{Y} - Y)$ is the drift of the state variable; therefore $m_0 = m_1\bar{Y}$ in what follows. If the entropy constraint takes the form $h' \cdot h \leq \frac{2\eta}{Y} Y$, then optimal Girsanov kernels are :

$$\begin{aligned}
h_1^* &= \sqrt{\frac{2\eta}{Y}} \frac{l\rho - nV_Y}{\sqrt{(l\rho - nV_Y)^2 + l^2(1-\rho^2)}} \\
h_2^* &= \sqrt{\frac{2\eta}{Y}} \frac{l\sqrt{1-\rho^2}}{\sqrt{(l\rho - nV_Y)^2 + l^2(1-\rho^2)}}
\end{aligned}$$

and the HBJ equation we have to solve is

$$(m_0 - m_1 Y) V_Y - Y \sqrt{\frac{2\eta}{Y}} \sqrt{(l\rho - nV_Y)^2 + l^2(1-\rho^2)} + \frac{n^2 Y}{2} V_{YY} - Y \left(g_1 - \frac{l^2}{2} \right) - \delta V = 0$$

Let us consider the solution V of the following ordinary differential equation

$$(l\rho - nV_Y)^2 + l^2(1-\rho^2) - k^2 = 0$$

with k constant to be determined. This solution reads

$$V(Y) = Y \frac{\left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2}\right)}{n} + C$$

where C is a constant to be determined as well. Substituting into the HBJ equation we obtain

$$Y \left(\frac{l^2}{2} - \sqrt{\frac{2\eta}{Y}} k - \frac{\beta}{n} \left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) - g_1 - \left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) \frac{m_1}{n} \right) - \left(\beta C - \left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) \frac{m_0}{n} \right) = 0$$

Therefore setting

$$C = \frac{\left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) m_0}{n\beta}$$

and k solution of the quadratic equation

$$\frac{l^2}{2} - \sqrt{\frac{2\eta}{Y}} k - \frac{\beta}{n} \left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) - g_1 - \left(\rho l - \sqrt{k^2 - (-1 + \rho^2) l^2} \right) \frac{m_1}{n} = 0 \quad (2.51)$$

the function $V(Y)$ solves the ODE. The optimal Girsanov kernels are

$$\begin{aligned} h_1^* &= \sqrt{\frac{2\eta Y}{Y}} \frac{\sqrt{k^2 - (-1 + \rho^2) l^2}}{k} \\ h_2^* &= \sqrt{\frac{2\eta Y}{Y}} \frac{l\sqrt{1 - \rho^2}}{k} \end{aligned}$$

Geometric Ornstein Uhlenbeck

In this case a similar analysis holds provided we impose the following state dependent instantaneous entropy constraint:

$$h' \cdot h \leq 2\eta Y^2$$

Optimal controls are

$$\begin{aligned} h_1^* &= Y\sqrt{2\eta} \frac{l\rho - nV_Y}{\sqrt{(l\rho - nV_Y)^2 + l^2(1 - \rho^2)}} \\ h_2^* &= Y\sqrt{2\eta} \frac{l\sqrt{1 - \rho^2}}{\sqrt{(l\rho - nV_Y)^2 + l^2(1 - \rho^2)}} \end{aligned}$$

and the HBJ equation is (we keep the same parameters)

$$(m_0 + m_1 Y)V_Y - Y^2 \sqrt{2\eta} \sqrt{(l\rho - nV_Y)^2 + l^2(1 - \rho^2)} + \frac{n^2 Y^2}{2} V_{YY} - g_1 Y - \frac{l^2}{2} Y^2 - \delta V = 0$$

Let us consider the solution V of the following ordinary differential equation

$$(l\rho - nV_Y)^2 + l^2(1 - \rho^2) - (\lambda V_Y - k)^2 = 0$$

with k and λ constants to be determined. This solution reads

$$V(Y) = \frac{\left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1 - \rho^2) + \lambda^2(-1 + 2\rho^2))l^2}\right)}{\lambda^2 - n^2} Y + C$$

where C is a constant to be determined as well. Substituting into the HBJ equation we easily separate variables and conclude that with

$$C = \frac{m_0 \left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1 - \rho^2) + \lambda^2(-1 + 2\rho^2))l^2}\right)}{(\lambda^2 - n^2)\delta}$$

and (k, λ) solution of the following system of quadratic equations

$$\begin{aligned} & \frac{\lambda\sqrt{2\eta} \left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1 - \rho^2) + \lambda^2(-1 + 2\rho^2))l^2}\right)}{(\lambda^2 - n^2)} - k\sqrt{2\eta} + \frac{l^2}{2} = 0 \\ & - \left(\frac{\beta \left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1 - \rho^2) + \lambda^2(-1 + 2\rho^2))l^2}\right)}{\lambda^2 - n^2} \right) - g_1 + \\ & \frac{\left(\lambda k - n\rho l + \sqrt{n^2 k^2 - 2n\lambda k\rho l + (n^2(1 - \rho^2) + \lambda^2(-1 + 2\rho^2))l^2}\right) m_1}{\lambda^2 - n^2} = 0 \end{aligned} \quad (2.52)$$

the function $V(Y)$ solves the ODE.

An additional example

The HBJ equation arising in this case is

$$\begin{aligned} & (m_0 + m_1 Y - n\kappa_1 \sqrt{Y})V_Y - \sqrt{2\eta Y} \sqrt{(\sigma\rho - V_Y)^2 - l^2(1 - \rho^2)} + \frac{n^2 Y}{2} V_{YY} - \\ & Y \left(g_1 - \frac{l^2}{2} \right) + l\sqrt{Y}(\kappa_1 \rho + \kappa_2 \sqrt{1 - \rho^2}) - \delta V = 0 \end{aligned} \quad (2.53)$$

We can provide an explicit solution for this equation along the lines of the previous examples.

Let us consider the solution V of the following ordinary differential equation

$$(l\rho - V_Y)^2 - l^2(1 - \rho^2) - (\lambda V_Y - k)^2 = 0$$

with k and λ constants to be determined. This solution reads

$$V(Y) = \frac{\left(\lambda k - \rho l + \sqrt{k^2 - 2\lambda k\rho l + (1 - \rho^2 + \lambda^2(-1 + 2\rho^2))l^2}\right)}{\lambda^2 - 1} Y + C$$

where C is a constant to be determined as well. Substituting into the HBJ equation we easily separate variables and conclude that with

$$C = \frac{m_0 \left(\lambda k - \rho l + \sqrt{k^2 - 2 \lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2 \rho^2)) l^2} \right)}{(\lambda^2 - 1) \delta}$$

and (k, λ) solution of the following system of quadratic equations

$$\left\{ \begin{array}{l} \frac{\lambda \sqrt{2\eta} \left(\lambda k - \rho l + \sqrt{k^2 - 2 \lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2 \rho^2)) l^2} \right)}{(\lambda^2 - 1)} - k \sqrt{2\eta} + \\ n \kappa_1 \frac{\left(\lambda k - \rho l + \sqrt{k^2 - 2 \lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2 \rho^2)) l^2} \right)}{(\lambda^2 - 1)} - \sigma(\kappa_1 \rho + \kappa_2 \sqrt{1 - \rho^2}) = 0 \\ - \left(\frac{\beta \left(\lambda k - \rho l + \sqrt{k^2 - 2 \lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2 \rho^2)) l^2} \right)}{-1 + \lambda^2} \right) - g_1 + \\ \frac{\left(\lambda k - \rho l + \sqrt{k^2 - 2 \lambda k \rho l + (1 - \rho^2 + \lambda^2 (-1 + 2 \rho^2)) l^2} \right) m_1}{-1 + \lambda^2} = 0 \end{array} \right. \quad (2.54)$$

the function $V(Y)$ solves the ODE.

Multidimensional Extensions

Multidimensional extension of the previous examples are promptly obtained for those specifications where the problem displays a separable structure (similarly to Longstaff & Schwartz (1992)). We have k state variables Y_i and uncertainty is driven by a $(k + 1)$ -dimensional standard Brownian motion $Z(t)$. For the model described in (??) and its extension outlined in the previous section, this amounts to choosing the diagonal matrices U and R driving the state-dependence of volatilities in the following way:

$$u_{0i} = r_{0i} = 0 \quad r_i = u_i = u \quad i = 1, 2 \dots k + 1$$

Therefore $U = R = (u \cdot Y) I_{k+1}$ In what follows we will give details of the three cases whose one-dimensional analog has been treated.

Extended affine dynamics

We remind that the reference probability measure under which the relevant coefficients are affine in the state variables has been changed to Q by means of a the constant Girsanov kernel $\kappa = (\kappa_1 \kappa_2 \dots \kappa_{k+1})$. It is easy to see that optimal controls are:

$$h^*(t) = \sqrt{2\eta} \frac{-N' \nabla V(Y) + L'}{\sqrt{(-N' \nabla V(Y) + L')' (-N' \nabla V(Y) + L')}}}$$

and the corresponding Hamilton-Bellman-Jacobi equation for the infinite-time horizon case reads (in the notation introduced in (??)):

$$\nabla V(Y)'(M_0 + M_1 Y - \sqrt{u \cdot Y} N \kappa) - \sqrt{2\eta(u \cdot Y)} \sqrt{(-N' \nabla V(Y) + L')' (-N' \nabla V(Y) + L')} + \frac{(u \cdot Y)}{2} \text{trace} [N' \nabla^2 V(Y) N] - \beta V(Y) - \left(g_0 + g_1 \cdot Y - \sqrt{u \cdot Y} L \cdot \kappa - \frac{(u \cdot Y)}{2} L L' \right) = 0$$

The solution of the PDE above is then additively separable:

$$V(Y) = C + \sum_{i=1}^k v_i(Y_i)$$

Let $v_i(Y_i)$ solve the ordinary differential equation

$$\left(\sum_{j=1}^{k+1} n_{ij}^2 \right) v_i'(Y_i)^2 - 2 v_i'(Y_i) \left(\sum_{j=1}^{k+1} n_{ij} l_j \right) + l_i^2 - (K_i v_i'(Y_i) - H_i) = 0$$

with H_i and K_i constants to be determined. The solution is

$$v_i(Y_i) = C_i + \frac{Y_i \left(2 \left(\sum_{j=1}^{k+1} n_{ij} l_j \right) + K_i - \sqrt{(2 \sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4 \left(\sum_{j=1}^{k+1} n_{ij}^2 \right) (H_i + l_i^2)} \right)}{2 \sum_{j=1}^{k+1} n_{ij}^2}$$

substituting the last solutions into the equation and separating variables we conclude that with

$$C = \sum_{i=1}^k C_i = \frac{1}{\beta} \sum_{i=1}^k \frac{m_{0i} \left(2 \left(\sum_{j=1}^{k+1} n_{ij} l_j \right) + K_i - \sqrt{(2 \sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4 \left(\sum_{j=1}^{k+1} n_{ij}^2 \right) (H_i + l_i^2)} \right)}{2 \sum_{j=1}^{k+1} n_{ij}^2} - g_0$$

and $(H = \sum_{i=1}^k H_i, K_1, K_2, \dots, K_k)$ solving the nonlinear system of $k+1$ equations in $k+1$ unknowns

$$\begin{cases} \sqrt{2\eta} \sqrt{l_{k+1} - \sum_{p=1}^k \sum_{p \neq q=1}^k \left(\sum_{j=1}^{k+1} n_{pj} n_{qj} \right) b_p(K_p, H_p) b_q(K_q, H_q) + \sum_{i=1}^k K_i b_i(K_i, H_i) - H + (N - L) \cdot \kappa} = 0 \\ -LL'u' + g_1' + \beta \mathbf{b}(\mathbf{K}, \mathbf{H}) - M_1' \mathbf{b}(\mathbf{K}, \mathbf{H}) = 0 \end{cases}$$

