

# Yan Theorem in $L^\infty$ with Applications to Asset Pricing

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**Abstract** We prove an  $L^\infty$  version of Yan theorem and deduce from it a necessary condition for the absence of free lunches in a model of financial markets in which asset prices are a continuous  $\mathbb{R}^d$  valued process and only simple investment strategies are admissible. Our proof is based on a new separation theorem for convex sets of finitely additive measures.

**Key words** Arbitrage, Free lunch, Fundamental theorem of asset pricing, Martingale measure, Yan theorem.

**JEL Classification:** G12

**Mathematics Subject Classification (2000):** : 91B28, 60G44, 60H30

## 1 Introduction

In a well known paper [23], Yan proved a result concerning some convex subsets of  $L^1$  which turned out to wield noteworthy influence on much of the mathematical finance literature that followed. Given a probability  $P$  and a convex set  $\mathcal{K} \subset L^1$  containing 0 the theorem provides a necessary and sufficient condition for the existence of a probability  $Q$  equivalent to  $P$  and with respect to which the expected value of elements of  $\mathcal{K}$  is uniformly bounded from above, i.e.  $Q[\mathcal{K}] < a < \infty$ . Several versions of this theorem have later been proved in the literature. Ansel and Stricker [1] obtained a first generalization to  $L^p$ ,  $1 \leq p < \infty$ ; furthermore they illustrated the far reaching implications

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for mathematical finance, especially arbitrage theory (further developed in [22]). Jouini, Napp and Schachermayer [11] have recently obtained a proof for locally convex spaces satisfying some special conditions. In this paper we focus on  $L^\infty$  and show that the original claim of Yan remains true also in this context – see Theorem 2 below. We then apply this conclusion to prove a version of the Fundamental Theorem of Asset Pricing (*FTAP*) for a continuous price process. Our argument relies on a preliminary result, Theorem 1, concerning the separation of convex sets of finitely additive measures and from which the theorem of Yan readily follows. This further theorem, being of some generality, may be of its own interest.

In the literature there have been several different proofs of the *FTAP* (see, among others, [1], [7] and [14] for the case of continuous prices process and [6] for the locally bounded case), a well known and much useful result stating, in rough terms, that if financial markets are free of arbitrage opportunities, then there exists a probability measure with respect to which asset returns are martingales. The existing versions of this theorem differ one from another with respect to the class of trading strategies considered as admissible and to the different definitions of an arbitrage opportunity adopted. In many papers the latter concept is reinforced into that of a so-called free lunch, whose definition is directly inspired by the condition originally proposed by Yan (and, independently, by Kreps [13]) and which requires the choice of a reference topological space. In [1] the  $L^p$  definition is adopted relatively to simple investment strategies whereas in [6] free lunches are defined with reference to  $L^\infty$  (so-called *free-lunches-with-vanishing-risk*) but the full fledge of stochastic integration is exploited by admitting general investment strategies. [7] and [14] focus on arbitrage opportunities with continuous price process and general integrands as portfolios.

In section 3 we apply our version of Yan theorem to a model of financial markets in which only simple investment strategies are admitted but free lunches, defined with reference to  $L^\infty$ , are ruled out. This makes our results akin to those established in [6, section 7] (see the more detailed comments below). It should be remarked that absence of free lunches in  $L^\infty$  represents a much weaker constraint on markets than the corresponding condition formulated in the  $L^p$  framework. It therefore guarantees poorer mathematical properties, which explains the interest for a corresponding version of Yan characterization. In fact we prove that the absence of free lunches implies the existence of a strictly positive local martingale  $Z$  that, if adopted as a discount factor, transforms asset prices and returns into local martingales. We also prove that, when focusing on arbitrage opportunities rather than free lunches, the same conclusion obtains save that the intervening discount factor need not be strictly positive. In either case, however, the mere existence of the process  $Z$  does not provide a sufficient condition for excluding arbitrage opportunities.

Eventually in section 4 we comment on the financial interpretation of the results obtained; section 5 concludes.

## 2 Yan Theorem.

Let  $(\Omega, \mathcal{F}, P)$  be a standard probability space and  $\mathfrak{M}$  the space of bounded, finitely additive measures on  $\mathcal{F}$  vanishing on  $P$  null sets (usually denoted by  $ba(\Omega, \mathcal{F}, P)$ , as in [8]). By  $f \in L_{++}^\infty$  we mean  $f \in L_+^\infty$  and  $P(f) > 0$  while we shall speak of a *strictly positive* measure  $m$  (in symbols  $m \in \mathfrak{M}_{++}$ ) if  $m$  is a positive set function (in symbols  $m \in \mathfrak{M}_+$ ) and  $m(f) > 0$  for any  $f \in L_{++}^\infty$ . Since  $\mathfrak{M}$  is the

topological dual of  $L^\infty$  [8, theorem IV.8.16, p. 296], we denote by  $\tau$  the weak\* topology on  $\mathfrak{M}$  and denote by  $\mathcal{M}^\tau$  the  $\tau$  closure of any subset  $\mathcal{M}$  of  $\mathfrak{M}$ . As a matter of notation, we find it convenient not to distinguish between a set and its indicator.

We recall the decomposition of Yosida and Hewitt [24, theorem 1.24, p. 52] or [3, theorem 10.2.1, p. 241]: for each  $m \in \mathfrak{M}_+$  there exists a unique way of writing  $m = m^c + m^\perp$  with  $m^c, m^\perp \in \mathfrak{M}_+$ ,  $m^c$  countably additive and absolutely continuous with respect to  $P$  and  $m^\perp$  is purely finitely additive i.e. such that for any  $\epsilon > 0$  there exists  $F \in \mathcal{F}$  such that  $m^\perp(F) = 0$  and  $P(F^c) < \epsilon$  [24, theorem 1.19, p. 50] or [3, theorem 10.3.3, p. 244]. We remark that if  $m \in \mathfrak{M}_+$  and  $F \in \mathcal{F}$  is such that  $P(F) > 0 = m^c(F)$  then, by orthogonality, we can find  $F' \subset F$ ,  $F' \in \mathcal{F}$  such that  $P(F') > P(F)(1 - \epsilon)$  and  $m(F') = 0$ . In other words, if  $m \in \mathfrak{M}_{++}$  then  $m^c$  is equivalent to  $P$ ; it is obvious that the reverse is also true.

We start this section with the following theorem of the alternative, reminding of another war horse in mathematical finance (Farkas lemma) and therefore perhaps of its own interest.

**Theorem 1** *Let  $\mathcal{M}$  be a convex subset of  $\mathfrak{M}_+$  which is relatively  $\tau$  compact. Then either one of the following, mutually exclusive properties holds:*

- (i).  $P(f) > 0 = \sup \{m(f) : m \in \mathcal{M}\}$  for some  $f \in L_{++}^\infty$ ;
- (ii).  $\mathcal{M}^\tau$  admits a strictly positive element.