where $\mathbf{b}(\mathbf{K}, \mathbf{H}) = (b_1(K_1, H_1), b_2(K_2, H_2), \dots, b_k(K_k, H_k))'$ and

$$b_i(K_i, H_i) = \frac{2 \left(\sum_{j=1}^{k+1} n_{ij} l_j \right) + K_i - \sqrt{(2 \sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4 \left(\sum_{j=1}^{k+1} n_{ij}^2 \right) (H_i + l_i^2)}}{2 \sum_{j=1}^{k+1} n_{ij}^2}$$

the function $V(Y)$ solves the partial differential equation

Affine dynamics with state-dependent constraint

The constraint on the entropy's instantaneous rate of growth becomes

$$h' \cdot h \leq 2\eta(u \cdot Y) \quad (2.55)$$

In light of this, the optimal Girsanov kernels are

$$h^*(t) = \sqrt{(2\eta)(u \cdot Y)} \frac{-N' \nabla V(Y) + L'}{\sqrt{(-N' \nabla V(Y) + L')' (-N' \nabla V(Y) + L')}}$$

whereas the corresponding Hamilton-Bellman-Jacobi equation for the infinite-time horizon case reads (in the notation introduced in (??)):

$$\begin{aligned} \nabla V(Y)'(M_0 + M_1 Y) - (u \cdot Y) \sqrt{2\eta} \sqrt{(-N' \nabla V(Y) + L')' (-N' \nabla V(Y) + L')} + \\ \frac{(u \cdot Y)}{2} \text{trace} [N' \nabla^2 V(Y) N] - \beta V(Y) - \left(g_0 + g_1 \cdot Y - \frac{(u \cdot Y)}{2} L L' \right) = 0 \end{aligned} \quad (2.56)$$

The solution of the PDE above is in fact then additively separable:

$$V(Y) = C + \sum_{i=1}^k v_i(Y_i)$$

Let $v_i(Y_i)$ solve the ordinary differential equation

$$\left(\sum_{j=1}^{k+1} n_{ij}^2 \right) v_i'(Y_i)^2 - 2 v_i'(Y_i) \left(\sum_{j=1}^{k+1} n_{ij} l_j \right) + l_i^2 - H_i = 0$$

with H_i constant to be determined. The solution is

$$v_i(Y_i) = C_i + \frac{Y_i \left(\left(\sum_{j=1}^{k+1} n_{ij} l_j \right) - \sqrt{\left(\sum_{j=1}^{k+1} n_{ij} l_j \right)^2 - \left(\sum_{j=1}^{k+1} n_{ij}^2 \right) (H_i - l_i^2)} \right)}{\sum_{j=1}^{k+1} n_{ij}^2}$$

substituting the last solutions into the equation and separating variables we conclude that with

$$C = \sum_{i=1}^k C_i = \frac{1}{\beta} \sum_{i=1}^k \frac{m_{0i} \left(\left(\sum_{j=1}^{k+1} n_{ij} l_j \right) - \sqrt{\left(\sum_{j=1}^{k+1} n_{ij} l_j \right)^2 - \left(\sum_{j=1}^{k+1} n_{ij}^2 \right) (H_i - l_i^2)} \right)}{\sum_{j=1}^{k+1} n_{ij}^2} - g_0$$

and $(H_1, H_2 \dots H_k)$ solving the nonlinear system of k equations in k unknowns

$$u' \sqrt{2\eta} \sqrt{l_{k+1} - \sum_{p=1}^k \sum_{p \neq q=1}^k \left(\sum_{j=1}^{k+1} n_{pj} n_{qj} \right) b_p(K_p, H_p) b_q(K_q, H_q)} + \sum_{i=1}^k H_i - L L' u' + g_1' + \beta \mathbf{b}(\mathbf{H}) - M_1' \mathbf{b}(\mathbf{H}) = 0$$

where $\mathbf{b}(\mathbf{H}) = (b_1(H_1), b_2(H_2), \dots, b_k(H_k))'$ and

$$b_i(H_i) = \frac{(\sum_{j=1}^{k+1} n_{ij} l_j) - \sqrt{(\sum_{j=1}^{k+1} n_{ij} l_j)^2 - (\sum_{j=1}^{k+1} n_{ij}^2) (H_i - l_i^2)}}{\sum_{j=1}^{k+1} n_{ij}^2}$$

the function $V(Y)$ solves the partial differential equation.

Geometric Ornstein-Uhlenbeck with state-dependent constraint

As for the previous case we have to modify the constraint on the entropy's instantaneous rate of growth:

$$h' \cdot h \leq 2\eta (u \cdot Y)^2$$

Keeping in mind that in the current setting:

$$\sigma(Y) = LU^2(Y)L' \quad \Xi(Y) = NR^2(Y)N'$$

and $L = U = (u \cdot Y)I_{k+1}$, the optimal Girsanov kernels are

$$h^*(t) = (u \cdot Y)\sqrt{2\eta} \frac{-N'\nabla V(Y) + L'}{\sqrt{(-N'\nabla V(Y) + L')'(-N'\nabla V(Y) + L')}}}$$

whereas the corresponding Hamilton-Bellman-Jacobi equation for the infinite-time horizon case reads (in the notation introduced in (??)):

$$\begin{aligned} \nabla V(Y)'(M_0 + M_1 Y) - (u \cdot Y)^2 \sqrt{2\eta} \sqrt{(-N'\nabla V(Y) + L')'(-N'\nabla V(Y) + L')} + \\ \frac{(u \cdot Y)^2}{2} \text{trace}[N'\nabla^2 V(Y)N] - \beta V(Y) - \left(g_0 + g_1 \cdot Y - \frac{(u \cdot Y)^2}{2} L L'\right) = 0 \end{aligned}$$

Not surprisingly the solution of the PDE above is again additively separable:

$$V(Y) = C + \sum_{i=1}^k v_i(Y_i)$$

Let $v_i(Y_i)$ solve the ordinary differential equation

$$\left(\sum_{j=1}^{k+1} n_{ij}^2\right) v_i'(Y_i)^2 - 2 v_i'(Y_i) \left(\sum_{j=1}^{k+1} n_{ij} l_j\right) + l_i^2 - (K_i v_i'(Y_i) - H_i) = 0$$

with H_i and K_i constants to be determined. The solution is

$$v_i(Y_i) = C_i + \frac{Y_i \left(2(\sum_{j=1}^{k+1} n_{ij} l_j) + K_i - \sqrt{(2\sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4(\sum_{j=1}^{k+1} n_{ij}^2) (H_i + l_i^2)} \right)}{2 \sum_{j=1}^{k+1} n_{ij}^2}$$

substituting the last solutions into the equation and separating variables we conclude that with

$$C = \sum_{i=1}^k C_i = \frac{1}{\beta} \sum_{i=1}^k \frac{m_{0i} \left(2(\sum_{j=1}^{k+1} n_{ij} l_j) + K_i - \sqrt{(2\sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4(\sum_{j=1}^{k+1} n_{ij}^2)(H_i + l_i^2)} \right)}{2\sum_{j=1}^{k+1} n_{ij}^2} - g_0$$

and $(H = \sum_{i=1}^k H_i, K_1, K_2 \dots K_k)$ solving the nonlinear system of $k+1$ equations in $k+1$ unknowns

$$\begin{cases} \sqrt{2\eta} \sqrt{l_{k+1} - \sum_{p=1}^k \sum_{p \neq q=1}^k (\sum_{j=1}^{k+1} n_{pj} n_{qj}) b_p(K_p, H_p) b_q(K_q, H_q) + \sum_{i=1}^k K_i b_i(K_i, H_i) - H - LL' u'} = 0 \\ g'_1 + \beta \mathbf{b}(\mathbf{K}, \mathbf{H}) - M'_1 \mathbf{b}(\mathbf{K}, \mathbf{H}) = 0 \end{cases}$$

where $\mathbf{b}(\mathbf{K}, \mathbf{H}) = (b_1(K_1, H_1), b_2(K_2, H_2), \dots, b_k(K_k, H_k))'$ and

$$b_i(K_i, H_i) = \frac{2(\sum_{j=1}^{k+1} n_{ij} l_j) + K_i - \sqrt{(2\sum_{j=1}^{k+1} n_{ij} l_j + K_i)^2 - 4(\sum_{j=1}^{k+1} n_{ij}^2)(H_i + l_i^2)}}{2\sum_{j=1}^{k+1} n_{ij}^2}$$

the function $V(Y)$ solves the partial differential equation.

Appendix B

Constant degree of pessimism: an additional example

In order to analyze a widely used factor specification which is not included in (??), we discuss here a one factor model whose returns on the technology and state variable display geometric Ornstein-Uhlenbeck dynamics:

$$\alpha(Y) = g_0 + g_1 Y; \quad \sigma(Y) = lY; \quad \Lambda(Y) = m(Y - m_0); \quad \Xi(Y) = nY \quad (2.57)$$

Without loss of generality, we set $g_0 = 0$. As was the case for the previous examples, we will succeed in solving the model for a particular specification within the chosen class and afterwards rely on perturbation theory to gain further insight. Our 'zero-th order' specification is indeed interesting and widely used in its turn: Y follows a geometric brownian motion, that is, $m_0 = 0$.

The methods needed to attack the HBJ equation arising from the agents' consumption-investment problem are similar to those involved in the bond pricing equation, the relevant difference being the non-homogeneity to be treated along the lines of the previous model; a similar consideration applies to the perturbation analysis, therefore we concentrate on the latter problem and briefly mention the solution of the former.

The equilibrium short rate is given by :

$$r(t) = \left(g_1 - l \sqrt{2\eta} \right) Y - l^2 Y^2$$

whereas the market price of risk is an affine function of Y :

$$\theta_{h^*}(t) = lY + \sqrt{2\eta}$$

Although the short rate is a quadratic form in the state variable, the dynamics of the latter under the risk neutral measure, that is

$$dY = \left[(m - n\sqrt{2\eta}) Y + l n Y^2 \right] dt + n Y dZ^*(t)$$

prevents the model from falling into the exponential quadratic class studied, for instance, in Leippold and Wu (2002). Let us then consider the fundamental bond pricing equation and apply the change of variables $X = (2/n^2)Y$ and $u = (n^2/2)\tau$

$$\begin{aligned} -P_u(u, X) + X^2 P_{X X}(u, X) + \left(\frac{2(m - n\sqrt{2\eta})}{n^2} X - l n X^2 \right) P_X(u, X) \\ - \left[\left(g_1 - l \sqrt{2\eta} \right) X - \frac{n^2 l^2}{2} X^2 \right] P(u, X) = 0 \end{aligned} \quad (2.58)$$

with $P(0, X) = 1$. If we consider a solution of the form $P(u, X) = F(u, X) + G(X)$, then direct

substitution mandates that the summands on the r.h.s. solve

$$X^2 G_{XX}(u, X) + \left(\frac{2(m - n\sqrt{2\eta})}{n^2} X - l n X^2 \right) G_X(u, X) - \left[(g_1 - l \sqrt{2\eta}) X - \frac{n^2 l^2}{2} X^2 \right] G(u, X) = 0$$

and

$$\begin{aligned} -F_u(u, X) + X^2 F_{XX}(u, X) + \left(\frac{2(m - n\sqrt{2\eta})}{n^2} X - l n X^2 \right) F_X(u, X) \\ - \left[(g_1 - l \sqrt{2\eta}) X - \frac{n^2 l^2}{2} X^2 \right] F(u, X) = 0 \end{aligned}$$

$$F(0, X) = 1 - G(X).$$

In order to solve the first (ordinary) differential equation, we cast it into the confluent hypergeometric equation

$$z f''(z) + (\gamma - z) f'(z) + \alpha f(z) = 0$$

by means of the ansatz $G(X) = e^{g(X)} X^s f(X)$, for suitable $g(X)$ and s , and the change of variable $z = nl(1 - 2i)X$; eventually we obtain:

$$G(X) = e^{\frac{(1-i)nl}{2} X} H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m - 2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)}, \frac{2(m - 2n\sqrt{2\eta})}{n^2}, nl(1-2i) X \right)$$

where $H_1(\cdot)$ is the confluent first order hypergeometric function.

As of the second (partial) differential equation, we may separate the variables as $F(u, X) = p(u)q(X)$ and without loss of generality choose as constant of separation

$$-\frac{1}{4} \left(1 - \frac{2(m - 2n\sqrt{2\eta})}{n^2} + \phi \right)^2 + \frac{1}{4} \left(\frac{2(m - 2n\sqrt{2\eta})}{n^2} - 1 \right)^2$$

for a parameter $\phi \in \mathbb{R}$. We obtain

$$\begin{aligned} F(u, X) = e^{\frac{(1-i)nl}{2} X} \times \\ \int_{\mathbb{R}} H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m - 2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)} + \frac{\phi i}{1-2i}, \frac{2(m - 2n\sqrt{2\eta})}{n^2}, nl(1-2i) X \right) \times \\ X^{\frac{\phi}{2}} e^{\left[-\frac{1}{4} \left(1 - \frac{2(m - 2n\sqrt{2\eta})}{n^2} + \phi \right)^2 + \frac{1}{4} \left(\frac{2(m - 2n\sqrt{2\eta})}{n^2} - 1 \right)^2 \right] u} \mu^*(\phi) d\phi \end{aligned}$$

where $\mu^*(\phi)$ is determined by the initial condition $F(0, X) = 1 - G(X)$ and therefore solves

$$\begin{aligned} \int_{\mathbb{R}} H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m - 2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)} + \frac{\phi i}{1-2i}, \frac{2(m - 2n\sqrt{2\eta})}{n^2}, nl(1-2i) X \right) \times \\ X^{\frac{\phi}{2}} \mu(\phi) d\phi = e^{-\frac{(1-i)nl}{2} X} - \\ H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m - 2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)}, \frac{2(m - 2n\sqrt{2\eta})}{n^2}, nl(1-2i) X \right) \quad (2.59) \end{aligned}$$

Let us summarize this solution in the following proposition.

Proposition 15 *Let the return on the production opportunity evolve according to (2.57) and the state variable Y be described by a geometric brownian motion process. Then the price of a zero-coupon bond with time to maturity τ is given by the following function:*

$$P(\tau, Y) = F(\tau, Y) + G(Y)$$

$$F(u, X) = e^{\frac{(1-i)nl}{n^2} Y} \times \int_{\mathbb{R}} H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m-2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)} + \frac{\phi i}{1-2i}, \frac{2(m-2n\sqrt{2\eta})}{n^2}, \frac{2l(1-2i)}{n} Y \right) \times \left(\frac{2Y}{n^2} \right)^{\frac{\phi}{2}} e^{\left[-\frac{1}{4} \left(1 - \frac{2(m-2n\sqrt{2\eta})}{n^2} + \phi \right)^2 + \frac{1}{4} \left(\frac{2(m-2n\sqrt{2\eta})}{n^2} - 1 \right)^2 \right] \frac{n^2}{2} \tau} \mu^*(\phi) d\phi$$

$$G(Y) = e^{\frac{(1-i)nl}{n^2} Y} \times H_1 \left(\frac{-(g_1 + l\sqrt{2\eta})}{nl(1-2i)} - \frac{(m-2n\sqrt{2\eta})}{n^2} \frac{(1-i)}{(1-2i)} + \frac{\phi i}{1-2i}, \frac{2(m-2n\sqrt{2\eta})}{n^2}, \frac{2l(1-2i)}{n} Y \right)$$

where $\mu^*(\phi)$ is given implicitly by (2.59).

Appendix C

In this Appendix³⁶ we relax the assumption of a logarithmic felicity function for the representative investor, which enhanced our chances of accomplishing explicit insights into equilibrium quantities. We assume a finite time horizon $[0, T]$ and neglect discounting at the rate δ . We briefly remind that we consider a range of absolutely continuous probability measures assumed to be likely data generating processes. According to Girsanov theorem $Z_h(t) = Z(t) + \int_0^t h(s)ds$ is a standard brownian motion under the model contamination $P^h(\cdot) = \mathbb{E}[\mathcal{E}(-\int h dZ)\mathbf{1}(\cdot)]$. Admissibility of P^h is defined by means of the instantaneous entropy bound $\frac{1}{2}h'h \leq 2\eta(Y)$ on the Girsanov kernel, and under any such belief the opportunity set is posited to follow the dynamics (??).With

$$\Sigma(Y) = \begin{bmatrix} \sigma(Y) \\ \vartheta(Y) \end{bmatrix} \begin{matrix} 1 \times (k+1) \\ k \times (k+1) \end{matrix}$$

and

$$\theta_h = \Sigma(Y)^{-1} \begin{pmatrix} \alpha - r \\ \beta - r \bar{1}_k \end{pmatrix} + h$$

the state-price density $\xi_h(t)$ is governed by the SDE

$$\frac{d\xi_h(t)}{\xi_h(t)} = -r(t)dt - \theta_h(Y)dZ(t)$$

Consider the budget constraint

$$\xi_h(t)W(t) + \int_0^t \xi_h(s)c(s)ds = x + \int_0^t W(s)\xi_h(s) [\Sigma(Y)' \pi(s) - \theta_h(Y)]' \cdot dZ(s) \quad (2.60)$$

The l.h.s. of (2.60) is clearly a positive P -local martingale³⁷, hence a P -supermartingale, therefore for every consumption plan $c(t)$ satisfying (2.60) (for some $\pi(t)$) within an admissible model contamination, we have:

$$\mathbb{E} \left[\int_0^T \xi_h(s)c(s)ds \right] \leq \mathbb{E} \left[\xi_h(T)W(T) + \int_0^T \xi_h(s)c(s)ds \right] \leq x \quad (2.61)$$

Conversely, it can be shown that if

$$\mathbb{E} \left[\int_0^T \xi_h(s)c(s)ds \right] = x$$

then there exists a strategy $\pi(t)$ such that $c(t)$ satisfies (2.60); therefore the latter admits the static formulation (2.61) and the ambiguity averse representative investor solves the max-min expected

³⁶See Cox and Huang (1988), Pliska (1986) or Karatzas, Lehoczky and Shreve (1987) for details on the martingale approach to consumption-investment problems. The derivation of the hedging demand component of optimal portfolio based on Malliavin calculus can be found, for instance, in Karatzas and Shreve (1998) and especially Detemple, Garcia and Rindisbacher (2003)

³⁷In order to rule out doubling strategies, following Dybvig and Huang (1988) we require the nonnegativity of the corresponding wealth process for a π to be admissible.

utility maximization:

$$\begin{aligned} \sup_{c, \pi} \quad & \inf_{h: (h', h) \leq 2\eta(Y)} \mathbb{E} \left[\int_0^T U(c(s)) ds \right] \\ \text{s.t.} \quad & \mathbb{E} \left[\int_0^T \xi_h(s) c(s) ds \right] \leq x \end{aligned} \quad (2.62)$$

Once we interchange the order of maximization and minimization in (2.62) we realize that standard Lagrangian theory mandates the following condition for the innermost consumption-investment problem:

$$c^*(t) = \mathcal{I}(\psi \xi_h(t)) \quad (2.63)$$

where $\mathcal{I}(\cdot)$ denotes inverse marginal utility and ψ is the unique positive Lagrange multiplier such that $x = \mathbb{E} \left[\int_0^T \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \right]$.

By definition of financial wealth

$$\xi_h(t) W(t) = \mathbb{E} \left[\int_t^T \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \middle| \mathcal{F}_t \right] \quad (2.64)$$

therefore, if $\tilde{f}(t, T)$ denotes the Levy-martingale $\mathbb{E} \left[\int_0^T \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \middle| \mathcal{F}_t \right]$, then

$$\xi_h(t) W(t) + \int_0^t \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds = \tilde{f}(t, T)$$

and we just need to compare the l.h.s. of (2.60) with the stochastic integral representation of $\tilde{f}(t, T)$, $d\tilde{f}(t, T) = \phi(t)' dZ_h$, and recall the uniqueness of the (special) semimartingale representation, to conclude

$$\pi(t) = \Sigma(Y)^{\prime -1} \theta_h(Y) + \Sigma(Y)^{\prime -1} \frac{\phi(t)}{W(t) \xi_h(t)} \quad (2.65)$$

The first of the following chain of equalities is dictated by the Clark-Hansmann-Ocone formula. $\mathcal{D}_t \cdot$ denotes the Malliavin differential operator.

$$\begin{aligned} \phi(t) &= \mathcal{D}_t \tilde{f}(t, T) \\ &= \mathbb{E} \left[\int_t^T \mathcal{D}_t [\xi_h(s) \mathcal{I}(\psi \xi_h(s))] ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_t^T \left[\frac{\psi \xi_h(s)}{\frac{\partial^2 U(\mathcal{I}(\psi \xi_h(s)))}{\partial y^2}} + \mathcal{I}(\psi \xi_h(s)) \right] \mathcal{D}_t \xi_h(s) ds \middle| \mathcal{F}_t \right] \end{aligned}$$

But the chain rule for Malliavin calculus mandates, for $s > t$

$$\mathcal{D}_t \xi_h(s) = -\theta'_h(Y) \xi_h(s) - H(t, s)' \xi_h(s)$$

with

$$H(t, s) = \int_t^s \mathcal{D}_t \left(r_h(u) + \frac{|\theta_h(Y)|^2}{2} \right) du + \int_t^s \mathcal{D}_t \theta'_h(Y) dZ_h(u)$$

therefore For $\mathcal{R} = -\frac{U_c}{cU_{cc}}$ denoting the (state dependent) coefficient of relative risk aversion, we have

$$\begin{aligned} \phi(t) &= \theta_h(Y) \mathbb{E} \left[\int_t^T \left[\frac{1}{\mathcal{R}(s)} - 1 \right] \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \middle| \mathcal{F}_t \right] + \\ &\quad \mathbb{E} \left[\int_t^T \left[\frac{1}{\mathcal{R}(s)} - 1 \right] \xi_h(s) \mathcal{I}(\psi \xi_h(s)) H(t, s) ds \middle| \mathcal{F}_t \right] \\ &= -\theta_h(Y) W(t) \xi_h(t) + \theta_h(Y) \xi_h(t) \mathbb{E} \left[\int_t^T \frac{1}{\mathcal{R}(s)} \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \middle| \mathcal{F}_t \right] + \\ &\quad \mathbb{E} \left[\int_t^T \left[\frac{1}{\mathcal{R}(s)} - 1 \right] \xi_h(s) \mathcal{I}(\psi \xi_h(s)) H(t, s) ds \middle| \mathcal{F}_t \right] \end{aligned}$$

Upon substitution in (2.65) we obtain the optimal unconstrained (i.e. non equilibrium) policy of the representative agent

$$\begin{aligned} \pi(t) = \Sigma'^{-1} \theta_h(t) &\frac{\mathbb{E} \left[\int_t^T \frac{1}{\mathcal{R}(s)} \xi_h(s) \mathcal{I}(\psi \xi_h(s)) ds \middle| \mathcal{F}_t \right]}{\xi_h(t) W(t)} + \\ &\Sigma(Y)^{\prime-1} \frac{\mathbb{E} \left[\int_t^T \left[\frac{1}{\mathcal{R}(s)} - 1 \right] \xi_h(s) \mathcal{I}(\psi \xi_h(s)) H(t, s) ds \middle| \mathcal{F}_t \right]}{\xi_h(t) W(t)} \end{aligned} \quad (2.66)$$

At this point we have to address both conditions appearing in the definition of the equilibrium by carrying out the previous consumption-investment program once the trading strategy of our agent has been constrained to lie in the set dictated by 2).