*Proof* (ii) contradicts (i) as  $\sup \{m(f) : m \in \mathcal{M}\} = \sup \{m(f) : m \in \mathcal{M}^\tau\}$  for each  $f \in L^\infty$ . Thus we only need to prove that (i) holds when (ii) fails. Remark that endowing  $\mathcal{M}$  with the  $\tau$  topology makes it into a Hausdorff, locally convex, topological vector space [20, proposition 21, p. 240].

For  $m \in \mathfrak{M}$ , let  $P_m$  be the component of  $P$  orthogonal to  $m^c$  in the Lebesgue decomposition of  $P$  and denote  $\mathcal{S}(m) = \{F \in \mathcal{F} : P_m(F^c) = m^c(F) = 0\}$  and

$$\eta = \inf \{P(F) : F \in \mathcal{S}(m), m \in \mathcal{M}^\tau\}$$

Let  $\langle m_r \rangle_{r \in \mathbb{N}}$  and  $\langle F_r \rangle_{r \in \mathbb{N}}$  be sequences in  $\mathcal{M}^\tau$  and  $\mathcal{F}$  respectively such that  $F_r \in \mathcal{S}(m_r)$  for  $r \geq 1$  and  $\eta = \lim_r P(F_r)$ . Define  $F_0 = \bigcap_r F_r$  and  $m_0 = \sum_r 2^{-r} m_r$ : then  $P(F_0) \leq \eta$ . If  $G_r \in \mathcal{F}$ ,  $m_r^\perp(G_r) = 0$  and  $P(G_r^c) < \epsilon 2^{-r}$  for  $r \geq 1$  and if we set  $G = \bigcap_r G_r$ , then  $P(G^c) \leq \epsilon$  while  $\sum_r 2^{-r} m_r^\perp(G) = 0$ : this proves that  $\sum_r 2^{-r} m_r^\perp \in \mathfrak{M}_+$  is purely finitely additive. Since  $\sum_r 2^{-r} m_r^c \in \mathfrak{M}_+$  is countably additive and the Yosida and Hewitt decomposition unique, we conclude that  $m_0^c = \sum_r 2^{-r} m_r^c$ . Clearly,  $m_0^c(F_0) = 0$ . If  $E \in \mathcal{F}$  and  $m_0^c(E) = 0$ , then  $m_r^c(E) = 0$  for each  $r$  so that  $P(EF_0^c) \leq \sum_r P(EF_r^c) = 0$ . In other words  $P \ll m_0^c$  in restriction to  $F_0^c$  so that  $P_{m_0}(F_0^c) = 0$  or, equivalently,  $F_0 \in \mathcal{S}(m_0)$ . However, since  $\mathcal{M}^\tau$  is convex and closed,  $m_0 \in \mathcal{M}^\tau$  and it then follows that  $P(F_0) \geq \eta$ . We have thus shown that  $\eta$  is actually attained so that if (ii) fails then  $\eta > 0$ .

Let  $m \in \mathcal{M}^\tau$  and  $n \in \mathbb{N}$ . Since  $m^\perp$  and  $P$  are orthogonal there exists a  $\mathcal{F}$  measurable subset  $F_m^n$  of  $F_m \in \mathcal{S}(m)$  such that  $m(F_m^n) = 0$  and  $P(F_m^n) > \eta(1 - 2^{-n})$  and by the axiom of choice we obtain a collection  $\{F_m^n : m \in \mathcal{M}^\tau, n \in \mathbb{N}\}$  of sets with this property. Define the set

$$\mathcal{U}_m^n = \{m' \in \mathcal{M}^\tau : m'(F_m^n) < 2^{-n}\}$$

As  $\mathcal{U}_m^n$  contains  $m$ ,  $\{\mathcal{U}_m^n : m \in \mathcal{M}^\tau\}$  is an open cover of the compact set  $\mathcal{M}^\tau$ . There exists then a finite collection  $\{\varphi_i : i = 1, \dots, I\}$  of continuous maps  $\varphi_i : \mathcal{M}^\tau \rightarrow [0, 1]$  each vanishing outside  $\mathcal{U}_{m_i}^n$

for some  $m_i \in \mathcal{M}^\tau$  and such that  $\sum_{i=1}^I \varphi_i(m) = 1$  for each  $m \in \mathcal{M}^\tau$  [20, proposition 16, p. 200]. Define the functions  $h_n : \mathcal{M}^\tau \rightarrow L_+^\infty$  and  $\rho_n : \mathcal{M}^\tau \times \mathcal{M}^\tau \rightarrow [0, 1]$  implicitly as

$$h_n(m) = \sum_{i=1}^I \varphi_i(m) F_{m_i}^n \quad \text{and} \quad \rho_n(m', m) = m'(h_n(m))$$

It is immediate that  $h_n$  is continuous and that therefore so is  $m \rightarrow \rho_n(m', m)$ ; moreover,  $m' \rightarrow \rho_n(m', m)$  is linear. By a theorem of Ky Fan [9, theorem 1, p. 103], it follows that there exists  $m_n \in \mathcal{M}^\tau$  such that

$$\sup_{m \in \mathcal{M}^\tau} \rho_n(m_n, m) \leq \sup_{m \in \mathcal{M}^\tau} \rho_n(m, m)$$

However, by construction if  $m \in \mathcal{M}^\tau$  and  $\varphi_i(m) > 0$  then  $m \in \mathcal{U}_{m_i}^n$  i.e.  $m(F_{m_i}^n) < 2^{-n}$  so that

$$\rho_n(m, m) = \sum_{i=1}^I \varphi_i(m) m(F_{m_i}^n) < 2^{-n}$$

Let  $h_n = h_n(m_n)$ . We have thus obtained a sequence  $\langle h_n \rangle_{n \in \mathbb{N}}$  such that  $m(h_n) < 2^{-n}$  for every  $m \in \mathcal{M}^\tau$  while

$$P(h_n) = \sum_{i=1}^I \varphi_i(m_n) P(F_{m_i}^n) \geq \sum_{i=1}^I \varphi_i(m_n) \eta (1 - 2^{-n}) = \eta (1 - 2^{-n})$$

Replacing  $h_n$  by an appropriate convex combination  $f'_n = \sum_{k=0}^{K_n} \alpha_k^n h_{n+k}$ , we obtain, by Komlos lemma [6, lemma A1.1, p. 515], that the sequence  $\langle f'_n \rangle_{n \in \mathbb{N}}$  admits a  $P$  a.s. limit  $f'$  i.e., by Egoroff theorem [8, theorem II.6.12, p. 149], that it converges to  $f'$  *uniformly* outside some  $F \in \mathcal{F}$  such that  $P(F^c) < \eta\delta$ , for  $\delta$  arbitrarily small. Let  $f_n = f'_n F$  and  $f = f' F$ . Given that  $0 \leq f_n \leq f'_n \leq 1$ , then for  $m \in \mathcal{M}^\tau$ ,

$$m(f) = \lim_n m(f_n) \leq \liminf_n m(f'_n) = \liminf_n \sum_{k=0}^{K_n} \alpha_k^n m(h_{n+k}) < \lim_n 2^{-n} = 0$$

while

$$P(f) = \lim_n P(f_n) \geq \lim_n P(f'_n) - P(F^c) = \lim_n \sum_{k=0}^{K_n} \alpha_k^n P(h_{n+k}) - P(F^c) \geq \eta(1 - \delta)$$

and (i) follows.