With $b = 1 - \omega - \bar{\Gamma}'_k v$ denoting investment in the bond, we have

$$\pi = \begin{bmatrix} 1 & 1 \times k \\ & 0 \end{bmatrix}' \Longleftrightarrow \begin{bmatrix} b & 1 \times k \\ & v \end{bmatrix}' \in L \equiv \begin{bmatrix} 0 & 1 \times k \\ & 0 \end{bmatrix}'$$

The support function of $-L$

$$\mathcal{S}(r, \beta) = \sup_{[b, v] \in L} -(br + v' \cdot \beta)$$

is a convex, lower semi-continuous function, finite on its effective domain

$$\tilde{L} \equiv \{(r, \beta) : \delta(r, \beta) < \infty\}$$

In our case, $\mathcal{S}(r, \beta) = 0$ and $\tilde{L} \equiv \mathbb{R}^{k+1}$.

In line with the economic intuition about the interest rate process, we assume the latter, hence the support function, to be essentially bounded. We may then regard the instantaneous expected returns on the bond (r) and on the financial assets as dynamic Kuhn-Tucker multipliers meant to address the inability of the agent to exploit these investment opportunities. According to this interpretation,

the available opportunity set (the linear technology) would be fictitiously completed with securities (a bond and k financial assets) whose expected returns processes are chosen in such a way that the representative agent finds it optimal not to exploit them. By means of duality techniques, the equilibrium interest rate and the equilibrium risk premia of any contingent claim in zero net supply issued in the market, are easily obtained as solutions of the convex control problem

$$J_h^*(x, Y) = \inf_{(r, \beta) \in \tilde{L}} J_h(x, Y) = \inf_{(r, \beta) \in \tilde{L}} \mathbb{E} \left[\int_0^T U(\mathcal{I}(\psi \xi_h(s))) ds \right] \quad (2.67)$$

s.t. dynamics of Q , S and state variables

whose necessary conditions for optimality, according to the martingale approach, clearly mandate $\omega = 1$ and $v(t) = 0\bar{1}_k$, $\bar{1}_k$ being a k -dimensional vector of ones. Denoting by $\xi_h^*(t)$ the state price density evaluated at the equilibrium risk premiums of the financial assets ($\theta_h^*(t)$), the solution of

$$J(x, Y) = \inf_{h: (h', h) \leq 2k} J_h^*(x, Y) = \inf_{h: (h', h) \leq 2k} \mathbb{E} \left[\int_0^T U(\mathcal{I}(\psi \xi_h^*(s))) ds \right]$$

s.t. dynamics of the state variables

delivers the optimal belief.

Utility from terminal wealth. Felicity of power type.

Let us consider the case in which the representative agent derives utility from the terminal level of financial wealth and displays a felicity function of power type. According to the previous section we need to solve

$$\inf_{h: h' \cdot h \leq 2\eta(Y)} \inf_{(r, \beta)} \sup_{\pi \in \mathbb{R}} \mathbb{E} \left[\frac{W(T)^\gamma - 1}{\gamma} \right]$$

subject to the static budget constraint $\mathbb{E}[\xi_h(T)W(T)] \leq x$ and the dynamics of the state variables under the reference belief. It is easy to see that the optimality conditions for the consumption-investment problem imply $c^* = 0$ and

$$W^*(T) = (\psi \xi_h(T))^{\frac{1}{\gamma-1}}$$

where the Lagrange multiplier ψ is identified as $\left(x/\mathbb{E}[\xi_h(T)^{\frac{\gamma}{\gamma-1}}]\right)^{\gamma-1}$. The arguments involved above may be promptly adapted to derive the analog of (2.66) for the current framework. Using notation introduced in the previous section we may write

$$J_h^*(x, Y) = \inf_{(r, \beta) \in \tilde{L}} \frac{x^\gamma}{\gamma} \left(\mathbb{E} \left[\xi_h(T)^{\frac{\gamma}{\gamma-1}} \right] \right)^{1-\gamma} - \frac{1}{\gamma}$$

If we assume $\gamma \leq 1$ then for any $t \in [0, T]$ the equilibrium interest rate and returns on financial assets, and the optimal Girsanov kernel h^* are optimal controls of the program

$$\begin{aligned} \frac{1}{\xi_h(t)^{\frac{\gamma}{\gamma-1}}} \operatorname{ess\,inf}_{h:h' \cdot h \leq 2\eta(Y)_{(r,\beta)} \in \tilde{L}} \operatorname{ess\,inf}_{(r,\beta) \in \tilde{L}} \mathbb{E} \left[\xi_h(T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right] &= \operatorname{ess\,inf}_{h:h' \cdot h \leq 2\eta(Y)_{(r,\beta)} \in \tilde{L}} \operatorname{ess\,inf}_{(r,\beta) \in \tilde{L}} \mathbb{E}^\gamma \left[e^{\int_t^T \left(-\frac{\gamma}{\gamma-1} r(s) + \frac{\gamma}{2(\gamma-1)^2} \theta'_h(s) \theta_h(s) \right) ds} \middle| \mathcal{F}_t \right] \\ \text{s.t. } dY(t) &= \left[\Lambda(Y) + \Xi(Y) \left(h(t) - \frac{\gamma}{\gamma-1} \theta_h(t) \right) \right] dt + \Xi(Y) dZ_\gamma(t) \end{aligned} \quad (2.68)$$

where $E^\gamma[\cdot]$ denotes expectation with respect to the probability measure

$$P^\gamma(\cdot) = \mathbb{E} \left[\mathcal{E} \left(\int_0^T \frac{\gamma}{\gamma-1} \theta_h(s) dZ(s) \right) \mathbf{1}(\cdot) \right]$$

and $Z_\gamma(t) = Z(t) + \int_0^t \frac{\gamma}{\gamma-1} \theta_h(s) ds$ is a standard brownian motion under this measure. Under suitable regularity conditions on the coefficients the value function of the innermost control problem, say $\hat{J}_h(t, Y)$, is a classical solution of the HBJ equation

$$\begin{aligned} \frac{\partial \hat{J}_h}{\partial t} + \inf_{(r,\beta) \in \tilde{L}} \left\{ \left[\Lambda + \Xi \left(h - \frac{\gamma}{\gamma-1} \theta_h \right) \right]' \frac{\partial \hat{J}_h}{\partial Y} + \frac{1}{2} \operatorname{trace} \left[\Xi \Xi' \frac{\partial \hat{J}_h}{\partial Y \partial Y'} \right] \right. \\ \left. + \hat{J}_h \left[-\frac{\gamma}{\gamma-1} r + \frac{\gamma}{2(\gamma-1)^2} \theta'_h \theta_h \right] \right\} = 0 \end{aligned} \quad (2.69)$$

in $[0, T] \times \mathbb{R}^k$, with the terminal condition $\hat{J}_h(T, Y) = 1$.

Remark. In order to emphasize the impact of model uncertainty, let us briefly consider the non ambiguity averse case arising when $\eta(Y) = 0$. The equation above then reduces to

$$\frac{\partial J}{\partial t} + \inf_{(r,\beta) \in \tilde{L}} \left\{ \left[\Lambda - \Xi \frac{\gamma}{\gamma-1} \theta \right]' \frac{\partial J}{\partial Y} + \frac{1}{2} \operatorname{trace} \left[\Xi \Xi' \frac{\partial J}{\partial Y \partial Y'} \right] + J \left[-\frac{\gamma}{\gamma-1} r + \frac{\gamma}{2(\gamma-1)^2} \theta' \theta \right] \right\} = 0 \quad (2.70)$$

where $\theta = \theta_h - h$. Formal minimization of the expression within curly brackets leads to (recall from (??) in Appendix A the definitions of f and $\bar{\sigma}$)

$$\begin{cases} \frac{\gamma}{\gamma-1} \bar{\mathbf{I}}'_{k+1} \Sigma^{-1'} \Xi' \frac{\partial J}{\partial Y} - J \frac{\gamma}{(\gamma-1)^2} \bar{\mathbf{I}}'_{k+1} \Sigma^{-1'} \theta = \frac{\gamma}{\gamma-1} J \\ \frac{\gamma}{\gamma-1} \left[\frac{\bar{\sigma}}{\sigma} f \right]' \Xi' \frac{\partial J}{\partial Y} - J \frac{\gamma}{(\gamma-1)^2} \left[\frac{\bar{\sigma}}{\sigma} f \right]' \theta = 0 \bar{\mathbf{I}}_k \end{cases}$$

In light of the partitioned form of Σ^{-1} , the first equation implies, as a consequence of the remaining k s³⁸

$$\Sigma^{-1'} \Xi' \frac{1}{J} \frac{\partial J}{\partial Y} - \frac{1}{\gamma-1} \Sigma^{-1'} \theta = \begin{bmatrix} 1 \\ 0 \bar{\mathbf{I}}_k \end{bmatrix}$$

³⁸In light of 2.66 this equations imply that in equilibrium the representative investor should not invest either in the financial assets or in the money market.

or, after simple passages

$$\theta = \left(\Xi' \frac{1}{J} \frac{\partial J}{\partial Y} - \sigma' \right) (\gamma - 1) \quad (2.71)$$

We have obtained the following proposition

Proposition 16 *In the equilibrium economy populated by a representative agent with power utility, no ambiguity aversion, maximizing utility from terminal wealth and facing the opportunity set (??), the equilibrium interest rate and instantaneous excess returns on financial assets are given in terms of the value function J by*

$$\begin{aligned} r &= \alpha + \left(\sigma \sigma' - \sigma \Xi' \frac{1}{J} \frac{\partial J}{\partial Y} \right) (\gamma - 1) \\ \beta - r \bar{1}_k &= \left(\vartheta \sigma' - \vartheta \Xi' \frac{1}{J} \frac{\partial J}{\partial Y} \right) (1 - \gamma) \end{aligned} \quad (2.72)$$

Proof. Just rewrite (2.71) as

$$\begin{bmatrix} \alpha - r \\ \beta - r \bar{1}_k \end{bmatrix} = \Sigma \left(\Xi' \frac{1}{J} \frac{\partial J}{\partial Y} - \sigma' \right) (\gamma - 1)$$

and recall Σ 's block form (2.5). \square

Substitution of (2.71) and (2.72) leads to the nonlinear HBJ equation satisfied by the value function

$$\frac{\partial J}{\partial t} + [\Lambda + \gamma \Xi \sigma']' \frac{\partial J}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi \Xi' \frac{\partial J}{\partial Y \partial Y'} \right] - \frac{\gamma}{2} \frac{1}{J} \frac{\partial J'}{\partial Y} \Xi' \Xi \frac{\partial J}{\partial Y} - J \left[\frac{\gamma}{\gamma - 1} \alpha + \frac{\gamma}{2} \sigma \sigma' \right] = 0 \quad (2.73)$$

with $J(T, Y) = 1$. Let

$$G = J^{1-\gamma}$$

Then if J satisfies (2.73) one easily checks that $G(t, Y)$ solves the linear partial differential equation

$$\frac{\partial G}{\partial t} + [\Lambda + \gamma \Xi \sigma']' \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi \Xi' \frac{\partial G}{\partial Y \partial Y'} \right] + G \left[\gamma \alpha + \frac{\gamma(\gamma - 1)}{2} \sigma \sigma' \right] = 0 \quad (2.74)$$

with terminal condition $G(T, Y) = 1$. At this point we notice that the uniform ellipticity condition we imposed on $\Xi \Xi'$ and the uniform boundedness and continuity of coefficients meet the standard regularity conditions for this linear parabolic problem to possess a unique classical solution; a characterization of the latter then follows from Feynman-Kac theorem. theorem.

Proposition 17 *The unique solution of (2.74) is given by*

$$G(t, y) = \mathbb{E}^{-\gamma \sigma'} \left[e^{\int_t^T \left(\gamma \alpha + \frac{\gamma(\gamma - 1)}{2} \sigma \sigma' \right) ds} \middle| Y(t) = y \right]$$

where $E^{\gamma\sigma'}[\cdot]$ denotes expectation with respect to the probability measure

$$P^{-\gamma\sigma'}(\cdot) = \mathbb{E} \left[\mathcal{E} \left(\int_0^T -\gamma\sigma' \cdot dZ(s) \right) \mathbf{1}(\cdot) \right]$$

under which the state variables follow the dynamics

$$dY(t) = [\Lambda(Y) + \gamma\Xi(Y)\sigma(t)] dt + \Xi(Y) dZ_{-\gamma\sigma'}(t) \quad (2.75)$$

$Z_{-\gamma\sigma'}(t) = Z(t) - \int_0^t \gamma\sigma'(s)ds$ being a standard brownian motion under this measure. Furthermore the value function of the optimization problem of the representative agent is given by

$$J(t, y) = \left(\mathbb{E}^{-\gamma\sigma'} \left[e^{\int_t^T (\gamma\alpha + \frac{\gamma(\gamma-1)}{2}\sigma\sigma')ds} \middle| Y(t) = y \right] \right)^{\frac{1}{1-\gamma}} \quad (2.76)$$

Notice that as an additional consequence of the coefficients' boundedness the function G is bounded from below and J inherits its smoothness. Furthermore the strictly convex minimization problem appearing in (2.70) admits a unique solution. A standard verification theorem (for instance Fleming and Soner (1993), Theorem 3.1) then implies optimality of (r, β) as detailed above and that the value function of the problem is indeed J and characterized in the Proposition.

Before considering the general ambiguity averse case, let us mention two explicit examples.

Example. *Coefficients of the form (??).*

One easily checks that (2.75) is a multivariate diffusion of the square root type under the measure $P^{-\gamma\sigma'}$. Namely, according to the notation adopted in (??), let us write the dynamics of the state variable as

$$dY(t) = [M(\bar{Y} - Y(t)) + \gamma NR^{\frac{1}{2}} U^{\frac{1}{2}} L'] dt + NR^{\frac{1}{2}} dZ$$

We may hope to solve easily this specification if $U = R$. Notice that this is the only specification for which the usual partial equilibrium assumption of an affine risk neutral drift is fulfilled in equilibrium. In this case, with the additional notation

$$\varrho_0(\gamma) = \gamma g_0 + \frac{\gamma(\gamma-1)}{2} \sum_{j=1}^{k+1} L_j^2 u_{0j} \quad \varrho_i(\gamma) = \gamma g_{1i} + \frac{\gamma(\gamma-1)}{2} \sum_{j=1}^{k+1} L_j^2 u_{ji} \quad \mathcal{K}_{0i} = \gamma \sum_{j=1}^{k+1} L_j n_{ij} u_{0j} \quad \mathcal{K}_{ij} = \gamma \sum_{z=1}^{k+1} L_z n_{iz} u_{zj}$$

so that

$$\begin{aligned} \gamma\alpha + \frac{\gamma(\gamma-1)}{2} \sigma\sigma' &= \varrho_0(\gamma) + \varrho(\gamma) \cdot Y \\ \gamma\Xi\sigma' &= \mathcal{K}_0 + \mathcal{K}Y \end{aligned}$$

we may characterize explicitly the expectation involved in the value function G as an exponential affine function of Y (see Duffie and Kan (1996)):

$$G(t, T, Y) = \exp(A(T-t) - B(T-t)' \cdot Y)$$

$$\begin{aligned}\frac{dA(\tau)}{d\tau} &= -(\bar{Y}'M' + \mathcal{K}'_0)B(\tau) + \frac{1}{2} \sum_{i=1}^{k+1} (N'B(\tau))_i^2 u_{0i} - \varrho_0(\gamma) \\ \frac{dB(\tau)}{d\tau} &= -(M' - \mathcal{K}')B(\tau) - \frac{1}{2} \sum_{i=1}^{k+1} (N'B(\tau))_i^2 u_i + \varrho(\gamma)\end{aligned}$$

where $\tau = T - t$ and these ODEs satisfy the initial conditions $A(0) = 0$ and $B(0) = 0 \bar{\mathbf{1}}_k$. In light of the expression for G we obtain

$$\frac{1}{J} \frac{\partial J}{\partial Y} = \frac{1}{1 - \gamma} \frac{1}{J} \frac{\partial J}{\partial Y} = \frac{B(\tau)}{\gamma - 1}$$

so that the equilibrium short rate

$$r = g_0 + g_1 \cdot Y + LU(Y)L'(\gamma - 1) - LU(Y)N'B(\tau)$$

is affine in the state variables vector Y .

Example. *Quadratic models.*

As an additional example, we consider the class of quadratic models that has recently received attention in the literature. Let the instantaneous returns and volatility of the technology be affine in the state variables and let these be multivariate Ornstein-Uhlenbeck processes; then under $P^{-\gamma\sigma'}$ they still have multivariate Ornstein-Uhlenbeck dynamics³⁹:

$$\alpha(Y) = g_0 + g_1 \cdot Y \quad \sigma = Y' \begin{matrix} N' \\ k \times (k+1) \end{matrix}$$

$$dY(t) = [M(\bar{Y} - Y(t)) + \gamma NY]dt + dZ$$

If we define $\tilde{N} = N'N$ then we may characterize explicitly the expectation involved in the value function G as an exponential quadratic function of Y (see Leippold and Wu (2002) and references therein):

$$G(t, T, Y) = \exp(-A(T - t) - B(T - t)' \cdot Y - Y' C(T - t) Y)$$

$$\begin{aligned}\frac{dC(\tau)}{d\tau} &= \frac{\gamma(\gamma - 1)}{2} \tilde{N} - 2C(\tau)(M - \gamma N) - 2C(\tau)'C(\tau) \\ \frac{dB(\tau)}{d\tau} &= \gamma g_1 - (M - \gamma N)B(\tau) - 2C(\tau)B(\tau) - 2M\bar{Y} \\ \frac{dA(\tau)}{d\tau} &= \gamma g_0 + \text{trace}[C(\tau)] - \frac{B(\tau)'B(\tau)}{2} - B(\tau)'M\bar{Y}\end{aligned}$$

where $\tau = T - t$ and these ODEs satisfy the initial conditions $A(0) = 0, B(0) = 0 \bar{\mathbf{1}}_k$ and $C(0) = 0 \bar{\mathbf{1}}_k \bar{\mathbf{1}}_k'$. In light of the expression for G we obtain

$$\frac{1}{J} \frac{\partial J}{\partial Y} = \frac{2C(\tau)Y + B(\tau)}{1 - \gamma}$$

³⁹ $Z(t)$ has to be regarded as a new k -dimensional brownian motion obtained by suitable rescaling.

so that the equilibrium short rate

$$r = g_0 + g_1 \cdot Y + (\gamma - 1)Y' \tilde{N}Y + Y' N'(2C(\tau)Y + B(\tau))$$

is quadratic in the state variables vector Y .

Let us consider again the HJB equation (2.69). The following proposition is obtained by mimicking the line of reasoning adopted above and recalling the drift perturbation occurring under the reference measure with aversion for ambiguity:

Proposition 18 *Under suitable regularity conditions the unique solution of (2.69) is given by*

$$J(t, y) = \left(\mathbb{E}^{-\gamma\sigma'} \left[e^{\int_t^T (\gamma(\alpha + \sigma \cdot h) + \frac{\gamma(\gamma-1)}{2} \sigma \sigma') ds} \middle| Y(t) = y \right] \right)^{\frac{1}{1-\gamma}} \quad (2.77)$$

where $E^{\gamma\sigma'}[\cdot]$ denotes expectation with respect to the probability measure

$$P^{-\gamma\sigma'}(\cdot) = \mathbb{E} \left[\mathcal{E} \left(\int_0^T -\gamma\sigma' \cdot dZ(s) \right) \mathbf{1}(\cdot) \right]$$

under which the state variables follow the dynamics

$$dY(t) = [\Lambda(Y) + \Xi(Y)(\gamma\sigma(t)' + h(t))] dt + \Xi(Y) dZ_{-\gamma\sigma'}(t) \quad (2.78)$$

$Z_{-\gamma\sigma'}(t) = Z(t) - \int_0^t \gamma\sigma'(s) ds$ being a standard brownian motion under this measure. Furthermore the equilibrium interest rate and instantaneous excess returns on financial assets are given in terms of the value function \hat{J}_h by

$$\begin{aligned} r &= \alpha + \left(\sigma \sigma' - \sigma \Xi' \frac{1}{\hat{J}_h} \frac{\partial \hat{J}_h}{\partial Y} \right) (\gamma - 1) + \sigma \cdot h \\ \beta - r \bar{1}_k &= \left(\vartheta \sigma' - \vartheta \Xi' \frac{1}{\hat{J}_h} \frac{\partial \hat{J}_h}{\partial Y} \right) (1 - \gamma) + \vartheta h \end{aligned} \quad (2.79)$$

We emphasize the twofold effect of model uncertainty on equilibrium short rate and risk premia: the direct first order impact arising through the term proportional to h and the effect on the value function J due to the impact on the transition density with respect to which the expectation is computed.

It is evident from what stated so far that the task of selecting an optimal Girsanov kernel h^* amounts to solving the program⁴⁰

$$\begin{aligned} \arg \inf_{h: (h' h \leq 2\eta(Y))} \hat{J}_h(t, Y) &= \arg \sup_{h: (h' h \leq 2\eta(Y))} \mathbb{E} \left[e^{\int_t^T (\gamma(\alpha + \sigma \cdot h) + \frac{\gamma(\gamma-1)}{2} \sigma \sigma') ds} \middle| Y(t) = y \right] \\ \text{s.t. } dY(t) &= [\Lambda(Y) + \Xi(Y)(\gamma\sigma(t)' + h(t))] dt + \Xi(Y) dZ(t) \end{aligned}$$

⁴⁰We drop the superscript $\gamma\sigma'$ for ease of exposition.