Theorem 1 may be restated by saying that if the convex sets  $\mathcal{M}^\tau$  and  $\mathfrak{M}_{++}$  are disjoint then they may be separated via a linear functional which is not only  $\tau$  continuous but strictly positive too. The interest for this conclusion is that in the general case  $\mathfrak{M}_{++}$  will neither be closed nor will it contain an interior point so that the claim is somewhat stronger than the usual separation theorems<sup>1</sup>.

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<sup>1</sup> An attempt to obtain a version of this theorem with  $L^\infty$  replaced by the space of bounded functions  $\mathfrak{B}(\mathcal{F})$  was made in [4, lemma A.6].

A typical use of Theorem 1 arises when the set  $\mathcal{M}$  consists of finitely additive probabilities separating convex subsets of  $L^\infty$ . Among such situations, a most interesting one is the  $L^\infty$  version of the theorem of Yan in which  $\mathcal{K} \subset L^\infty$  is a convex set containing the origin,  $\mathcal{C} = \mathcal{K} - L_+^\infty$  and  $\bar{\mathcal{C}}$  is the closure of  $\mathcal{C}$  in the norm topology of  $L^\infty$ . Let also

$$\mathcal{M}_{\mathcal{K}} = \left\{ m \in \mathfrak{M}_+ : m(\Omega) = 1, \sup_{k \in \mathcal{K}} m(k) < \infty \right\}$$

and

$$\mathcal{M}_{\mathcal{K}}^1 = \{ m \in \mathfrak{M}_+ : m[\bar{\mathcal{C}}] \leq 1, \|m\| \leq 1 \}$$

**Theorem 2** *The following are equivalent:*

1. for every  $f \in L_{++}^\infty$  there exists  $c > 0$  such that  $cf \notin \bar{\mathcal{C}}$ ;
2. for every  $F \in \mathcal{F}$  such that  $P(F) > 0$  there exists  $c > 0$  such that  $cF \notin \bar{\mathcal{C}}$ ;
3.  $\mathcal{M}_{\mathcal{K}}$  admits a strictly positive element.

*Proof* (3  $\rightarrow$  1). Let  $m$  be a strictly positive element of  $\mathcal{M}_{\mathcal{K}}$ ,  $f \in L_{++}^\infty$  and  $c > 0$  be such that  $cf \in \bar{\mathcal{C}}$ :  $\sup_{k \in \mathcal{K}} m(k) = \sup_{x \in \bar{\mathcal{C}}} m(x) \geq cm(f) > 0$ . Therefore, if 1 fails so does 3. The implication (1  $\rightarrow$  2) is obvious.

(2  $\rightarrow$  3). Let  $cF \notin \bar{\mathcal{C}}$  and  $\phi_F$  be a continuous, non trivial linear functional on  $L^\infty$  separating  $\{cF\}$  and  $\bar{\mathcal{C}}$  [8, corollary V.2.11, p. 418] and let  $\bar{m}_F$  the element of  $\mathfrak{M}$  representing  $\phi_F$ .  $0 \in \mathcal{K}$  implies that  $\bar{m}_F[\bar{\mathcal{C}}] < a < c\bar{m}_F(F)$  for some  $a > 0$ ;  $-L_+^\infty \subset \bar{\mathcal{C}}$  implies that  $\bar{m}_F[-L_+^\infty]$  is a convex cone in  $(-\infty, a)$  i.e.  $\bar{m}_F[L_+^\infty] \geq 0$  so that  $\bar{m}_F \in \mathfrak{M}_+$  and  $\bar{m}_F(\Omega) > 0$  (as  $\phi_F$  is non trivial). Letting  $m_F = [(1+c)\|\bar{m}_F\|]^{-1} \bar{m}_F$  we conclude that  $m_F \in \mathcal{M}_{\mathcal{K}}^1$  and  $m_F(F) > 0$ . Thus  $\mathcal{M}_{\mathcal{K}}^1$  is non empty, convex and  $\tau$  compact [8, lemma I.5.7(a), p. 17 and theorem V.4.2, p. 424]. Moreover, it fails to possess property (i) of Theorem 1 and admits, as a consequence, a strictly positive element  $\bar{m}$ . The claim is established replacing  $\bar{m}$  by  $m = \|\bar{m}\|^{-1} \bar{m}$ .

The  $L^p$ ,  $1 \leq p < \infty$  versions of this theorem considered by Yan [23] and by Ansel and Stricker [1] rely crucially on the fact that in that framework separating measures admit a density, a property that does not carry through to  $\mathfrak{M}$  as the Radon Nikodym theorem fails in the absence of countable additivity. The minimax inequality exploited in the proof of Theorem 1 allows to overcome such difficulty.<sup>2</sup>

### 3 Applications to Mathematical Finance.

Let  $S = (S_t : t \in \mathbb{R}_+)$  be a continuous,  $\mathbb{R}^d$  valued process over the probability space  $(\Omega, \mathcal{F}, P)$  endowed with a filtration  $(\mathcal{F}_t : t \in \mathbb{R}_+)$  satisfying the usual assumptions of completeness and right continuity and, without loss of generality, assume  $\mathcal{F} = \sigma\left(\bigcup_{t \in \mathbb{R}_+} \mathcal{F}_t\right)$ . Denote by  $\mathcal{T}$  the set of all stopping times on the underlying filtration and, if  $\tau \in \mathcal{T}$ , let  $\mathcal{T}_\tau = \{v \in \mathcal{T} : P(v > \tau) = 1\}$ .  $S$  shall

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<sup>2</sup> After this paper was completed, I came across the work of Rokhlin [19] in which a form of Theorem 2 is proved for the special case in which  $\mathcal{C}$  is a convex cone by convex duality methods.

represent asset prices in discounted units. Let  $\Theta$  be the set of  $\mathbb{R}^d$  valued, simple processes  $\theta$  such that  $l_\theta \equiv \sup_{t \in \mathbb{R}_+} \left| \int_0^t \theta dS \right| \in L^\infty$ . Write for simplicity  $K^\theta = \left( \int_0^t \theta dS : t \in \mathbb{R}_+ \right)$ , the process describing the (discounted) returns of the investment strategy  $\theta \in \Theta$  and  $K(\Theta) = \{K^\theta : \theta \in \Theta\}$ . Observe that, by continuity, each of the components of the vector valued process  $S$  is locally in  $K(\Theta)$ . It should be remarked that the definition of the stochastic integral  $\int \theta dS$  is necessarily limited to the case in which  $\theta$  is a simple process, unless one is prepared to make more stringent assumptions on the nature of the price process  $S$ . Define the sets

$$\mathcal{K} = \{K_\infty^\theta : \theta \in \Theta\} \text{ and } \mathcal{C} = \mathcal{K} - L_+^\infty$$