The HBJ equation satisfied by the value function $J(t, Y)$ is

$$\frac{\partial J}{\partial t} + \sup_{h: h' h \leq 2\eta(Y)} \left\{ [\Lambda + \Xi(h + \gamma \sigma')]' \frac{\partial J}{\partial Y} + \frac{1}{2} \text{trace} \left[\Xi \Xi' \frac{\partial J}{\partial Y \partial Y'} \right] + J \left[\gamma (\alpha + \sigma \cdot h) + \frac{\gamma(\gamma - 1)}{2} \sigma \sigma' \right] \right\} = 0$$

with the terminal condition $J(T, Y) = 1$. Performing the (formal) optimization within curly brackets we obtain

$$h^*(\nu, Y) = -\frac{1}{\psi} \left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)$$

where

$$\psi = \frac{1}{\sqrt{2\eta(Y)}} \sqrt{\left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)' \left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)}$$

Therefore, the process

$$h^*(Y) = -\sqrt{2\eta(Y)} \frac{\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)'}{\sqrt{\left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)' \left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)}} \quad (2.80)$$

constitutes an optimal feed-back control, where J is a classical solution of the nonlinear partial differential equation

$$\begin{aligned} \frac{\partial J}{\partial t} + [\Lambda + \gamma \Xi \sigma']' \frac{\partial J}{\partial Y} - \sqrt{2\eta(Y)} \sqrt{\left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)' \left(\Xi(Y)' \frac{\partial J}{\partial Y} + J \gamma \sigma(Y)' \right)} + \\ \frac{1}{2} \text{trace} \left[\Xi \Xi' \frac{\partial J}{\partial Y \partial Y'} \right] + J \left[\gamma \alpha + \frac{\gamma(\gamma - 1)}{2} \sigma \sigma' \right] = 0 \end{aligned} \quad (2.81)$$

$$J(T, Y) = 1.$$

Example. *Time invariant pessimism*

The case characterized by time invariant pessimism ($\eta(Y) = \text{const.}$) and model coefficients of the form (??) still constitutes a considerable technical challenge. We provide a solution for both technological returns and state variables' returns displaying multivariate Ornstein-Uhlenbeck dynamics. Under $P^{-\gamma \sigma'}$ the state variables are still multivariate Ornstein-Uhlenbeck stochastic processes⁴¹:

$$\alpha(Y) = g_0 + g_1 \cdot Y \quad \sigma = \begin{matrix} \hat{\sigma} \\ 1 \times (k+1) \end{matrix}$$

$$dY(t) = [M(\bar{Y} - Y(t)) + \gamma \hat{\sigma}'] dt + dZ$$

⁴¹ As before we have scaled the Ornstein-Uhlenbeck process and expressed it in terms of a newly defined standard k -dimensional brownian motion $Z(t)$

The HJB equation to be solved reads

$$\begin{aligned} \frac{\partial J}{\partial t} + [M(\bar{Y} - Y(t)) + \gamma \hat{\sigma}']' \frac{\partial J}{\partial Y} - \sqrt{2\eta} \sqrt{\left(\frac{\partial J}{\partial Y} + J\gamma \hat{\sigma}' \right)' \left(\frac{\partial J}{\partial Y} + J\gamma \hat{\sigma}' \right)} + \\ \frac{1}{2} \text{trace} \left[\frac{\partial J}{\partial Y \partial Y'} \right] + J \left[\gamma (g_0 + g_1 \cdot Y) + \frac{\gamma(\gamma - 1)}{2} \hat{\sigma} \hat{\sigma}' \right] = 0 \end{aligned}$$

whose solution is easily seen to be of the form:

$$J = \exp(A(\tau) - B(\tau)' \cdot Y)$$

$$\begin{aligned} \frac{dB(\tau)}{d\tau} &= Y' M B(\tau) + \gamma g_1' \\ \frac{dA(\tau)}{d\tau} &= \sqrt{2\eta} \sqrt{(-B(\tau) + \gamma \hat{\sigma}')' (-B(\tau) + \gamma \hat{\sigma}')} - (\bar{Y}' M' + \gamma \hat{\sigma}) B(\tau) - \frac{B(\tau)' B(\tau)}{2} - \\ &\quad \gamma g_0 - \frac{\gamma(\gamma - 1)}{2} \hat{\sigma} \hat{\sigma}' \end{aligned}$$

$A(0) = 0$, $B(0) = 0$. Since

$$h^*(Y) = -\sqrt{2\eta} \frac{-B(\tau) + \gamma \hat{\sigma}'}{\sqrt{(-B(\tau) + \gamma \hat{\sigma}')' (-B(\tau) + \gamma \hat{\sigma}')}}}$$

and $\frac{1}{J} \frac{\partial J}{\partial Y} = -B(\tau)$, the short rate is affine in the state variable Y . Since the functional dependence of equilibrium quantities on the state variable is not dissimilar from its counterpart arising in the non ambiguity averse economy, time invariant pessimism in a multivariate gaussian framework implies ‘robustness’ with respect to ambiguity aversion.

As was the case for logarithmic felicity of intertemporal consumption, for coefficients of the general form (??) we may notice that in the single factor, complete market specification, where $h^* = -\sqrt{2\eta}$ the value function can be characterized analytically for the parametric restriction that makes the model fall within the quadratic class. In this case it is easy to see that $J = \exp(A(\tau)\sqrt{Y} + B(\tau)Y + C(\tau))$ for functions A, B, C solving suitable *ODEs*. Both effects due to model uncertainty induce a term proportional to \sqrt{Y} in the expressions for the short rate and risk premia. The equilibrium structure arising in the non ambiguity averse economy is not preserved in this (non robust) case.

Example. *Time varying pessimism*

Let us consider the case $\eta(Y) = \eta\sqrt{Y}$ and the single state variable model treated in 2.5. If $U = R$ the generalization to the multidimensional case can be carried out along the lines of (2.55). The solution of the HJB equation

$$\begin{aligned} J_t + [m_0 + (m_1 + \gamma n l \rho) Y] J_Y - Y \sqrt{\frac{2\eta}{Y}} \sqrt{(\gamma J l \rho - n J_Y)^2 + \gamma^2 J^2 l^2 (1 - \rho^2)} + \\ \frac{n^2 Y}{2} J_{YY} + Y \left(\gamma g_1 + \frac{\gamma(\gamma - 1)}{2} l^2 \right) J = 0 \end{aligned}$$

is of the form

$$J = \exp(A(\tau) - B(\tau)Y)$$

$$\begin{aligned}\frac{dB(\tau)}{d\tau} &= -B(\tau)(m_1 + \gamma n l \rho) + \frac{n^2}{2}B^2(\tau) - \sqrt{\frac{2\eta}{Y}} \sqrt{n^2 B^2(\tau) - 2\gamma n \sigma \rho l B(\tau) + l^2 \gamma \sigma^2} + \frac{\gamma}{\gamma - 1} g_1 + \frac{\gamma}{2} l^2 \\ \frac{dA(\tau)}{d\tau} &= -B(\tau)m_0\end{aligned}$$

$A(0) = 0$, $B(0) = 0$. Since

$$\begin{aligned}h_1^* &= \sqrt{\frac{2\eta}{Y}} Y \frac{l\rho + nB(\tau)}{\sqrt{n^2 B^2(\tau) - 2\gamma n \sigma \rho l B(\tau) + l^2 \gamma \sigma^2}} \\ h_2^* &= \sqrt{\frac{2\eta}{Y}} Y \frac{l\sqrt{1 - \rho^2}}{\sqrt{n^2 B^2(\tau) - 2\gamma n \sigma \rho l B(\tau) + l^2 \gamma \sigma^2}}\end{aligned}$$

and $\frac{1}{J} \frac{\partial J}{\partial Y} = -B(\tau)$, the short rate is still affine in the state variable Y .

The solutions of the additional cases of time varying pessimism treated in the paper may be recovered as follows. If we write $J(t, Y) = H(Y) + K(t, Y)$ then direct substitution suggests that $K(t, Y) = 1 - H(Y)$; we may then conjecture $K(t, Y) = p(t)q(Y)$ and mimic the solution approach pursued in Appendix B. Notice that if we posit $q(Y) = \exp(f(Y))$ and $H(Y) = \exp(g(Y))$ then f and g can be shown to be linear in Y by reasoning similarly to the case of logarithmic felicity, because the term proportional to the square of their gradient does not affect the separation of variables argument.

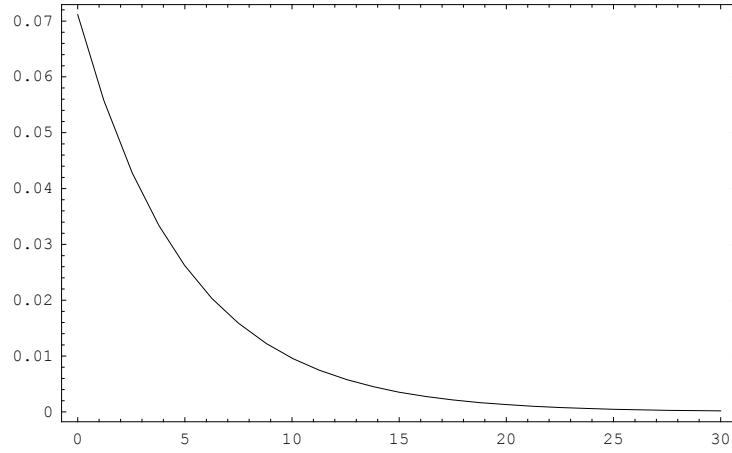


Figure 2.1: Two factor Gaussian model with time-varying pessimism. Volatility of instantaneous forward rates under no ambiguity aversion plotted against time to maturity. Parameters have been set to $\{g_1 = 0.3, g_2 = -0.7, n_2 = -0.29, n_1 = .01, m_1 = 0.1, m_2 = 0.2, q = 0.12, L_1 = .0076, L_2 = .05, L_3 = .005\}$

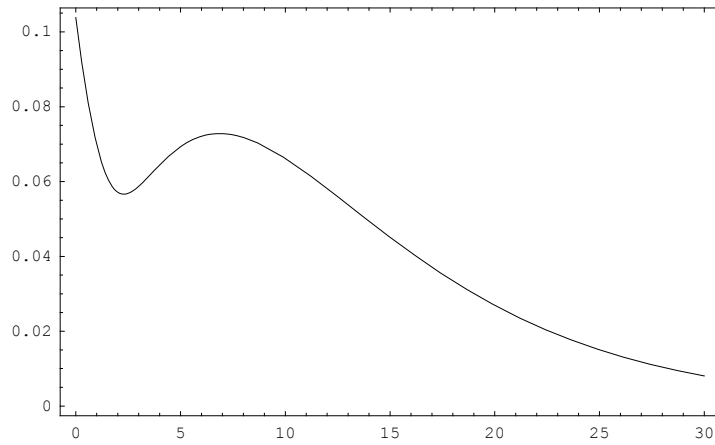


Figure 2.2: Two factor Gaussian model with time-varying pessimism. Volatility of instantaneous forward rates under ambiguity aversion ($\eta = 0.005$) plotted against time to maturity. Parameters have been set to $\{g_1 = 0.3, g_2 = -0.7, n_2 = -0.29, n_1 = .01, m_1 = 0.1, m_2 = 0.2, q = 0.12\}$

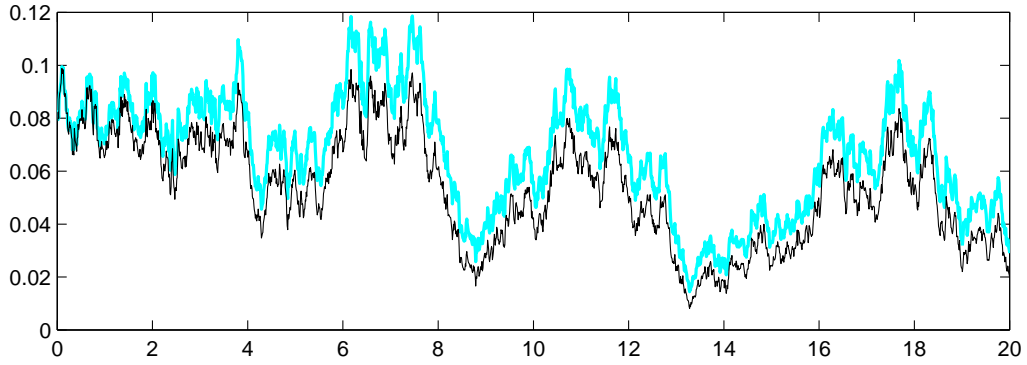


Figure 2.3: A typical path of the equilibrium short rate for $\eta = 0.01$ (black curve) compared to its counterpart implied by the model with Von Neumann-Morgenstern preferences. Parameters have been set to $\{\bar{Y} = 0.25, l = 0.106, n = 0.1764, \beta = 0.03, g_1 = 0.42, m_1 = 0.3, Y = 0.25\}$, $Y(0) = 0.193194$ ($\eta = 0$) and $Y(0) = 0.21$ ($\eta = 0.01$) which corresponds to $r(0) = 0.0789709$ in both cases.