Assume that there are no free lunches in the sense initially introduced by Delbaen and Schachermayer [6], i.e.

$$\bar{\mathcal{C}} \cap L_+^\infty = \{0\} \quad (1)$$

A weaker notion is that of absence of arbitrage opportunities, defined as

$$\mathcal{C} \cap L_+^\infty = \{0\} \quad (2)$$

Of course, since  $0 \in \mathcal{K}$ , then Theorem 2 establishes that if (1) holds there exists a strictly positive  $m \in \mathcal{M}_\mathcal{K}$ : this clearly implies that  $m[\bar{\mathcal{C}}] \leq 0$  and  $m[\mathcal{C}_0] = 0$  for any linear subspace  $\mathcal{C}_0$  of  $\mathcal{C}$  – such as  $\mathcal{K}$ . On the other hand, if (2) holds then, given that  $L_{++}^\infty$  has an internal point – e.g.  $\Omega_-$ , we conclude [8, theorem V.2.8, p. 417] that there exists a non null element  $m \in \mathfrak{M}$  such that  $m[\mathcal{C}] \leq 0 \leq m[L_+^\infty]$  and that can therefore be normalized to be a finitely additive probability. For the rest of this section  $m$  will be fixed.

The application of Yan theorem to financial modelling is therefore related to the absence of free lunches. It should be remarked that, differently from the  $L^p$  case treated in [1], the  $L^\infty$  definition of a free lunch introduced by Delbaen and Schachermayer did not foster the proof of a corresponding version of Yan theorem. In fact, given the extended set of trading strategies considered in [6] – available upon assuming the semimartingale nature of the price process – the norm and the weak\* closure of  $\mathcal{C}$  turn out to be equivalent, in the absence of free lunches. In a less perfect market, such as the one considered here, this remarkable property fails so that Theorem 2 above gains importance. The issue is now to show that, despite finite additivity, it is still possible to obtain a nice and tractable pricing rule.

Let  $\tau \in \mathcal{T}$  and denote by  $m_\tau$  the restriction of  $m$  to  $\mathcal{F}_\tau$  and by  $m_\tau^c + m_\tau^\perp$  its Yosida and Hewitt decomposition. Since  $m_\tau^c$  coincides with the restriction to  $\mathcal{F}_\tau$  of the outer measure generated by  $m_\tau$  [3, theorem 10.2.2, p. 242] – i.e.  $m_\tau^c(F) = \inf \{ \sum_n m(F_n) : F_n \in \mathcal{F}_\tau, F = \bigcup_n F_n \}$  – and given that  $F\{\tau \leq v\} \in \mathcal{F}_v$  when  $v \in \mathcal{T}$  and  $F \in \mathcal{F}_\tau$  we conclude

$$\begin{aligned} m_v^c(F; \tau \leq v) &= \inf \left\{ \sum_n m(G_n) : G_n \in \mathcal{F}_v, F\{\tau \leq v\} = \bigcup_n G_n \right\} \\ &\leq \inf \left\{ \sum_k m(F_k; \tau \leq v) : F_n \in \mathcal{F}_\tau, F = \bigcup_k F_k \right\} \\ &= m_\tau^c(F; \tau \leq v) \end{aligned} \quad (3)$$

– analogously,  $m_v^\perp(F; \tau \leq v) \geq m_\tau^\perp(F; \tau \leq v)$ . Furthermore, if  $v \in \mathcal{T}_\tau$

$$(m_\tau^c - m_v^c)(F) = (m_\tau - m_v)(F) + (m_v^\perp - m_\tau^\perp)(F) = (m_v^\perp - m_\tau^\perp)(F) \quad (4)$$

In order to take care of the rather delicate issue of “regularity” we introduce a modified decomposition of  $m_\tau$ . To this end remark that  $\mathcal{T}_\tau$  is a directed set if we let  $v' \succsim v$  whenever  $P(v' \leq v) = 1$ . We can define the set functions

$$m_{\tau+}^c = \lim_{v \in \mathcal{T}_\tau} m_v^c | \mathcal{F}_\tau \quad \text{and} \quad m_{\tau+}^\perp = \lim_{v \in \mathcal{T}_\tau} m_v^\perp | \mathcal{F}_\tau \quad (5)$$

so that  $m_\tau = m_{\tau+}^c + m_{\tau+}^\perp$ . (5) implies that  $m_{\tau+}^c$  and  $m_{\tau+}^\perp$  are positive set functions on  $\mathcal{F}_\tau$  and, by (3), that

$$m_\tau^c(F) \geq m_{\tau+}^c(F) \geq m_v^c(F) \geq m^c(F) \quad (6)$$

for each  $F \in \mathcal{F}_\tau$  and  $v \in \mathcal{T}_\tau$ .  $m_{\tau+}^c$  is then countably additive and absolutely continuous with respect to  $P$ . Let  $Y_\tau \in L^1(\Omega, \mathcal{F}_\tau, P)_+$  be the corresponding Radon Nikodym derivative. We list in the following lemma some useful properties.

**Lemma 1** *Let  $Y = (Y_t : t \in \mathbb{R}_+)$  and  $m_{\tau+}^c$  be defined as above.*

- (i).  *$Y$  is a positive supermartingale admitting a right continuous modification;*
- (ii). *if there are no free lunches i.e. (1) holds and if  $Y_\infty = \lim_t Y_t$ ,  $P$  a.s. then  $P(Y_\infty > 0) = 1$ ;*
- (iii). *for any sequence  $\langle \tau_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{T}$  such that  $\tau_{n+1} \in \mathcal{T}_{\tau_n}$ ,  $\sum_n \|m_{\tau_{n+1}}^c - m_{\tau_n}^c\| < 1$ .*

*Proof* Let  $F \in \mathcal{F}_t$ ,  $t < u$  and  $v \in \mathcal{T}_u$ . By (6),  $m_{t+}^c(F) \geq m_v^c(F)$  i.e.  $m_{t+}^c(F) \geq m_{u+}^c(F)$  from which it follows that  $Y$  is a positive supermartingale and admits an a.s. limit  $Y_\infty$  by Doob’s limit theorem. If  $\tau \in \mathcal{T}_t$ , then by (3)

$$m_\tau^c(\Omega) \leq m_{t+2^{-n}}^c(\tau > t + 2^{-n}) + m_\tau^c(\tau \leq t + 2^{-n}) \leq \lim_{u > t, u \downarrow t} m_{u+}^c(\Omega) + m_\tau^c(\tau \leq t + 2^{-n})$$

Since  $\lim_n m_\tau^c(\tau \leq t + 2^{-n}) = 0$  we conclude that  $m_{t+}^c(\Omega) \leq \lim_{u > t, u \downarrow t} m_{u+}^c(\Omega)$ . In other words, the function  $t \rightarrow P(Y_t)$  is right continuous so that  $Y$  admits a right continuous modification by virtue of a cornerstone result of Meyer [16, VI, T4, p. 95]. Let  $y$  be the Radon Nikodym derivative of  $m^c$  with respect to  $P$ : under (1),  $P(y = 0) = 0$ . By (3) and martingale convergence we obtain the inequality

$$Y_\infty = \lim_t Y_t \geq \lim_t P(y | \mathcal{F}_t) = y$$

from which the second claim readily follows. Let  $\langle \tau_n \rangle_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{T}$  with  $\tau_{n+1} \in \mathcal{T}_{\tau_n}$ . Then  $m_{\tau_{n+1}}^c \geq m_{\tau_n}^c$  and  $m_{\tau_{n+1}}^c(\Omega) \geq m_{\tau_n}^c(\Omega)$  so that

$$\sum_n \|m_{\tau_{n+1}}^c - m_{\tau_n}^c\| = \sum_n (m_{\tau_n}^c - m_{\tau_{n+1}}^c)(\Omega) \leq \sum_n (m_{\tau_n}^c - m_{\tau_{n+1}}^c)(\Omega) \leq m_0^c(\Omega) \leq 1$$

Without loss of generality we thus may and will assume that  $Y$  has right continuous paths. Let  $Y = M - A$  be the semimartingale decomposition of  $Y$ , into a positive local martingale  $M$  and an increasing, predictable, integrable process  $A$ . Denote by  $\mathcal{T}_Y$  the set of stopping times  $\tau$  such that the stopped process  $Y^\tau$  is of class  $D$ .

We proceed now to the explicit construction of a class of return processes. To this end fix  $\theta \in \Theta$  and define the sequence  $\mathcal{U}^n = \langle v_k^n \rangle_{k \in \mathbb{N}}$  of stopping times by letting  $v_0^n = 0$  and, for  $k \geq 1$ ,

$$v_k^n = \inf \left\{ t > v_{k-1}^n : \left| K_t^\theta - K_{v_{k-1}^n}^\theta \right| \geq 2^{-n} \text{ or } t > v_{k-1}^n + 2^{-n} \right\}$$

$\mathcal{U}^n$  is clearly an adapted subdivision and  $\langle \mathcal{U}^n \rangle_{n \in \mathbb{N}}$  a Riemann sequence, according to the terminology proposed in [10, p. 51]. As  $\langle v_k^n \rangle_{k \in \mathbb{N}}$  increases  $P$  a.s. to  $\infty$ , let  $I_n$  be an integer such that  $P(v_{I_n}^n > 2^n) > 1 - 2^{-n}$ .

For each  $k \geq 0$ , let  $F_k^n \in \mathcal{F}_{v_k^n}$  be such that (i)  $F_{k+1}^n \subset F_k^n$ , (ii)  $m_{v_k^n}^\perp(F_k^n) = 0$  and (iii)  $P(F_k^n) > 1 - \frac{2^{-n}}{1+2^{-k}}$  (so that  $P(\bigcap_k F_k^n) \geq 1 - 2^{-n}$ ). Let

$$\mathcal{C}_\theta = \left\{ a \sum_{k=1}^I F_{k-1}^n \left( K_{v_k^n \wedge \tau}^\theta - K_{v_{k-1}^n \wedge \tau}^\theta \right) : I, n \in \mathbb{N}, a \in \mathbb{R}, \tau \in \mathcal{T}_Y \right\} \quad (7)$$

Consider for the moment  $\tau \in \mathcal{T}_Y$  as fixed and, to improve notation, write  $\tau_k^n$  for  $v_k^n \wedge \tau$ . Observe that  $\{F_{k-1}^n; v_{k-1}^n < \tau\} \in \mathcal{F}_{\tau_{k-1}^n}$  and that  $m_{\tau_{k-1}^n}^\perp(F_{k-1}^n; v_{k-1}^n < \tau) \leq m_{v_{k-1}^n}^\perp(F_{k-1}^n) = 0$ . Therefore, by (4)

$$\begin{aligned} \left| m_{\tau_k^n}^\perp \left( F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right) \right| &\leq 2^{-n} m_{\tau_k^n}^\perp(F_{k-1}^n; v_{k-1}^n < \tau) \\ &= 2^{-n} \left( m_{\tau_k^n}^\perp - m_{\tau_{k-1}^n}^\perp \right) (F_{k-1}^n; v_{k-1}^n < \tau) \\ &= 2^{-n} \left( m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c \right) (F_{k-1}^n; v_{k-1}^n < \tau) \\ &\leq 2^{-n} \left[ \left( m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c \right) (F_{k-1}^n) + \left\| m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c \right\| \right] \\ &= 2^{-n} \left[ P \left( \left( Y_{\tau_{k-1}^n} - Y_{\tau_k^n} \right) F_{k-1}^n \right) + \left\| m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c \right\| \right] \end{aligned}$$

while on the other hand

$$m_{\tau_k^n}^c \left( F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right) = P \left( Y_{\tau_k^n} F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right)$$

As a consequence, if  $|a| = 1$

$$\begin{aligned} m \left( a F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right) &= \left( m_{\tau_k^n}^c + m_{\tau_k^n}^\perp \right) \left( a F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right) \\ &\geq a P \left( Y_{\tau_k^n} F_{k-1}^n \left( K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta \right) \right) \\ &\quad - 2^{-n} \left[ P \left( \left( Y_{\tau_{k-1}^n} - Y_{\tau_k^n} \right) F_{k-1}^n \right) + \left\| m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c \right\| \right] \end{aligned}$$



Since the sequence  $\langle F_k^n \rangle_{k \in \mathbb{N}}$  is decreasing, we further establish that

$$\begin{aligned} \sum_{k=1}^{I_n} (Y_{\tau_{k-1}^n} - Y_{\tau_k^n}) F_{k-1}^n &= \sum_{k=1}^{I_n-1} (Y_0 - Y_{\tau_k^n}) F_{k-1}^n F_k^{nc} + (Y_0 - Y_{\tau_{I_n}^n}) F_{I_n-1}^n \\ &\leq Y_0 \left( F_{I_n-1}^n \cup \bigcup_{k=1}^{I_n-1} F_{k-1}^n F_k^{nc} \right) \\ &\leq Y_0 \end{aligned}$$

i.e.  $\sum_{k=1}^{I_n} \left\{ (Y_{\tau_{k-1}^n} - Y_{\tau_k^n}) F_{k-1}^n + \|m_{\tau_{k-1}^n}^c - m_{\tau_k^n}^c\| \right\} \leq (Y_0 + 1)$  by Lemma 1(iii). Given that  $\mathcal{C}_\theta$  is a convex cone in  $\mathcal{C}$  and given the preceding developments we conclude

$$0 \geq m \left( \sum_{k=1}^{I_n} a F_{k-1}^n (K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta) \right) \geq a P \sum_{k=1}^{I_n} Y_{\tau_k^n} F_{k-1}^n (K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta) - 2^{-n} [P(Y_0) + 1]$$