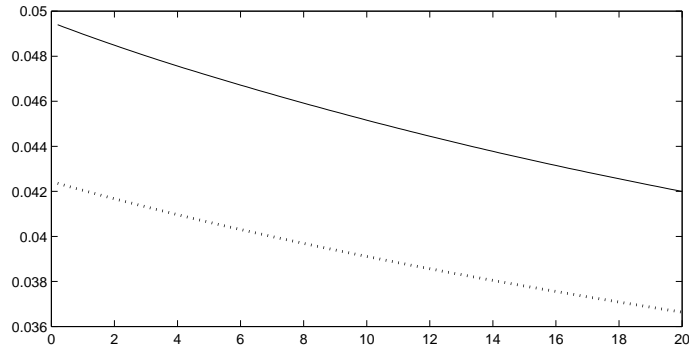


Figure 2.4: Comparison of equilibrium yield curves generated by the model with the same parameter set as the previous figure and $\eta = 0.001$ (dotted line), $\eta = 0$.

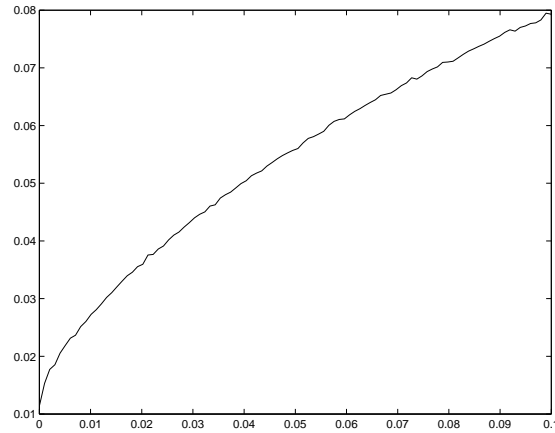


Figure 2.5: Price of a call option (y) expiring in one year on a zero coupon bond with time to maturity 3 years plotted against the ambiguity aversion parameter η (x). The parameter set is as in Figure (2.4).

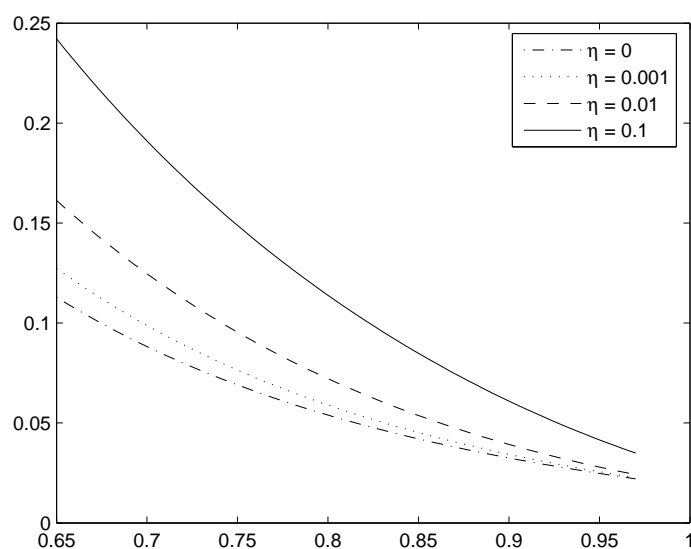


Figure 2.6: Black implied volatility curves for a call option that expires in one year on a zero coupon bond with time to maturity τ in Figure (2.4).

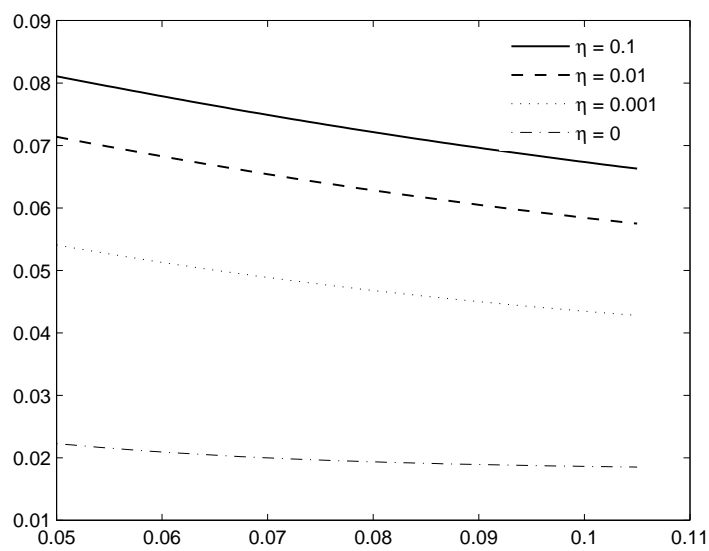


Figure 2.7: Black implied volatility curves for a floorlet on the 3month LIBOR (simple future interest rate) generated by the model. The floorlet expires in one year and the parameter set is as in Figure (2.4). Subplots correspond to different choices of η .

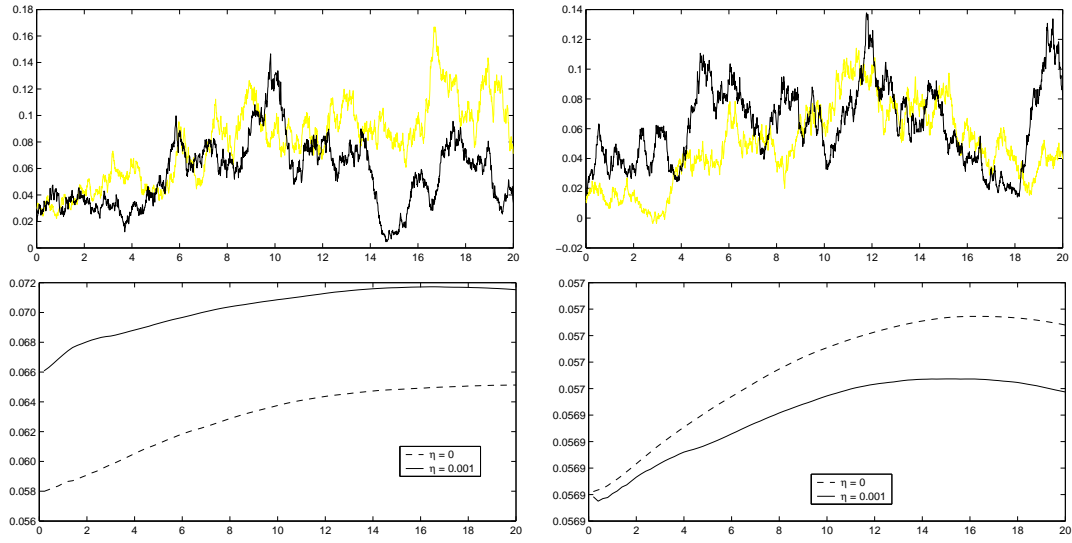


Figure 2.8: Two-factor Longstaff and Schwartz model with ambiguity on the factor driving the volatility of technological returns. Sample paths of short rates and equilibrium yield curves for $\eta = 0$ (black curves) and $\eta = 0.001$ (yellow curves). In the subplots on the first column $\rho = -0.5$, on the second column $\rho = 0.5$. Parameters have been set to $\{a = 0.18, \sigma = 0.134, f = 0.286, \delta = 0.03, \alpha = 0.22, c = 0.54, d = 0.5, \beta = 0, e = 0.237, l = 0.16\}$. l is the parameter of mean reversion in the dynamics of Y_2 , no longer constrained to $f^2/4$.

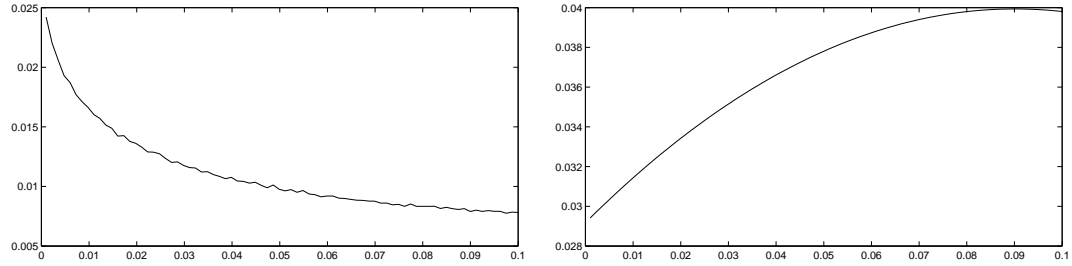


Figure 2.9: Ceteris Paribus, prices of call options on zero coupon bonds are decreasing in the ambiguity aversion parameter η (at least for small values) when ρ is negative (left panel), and increasing in η when ρ is positive (right panel). We report an example where the strike is 0.77 the maturity is one year and the underlying has time to maturity 3 years.

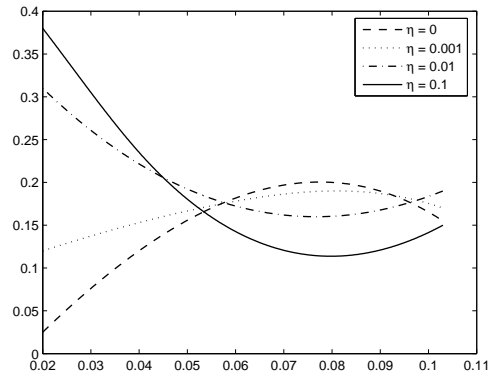


Figure 2.10: Implied Black caplet volatilities generated by the two factor model with $\rho = -0.5$. The parameter set is $\{a = 0.18, \sigma = 0.134, f = 0.286, \delta = 0.03, \alpha = 0.22, c = 0.54, d = 0.5, \beta = 0, e = 0.237, l = 0.16\}$. l is the parameter of mean reversion in the dynamics of Y_2

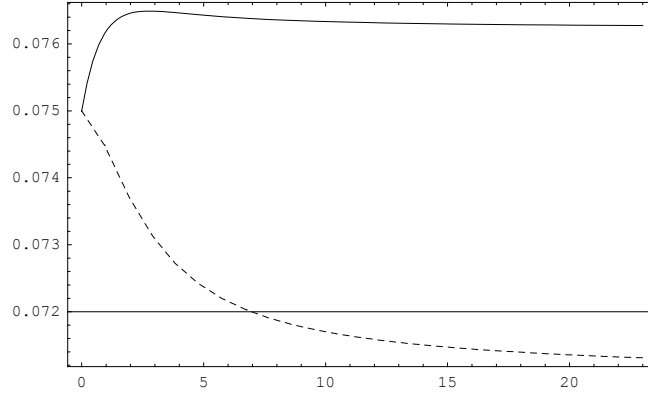


Figure 2.11: One factor square root specification with time-varying pessimism. Equilibrium yield curves for $\eta = 0.01$ (dashed line) and $\eta = 0$, with parameter set $\{\bar{Y} = 0.18, l = 0.134, \rho = 0.9, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7, r = 0.075\}$