In other words

$$0 = \lim_n P \sum_{k=1}^{I_n} Y_{\tau_k^n} F_{k-1}^n (K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta) \quad (8)$$

**Lemma 2** Let  $\sigma \in \mathcal{T}$ . Then, with the above notation

$$0 = P \left\{ Y_\sigma^\tau K_{\tau \wedge \sigma}^\theta + \int_0^{\tau \wedge \sigma} K^\theta dA \right\} \quad (9)$$

*Proof* Given that  $K_0^\theta = 0$  (by definition), the sum appearing in (8) may be rewritten as

$$Y_{\tau_{I_n}^n} K_{\tau_{I_n}^n}^\theta F_{I_n-1}^n + \sum_{k=1}^{I_n-1} Y_{\tau_k^n} K_{\tau_k^n}^\theta F_{k-1}^n F_k^{nc} - \sum_{k=1}^{I_n} F_{k-1}^n K_{\tau_{k-1}^n}^\theta (Y_{\tau_k^n} - Y_{\tau_{k-1}^n})$$

Furthermore,  $\left| P \left( Y_{\tau_k^n} K_{\tau_k^n}^\theta F_{k-1}^n F_k^{nc} \right) \right| \leq l_\theta P(M_\tau F_{k-1}^n F_k^{nc})$  and  $\left| P \left( F_{k-1}^n K_{\tau_{k-1}^n}^\theta (Y_{\tau_k^n} - Y_{\tau_{k-1}^n}) \right) \right| \leq l_\theta P(F_{I_n-1}^{nc} (A_{\tau_k^n} - A_{\tau_{k-1}^n}))$  imply

$$\left| P \sum_{k=1}^{I_n-1} Y_{\tau_k^n} K_{\tau_k^n}^\theta F_{k-1}^n F_k^{nc} \right| \leq l_\theta P \left( M_\tau \bigcup_{k=1}^{I_n-1} F_{k-1}^n F_k^{nc} \right) \leq l_\theta P(M_\tau F_{I_n-1}^{nc})$$

and

$$\left| P \sum_{k=1}^{I_n} F_{k-1}^{nc} K_{\tau_{k-1}^n}^\theta (Y_{\tau_k^n} - Y_{\tau_{k-1}^n}) \right| \leq l_\theta P \left( F_{I_n-1}^{nc} \sum_{k=1}^{I_n} (A_{\tau_k^n} - A_{\tau_{k-1}^n}) \right) \leq l_\theta P(M_\tau F_{I_n-1}^{nc})$$

respectively. Given that the sequence  $\langle Y_{\tau_{I_n}^n} K_{\tau_{I_n}^n}^\theta F_{I_n-1}^n \rangle_{n \in \mathbb{N}}$  is uniformly integrable, that  $\lim_n P(M_\tau F_{I_n-1}^{nc}) = 0$  and bounded convergence for the stochastic integral [10, theorem I.4.31(iii), p. 46], (8) translates

into

$$\begin{aligned}
0 &= \lim_n P \left\{ Y_{\tau_{I_n}^n} K_{\tau_{I_n}^n}^\theta F_{I_n-1}^n - \sum_{k=1}^{I_n} K_{\tau_{k-1}^n}^\theta \left( Y_{\tau_k^n} - Y_{\tau_{k-1}^n} \right) \right\} \\
&= \lim_n P \left\{ Y_{\tau_{I_n}^n} K_{\tau_{I_n}^n}^\theta F_{I_n-1}^n + \sum_{k=1}^{I_n} K_{\tau_{k-1}^n}^\theta \left( A_{\tau_k^n} - A_{\tau_{k-1}^n} \right) \right\} \\
&= P \left\{ Y_\tau K_\tau^\theta + \int_0^\tau K^\theta dA \right\}
\end{aligned}$$

(9) follows from the fact that  $\sigma \wedge \tau \in \mathcal{T}_Y$  whenever  $\sigma \in \mathcal{T}$  and  $\tau \in \mathcal{T}_Y$ .

The process  $YK^\theta + \int K^\theta dA$  is càdlàg, starts at 0 and, upon stopping at  $\tau$ , satisfies (9) for any stopping time  $\sigma$  i.e. [10, lemma 1.44, p. 11] it is a local martingale and therefore  $YK^\theta$  a semimartingale. Since  $Y$  is strictly positive the process  $Y^{-1}$  is well defined and, given that the inverse function is convex over the set  $]0, \infty[$ , it is itself a semimartingale [18, theorem VI.1.1, p. 221]. Then so is  $K^\theta$ , being the product of two semimartingales. Let then  $M^\theta + V^\theta$  be the semimartingale decomposition of  $K^\theta$  – with  $V^\theta$  predictable and of locally integrable variation and  $M^\theta$  a local martingale.

Exploiting the semimartingale nature of  $K^\theta$  and integration by parts we obtain

$$YK^\theta + \int K^\theta dA - \int K^\theta dM - \int Y_- dM^\theta = \int Y_- dV^\theta + \langle K^\theta, Y^c \rangle \quad (10)$$

It follows from (9) that on either side of (10) appears a local martingale (left hand side) which is predictable and of locally integrable variation (right hand side) and so is the process  $V^\theta + \langle K^\theta, \int Y_-^{-1} dM^c \rangle$  which must therefore vanish, by uniqueness of the Doob Meyer decomposition. Define the local martingale

$$Z = \mathcal{E} \int Y_-^{-1} dM \quad (11)$$

where  $\mathcal{E}$  is the exponential martingale of Doléans-Dade ( $\mathcal{L}$  will denote the stochastic logarithm).

What precedes is restated in the Proposition that follows in which we use the following definition, borrowed from [21]

**Definition 1** *A stochastic process  $X$  is a martingale density for  $S$  if  $X$  is a positive local martingale starting at  $X_0 = 1$  and such that  $XS$  is a local martingale;  $X$  is strictly positive if  $P(X_\infty > 0) = 1$ .*

In many contributions to finance (see [2], among others) a martingale density for the discounted price process  $S$  is termed *stochastic discount factor* or *market price of risk*.

**Proposition 1** *The absence of free lunches, as defined by (1), implies that the price process  $S$  is a semimartingale and that there exists a strictly positive martingale density for  $S$ .*

*Proof* That  $S$  is a semimartingale follows from the fact that  $K^\theta$  is a semimartingale for each  $\theta \in \Theta$  and that, as remarked above, the components of  $S$  are locally in  $K(\Theta)$ . By construction,  $Z$  is a positive local martingale starting at  $Z_0 = 1$ . To show that  $\{Z_\infty = 0\} \subset \{Y_\infty = 0\}$  recall that

$\langle U^c, A^c \rangle = 0$  for any semimartingale  $U$  [10, 4.49.(d), p. 52] and that  $M = Y + A$ . We then obtain from (11) that for each  $t \in \mathbb{R}_+ \cup \{\infty\}$

$$\begin{aligned}
Z_t &= \exp \left\{ \int_0^t Y_-^{-1} dM - \frac{1}{2} \int_0^t Y_-^{-2} d\langle M^c, M^c \rangle \right\} \prod_{s \leq t} e^{-Y_{s-}^{-1} \Delta M_s} (1 + Y_{s-}^{-1} \Delta M_s) \\
&\geq \exp \left\{ \int_0^t Y_-^{-1} dM - \frac{1}{2} \int_0^t Y_-^{-2} d\langle M^c, M^c \rangle \right\} \prod_{s \leq t} e^{-Y_{s-}^{-1} \Delta M_s} (1 + Y_{s-}^{-1} \Delta Y_s) \\
&= \exp \left\{ \int_0^t Y_-^{-1} dY - \frac{1}{2} \int_0^t Y_-^{-2} d\langle Y^c, Y^c \rangle + \int_0^t Y_-^{-1} dA^c \right\} \prod_{s \leq t} e^{-Y_{s-}^{-1} \Delta Y_s} (1 + Y_{s-}^{-1} \Delta Y_s) \quad (12) \\
&= Y_t \exp \left\{ \int_0^t Y_-^{-1} dA^c \right\} \\
&\geq Y_t
\end{aligned}$$

Therefore, if (1) holds  $Z$  is a strictly positive local martingale, by Lemma 1(ii). To see that  $Z$  is a martingale density we just exploit once again integration by parts, thus obtaining  $K^\theta Z = \int K^\theta dZ + \int Z_- dM^\theta$ .

Proposition 1 should be compared to [6, theorem 7.6(a), p. 509] where it is claimed that the absence of free lunches for simple integrands and over a bounded time interval implies the semimartingale property and the absence of free lunches for *general* integrands i.e. the existence of a probability measure  $Q$  equivalent to  $P$  and transforming asset returns into local martingales<sup>3</sup>. To strengthen this analogy, remark that in restriction to any *stochastic* interval  $[[0, \tau]]$  where  $\tau \in \mathcal{T}_Z$  (i.e.  $Z^\tau$  is of class  $D$ ) there exists an equivalent local martingale measure, simply defined as  $dQ_\tau = Z_\tau dP$ . However, we obtain the additional conclusion that if  $v \in \mathcal{T}_\tau \cap \mathcal{T}_Z$  then the corresponding local martingale measure  $Q_v$  relative to  $[[0, v]]$  satisfies  $Q_v|_{\mathcal{F}_\tau} = Q_\tau$ .

It is always an open question what is the “right” definition of an arbitrage opportunity. Although in the preceding proposition we considered the property that markets do not admit free lunches, the definition of an arbitrage opportunity implicit in (2) is definitely more sound in economic terms as it does not involve limit points of investment profits. We have already remarked that under (2) it is still possible to recover a separating, finitely additive probability measure  $m$ . In general this will not be strictly positive so that, letting  $Y$  have the same meaning as above, it is useful to define the stopping time

$$T = \inf \{t : Y_t = 0\}$$

An open issue is clearly that of assessing the probability of the event  $\{T < \infty\}$ .

**Corollary 1** *The absence of arbitrage opportunities, as defined by (2), implies that the price process  $S$  stopped at  $T$  is a semimartingale which admits a martingale density.*

*Proof* On re-reading what precedes it emerges clearly that (9) was derived without any reference to the property of  $Y$  being strictly positive. We thus still deduce from (9) that  $YK^\theta$  is a semimartingale

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<sup>3</sup> This claim is now recognized as being incorrect.

as well as the stopped process  $K^{\theta, T_k}$  where  $T_k = \inf \{t : Y_t < 2^{-k}\}$ . But then, since  $T$  is the a.s. limit of the increasing sequence  $\langle T_k \rangle_{k \in \mathbb{N}}$ ,  $K^{\theta, T}$  is a semimartingale [17, corollary, p. 54]. Observe that  $M = A$  on  $\{Y_- = 0\}$  and therefore  $\int \{Y_- = 0\} dM$  is a predictable local martingale of integrable variation and therefore null. The process  $Z$  in (11) is thus still well defined and by (12) it vanishes when  $Y$  does. Again by integration by parts we conclude that  $ZK^{\theta, T} = ZK^{\theta}$  is a local martingale.

It should not be too surprising that under (2) the martingale density may fail to be strictly positive. The corresponding situation in the context of martingale *measures* was illustrated in a well known example [6, example 7.7, p. 509] and then further considered, for the case of continuous price process, in [7]. In the latter reference, it is actually shown first that the absence of arbitrage opportunities implies the existence of a martingale density and, second, that this is associated to an absolutely continuous local martingale measure. However, both implications require that arbitrage opportunities be defined with respect to general integrands, not just simple processes. Corollary 1 is therefore of its own interest.

In order to further clarify the relationship between arbitrage opportunities, free lunches and martingale densities we conclude with the following result, already mentioned in [7], but of which we offer a new proof.

**Lemma 3** *If there are no arbitrage opportunities and there exists a strictly positive martingale density, then there are no free lunches.*

*Proof* We recall the following fact [6, proposition 3.5, p. 476]: under (2)  $\|K_{\infty}^{\theta} \wedge 0\| \geq \|K_{\tau}^{\theta} \wedge 0\|$  for each  $\tau \in \mathcal{T}$ . Let  $\langle \tau_n \rangle_{n \in \mathbb{N}}$  be a localizing sequence of stopping times along which  $Z$  is a uniformly integrable martingale. Then,

$$0 = \lim_n P(Z_{\tau_n} K_{\tau_n}^{\theta}) = \lim_n P(Z_{\tau_n} (K_{\tau_n}^{\theta} + \|K_{\infty}^{\theta} \wedge 0\|)) - \|K_{\infty}^{\theta} \wedge 0\| \geq P(Z_{\infty} K_{\infty}^{\theta}) - \|K_{\infty}^{\theta} \wedge 0\|$$

i.e.  $P(Z_{\infty} K_{\infty}^{\theta}) \leq \|K_{\infty}^{\theta} \wedge 0\|$ . If  $\langle x_n \rangle_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{C}$  (so that  $K_{\infty}^{\theta_n} \geq x_n$  for some  $\theta_n \in \Theta$ ) converging in  $L^{\infty}$  to  $x \geq 0$ , then

$$P(Z_{\infty} x) = \lim_n P(Z_{\infty} x_n) \leq \lim_n P(Z_{\infty} K_{\infty}^{\theta_n}) \leq \lim_n \|K_{\infty}^{\theta_n} \wedge 0\| \leq \lim_n \|x_n \wedge 0\| \leq \lim_n \|x - x_n\|$$

But then if  $P(Z_{\infty} = 0) = 0$  one is forced to conclude that  $P(x = 0) = 1$ .