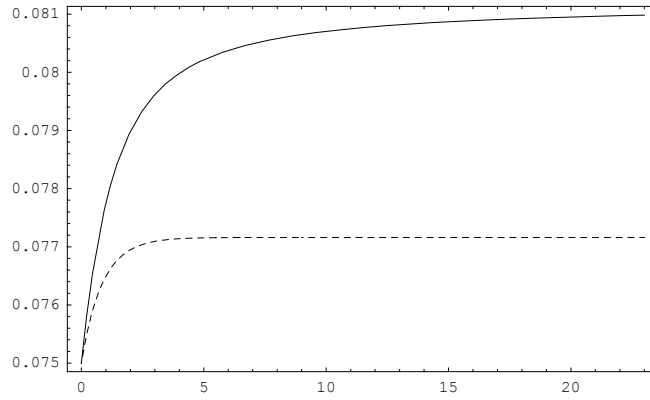


Figure 2.12: One factor square root specification with time-varying pessimism. Equilibrium yield curves for $\eta = 0.01$ (dashed line) and $\eta = 0$, with parameter set $\{\bar{Y} = 0.18, l = 0.134, \rho = 0, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7, r = 0.075\}$

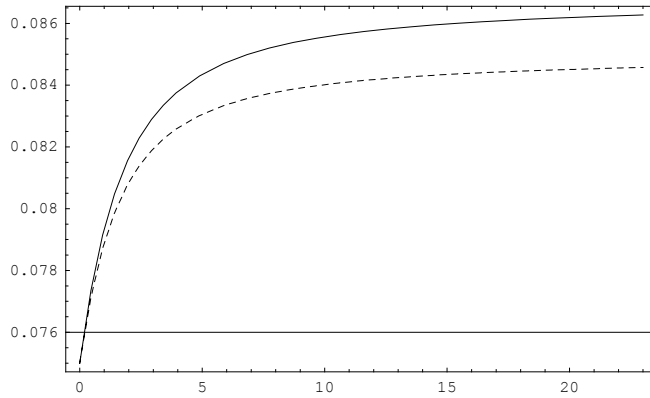


Figure 2.13: One factor square root specification with time-varying pessimism. Equilibrium yield curves for $\eta = 0.01$ (dashed line) and $\eta = 0$, with parameter set $\{\bar{Y} = 0.18, l = 0.134, \rho = -0.9, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7, r = 0.075\}$

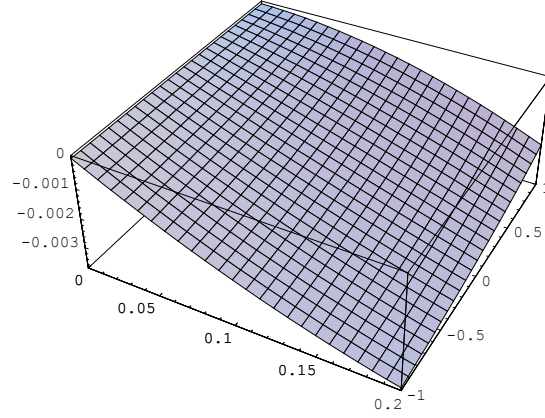


Figure 2.14: One factor square root specification with time-varying pessimism. Sensitivity of the mean reversion parameter $b(k)\bar{Y}m_1$ with respect to ρ plotted against ρ and the ambiguity aversion parameter η . Parameters have been set to $\{\bar{Y} = 0.18, l = 0.134, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7\}$

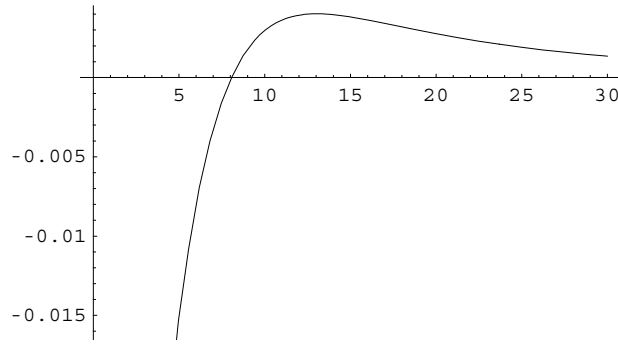


Figure 2.15: One factor square root specification with time-varying pessimism. Sensitivity of the slope of the yield curve (y) to the parameter η , for $\eta = 0$ and plotted against time to maturity (x). The parameter set is $\{\bar{Y} = 0.18, l = 0.134, \rho = -0.9, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7, r = 0.075\}$

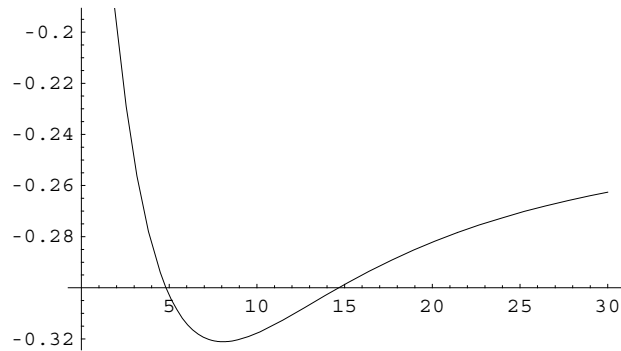


Figure 2.16: Sensitivity of yield to maturity (y) with respect to η , for $\eta = 0$ and plotted against time to maturity (x). The parameter set is $\{\bar{Y} = 0.18, l = 0.134, \rho = -0.9, n = 0.386, \delta = 0.03, g_1 = 0.5, m_1 = 0.7, r = 0.075\}$

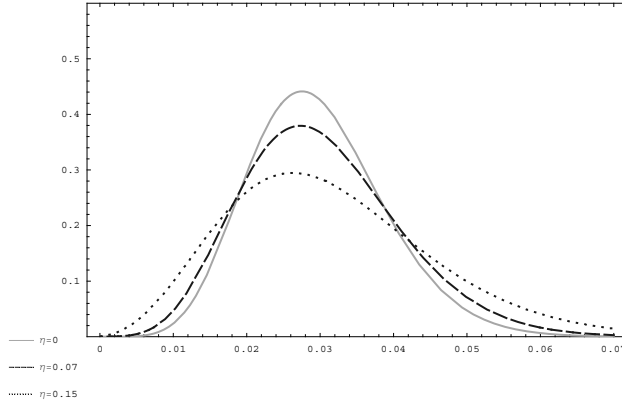


Figure 2.17: Equilibrium risk- neutral transition density of the short rate for different choices of η and the parameter set $\{\bar{Y} = 0.05, l = 0.134, \rho = -0.4, n = 0.154, \beta = 0.03, g_1 = 0.1, m_1 = 0.3, t = 3, T = 5, r = 0.05\}$

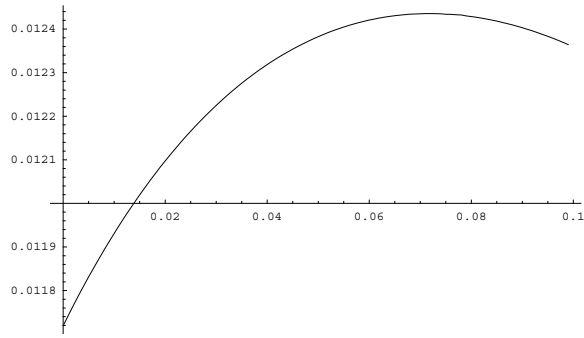


Figure 2.18: Price (y) of an out-of-the-money call option with strike $K = 0.9$ expiring in $T = 5$ on a zero coupon bond with maturity $s = 7$ for different choices of η (x) and parameter set $\{\bar{Y} = 0.05, l = 0.134, \rho = -0.4, n = 0.154, \beta = 0.03, g_1 = 0.1, m_1 = 0.3, t = 3, T = 5, r = 0.05\}$

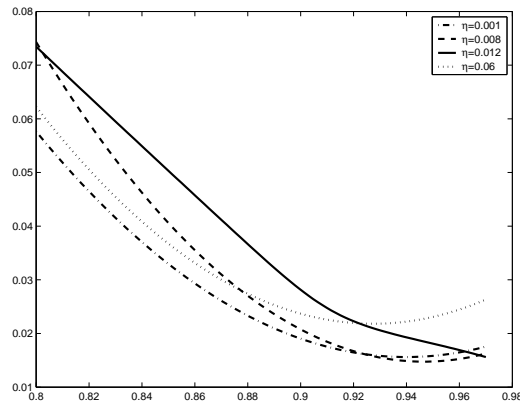


Figure 2.19: Black implied volatility curves for a call option with time to maturity of 5 years on a zero coupon bond with time to maturity 7 years. The current forward prices at expiry of the underline are (0.951212, 0.951345, 0.951444, 0.953023) for each of the values assumed by η . Parameters have been set to $\{\bar{Y} = 0.05, l = 0.134, \rho = -0.35, n = 0.154, \beta = 0.03, g_1 = 0.1, m_1 = 0.3, t = 3, T = 5, r = 0.05\}$

Chapter 3

Concluding Remarks

Allowing financial agents to possess a complete knowledge of the data generating process is a stringent modelling requirement. While acknowledging this fact, ambiguity aversion provides a still tractable modelling framework for addressing the lack of precision in the probabilistic description of the data generating model. In continuous-time, the requirement of absolute continuity of the multiple beliefs that the agent involves in his optimization problem implies that ambiguity essentially pertains the conditional mean of the financial opportunity set's evolution, in accordance with the intuition that first conditional moments are notoriously more difficult to identify than, for instance, second conditional moments.

The present thesis is an attempt to investigate whether two classical modelling framework like a Cox Ingersoll and Ross economy and a two-agent rational expectations equilibrium environment may benefit from a concern for ambiguity in terms of realistic predictions. Our findings support those of the recent strand of literature that deems ambiguity aversion as a candidate for assessing several 'puzzles' documented by the financial community. Indeed both Chapters document the ability of this preference ordering representation to affect equilibria quantities in a way that point towards empirical evidence, at least from the qualitative point of view.

One possible criticism to which this ambiguity set-up may be prone, is the observation that investors seem to dogmatically expect the worst while ignoring the information flow, thus remaining trapped in their invariant ambiguous probabilistic description of future contingencies. Whether model uncertainty preserves or even enhances its appealing once learning is accounted for is an issue that only very recently deserved the attention of the authors. The answers provided so far have emphasized that ambiguity may well fail to resolve asymptotically, so that "asset pricing relations under ambiguity aversion but no learning can be interpreted as the limit of an equilibrium learning process under ambiguity aversion" (Leippold, Trojani and Vanini (2004)¹). As of the second point implicit in the criticism above, that is, the dogmatic worst case optimizing behavior, the response to this question certainly relies on whether the ubiquitous identification of aversion for ambiguity with preference for robustness is an admissible one. The inspiration for the latter interpretation comes from the analogy with the methods of robust control exploited by the approach pioneered by

¹In this respect, see also Knox (2004) and Epstein and Schneider (2002).

Hansen, Sargent and coauthors, and to a lesser extent, from the analogy with the robust statistics' methodologies. While the multiplicity of beliefs considered may be interpreted as local contamination of a reference model, the optimization criterion implied by an ambiguity averse preference ordering does not deliver robust policies, i.e., policies that preserve stability of the indirect utility across these local contaminations. By construction, if Girsanov kernels are selected according to the criteria we have been analyzing, the indirect utility is increasing in the 'direction of the true' data generating model, provided it is represented by a Girsanov kernel in the admissible set. Whether an axiomatic theory of 'robust' decision making may be constructed with the priority of delivering smoothness of the value functions involved across possible models is an open question.

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