#### 4 Comments.

A martingale density, unless of class  $D$ , does not induce a martingale measure, rightly considered as a basic tool in much of the asset pricing literature. Further to that, even when strictly positive the existence of a martingale density is not sufficient, in general, to exclude arbitrage opportunities. For either reason, versions of the *FTAP* delivering the existence of an equivalent martingale measure may result more appealing than Proposition 1 above and in fact, to our knowledge, there has been no previous rigorous characterization of martingale densities on the basis of the no arbitrage principle. The existence of a martingale measure may be obtained either by enlarging the set of admissible

trading strategies, as in [6], or by adopting a definition of free lunch more restrictive than the one adopted above, as in [1] and [22].

It should be remarked, however, that, despite its desirable implications, the existence of a martingale measure places considerable constraints on the price process, particularly on volatility. An example of this is provided by the so called *strong non degeneracy* condition imposed, among others, in [5, p. 654], and consisting in a lower bound on asset volatility that guarantees the martingale nature of the market price of risk – or even its square integrability. However when we come to financial modelling, volatility is a key element in the explanation of some of the stylized facts of finance. As a consequence, although all financial models invariably admit a martingale density, those which admit a martingale measure are hardly the case.

Further restrictions to asset pricing models are implicit in the existence of a martingale measure  $Q$ . The pricing formula  $S_0 = Q(S_\infty)$ , which applies in all models in which the discounted price process is bounded, precludes the existence of pricing bubbles. However, when prices are positive and  $Z$  is only a martingale density, a straightforward application of Fatou lemma delivers  $S_0 \geq P(Z_\infty S_\infty)$ . As remarked in [15], it is reasonable to interpret  $P(Z_\infty S_\infty)$  as the fundamental value of the asset and, consequently, the quantity  $\beta(S) = S_0 - P(Z_\infty S_\infty)$  as the “bubble” part of the asset price. In [15], however, it is assumed that the martingale density is strictly positive but it is far from clear how this relates to the no arbitrage principle.

## 5 Conclusions.

After proving that Yan theorem remains valid after replacing  $L^p$ ,  $1 \leq p < \infty$ , by  $L^\infty$  we have considered a financial market characterized by a continuous vector process  $S$  describing asset prices in discounted units and absence of free lunches. These have been defined in terms of the  $L^\infty$  topology, analogously to [6]. In Proposition 1 we have proved that under these assumptions asset prices are necessarily semimartingales and that there exists a strictly positive martingale density, i.e. a local martingale  $Z$  such that  $Z_0 = 1$ ,  $P(Z_\infty > 0) = 1$  and that  $ZS$  is a local martingale. The novelty of this result is that it is formulated in terms of the density process  $Z$  rather than of a martingale measure which, under our assumptions, need not exist. Although the existence of a martingale measure is indeed a desirable property it is typically obtained by imposing burdensome constraints on the volatility process and in fact most models fail to satisfy it. Reformulating our problem in terms of arbitrage opportunities rather than free lunches allows the weaker conclusion that there exists a martingale density  $Z$  and that prices, stopped by the time  $Z$  expires, are semimartingales. Even when strictly positive, the existence of a martingale density is not sufficient to exclude arbitrage opportunities.

## References

1. Ansel J.P., Stricker C. Quelques Remarques sur un Théoreme de Yan, *Séminaire de Probabilité XXIV, Lecture Notes in Mathematics* **1426** (1990), 266-274
2. Back, K. Asset Pricing for General Processes, *Journal of Mathematical Economics*, 1991, 20(4): 371-395

3. Bhaskara Rao, K.P.S., Bhaskara Rao, M. Theory of Charges, Academic Press, London, 1983
4. Cassese, G. Asset Pricing with no Exogenous Probability, Mathematical Finance, 2007, to appear
5. Cvitanic, J., Karatzas, I. Hedging Contingent Claims with Constrained Portfolios, Annals of Applied Probability, 1993, 3(2): 652-681
6. Delbaen, F., Schachermayer, W. A General Version of the Fundamental Theorem of Asset Pricing, Mathematische Annalen, 1994, 300(1): 463-520
7. Delbaen, F., Schachermayer, W. The Existence of Absolutely Continuous Local Martingale Measures, Annals of Applied Probability, 1995, 5(4): 926-945
8. Dunford, N., Schwartz, J. T. Linear Operators. Part I, New York: Wiley 1988
9. Fan, K. A Minimax Inequality and Applications In: Inequalities III, ed. by O. Shisha, Academic Press, New York, 1972, 103-113
10. Jacod, J., Shiryaev, A. Limit Theorems for Stochastic Processes, Springer-Verlag, Berlin, 1987
11. Jouini, E., Napp, C., Schachermayer W. Arbitrage and State Price Deflators in a General Intertemporal Framework, Journal of Mathematical Economics, 2005, 41(): 722-734
12. Karatzas, I., Žitkovic, G. Optimal Consumption from Investment and Random Endowment in Incomplete Semimartingale Markets, Annals of Probability, 2003, 31(4), 1821-1858
13. Kreps, D. M. Arbitrage and Equilibrium in Economies with Infinitely Many Commodities, Journal of Mathematical Economics, 1981, 8(1): 15-35
14. Levental, S., Skorohod, A. V. A Necessary and Sufficient Condition for Absence of Arbitrage with Tame Portfolios, Annals of Applied Probability, 1995, 5(4): 906-925
15. Lowenstein, M., Willard, G. Rational Equilibrium Asset-Pricing Bubbles in Continuous Trading Models, Journal of Economic of Theory, 2000, 91(1): 17-58
16. Meyer, P. A. Probability and Potential, Blaisdell, Waltham, 1966
17. Protter, P. Stochastic Integration and Differential Equations, Berlin, Springer-Verlag, 2004
18. Revusz, D., Yor, M. Continuous Martingales and Brownian Motion, Berlin, Springer-Verlag, 1999
19. Rokhlin, D. B. The Kreps-Yan Theorem for  $L^\infty$ , International Journal of Mathematics and Mathematical Sciences, 2005, 2005(17): 2749-2756
20. Royden, H. Real Analysis, MacMillan, New York, 1988
21. Schweizer, M. Martingale Densities for General Asset Prices, Journal of Mathematical Economics, 1992, 21(4): 363-378
22. Stricker, C. Arbitrage et Lois de Martingale, Annales de l'Institut Henri Poincaré, 1990, 26(3): 451-460
23. Yan J. -A., Caractérisation d'une Classe d'Ensembles Convexes de  $L^1$  ou  $H^1$ , Séminaire de Probabilité XIV, Lecture Notes in Mathematics, 1980, 784: 220-222
24. Yosida, Y., Hewitt, E. Finitely Additive Measures, Transactions of the American Mathematical Society, 1952, 72(1): 46-66