

Robust and Accurate Inference for Generalized Linear
Models:
Complete Computations

by

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APPENDIX A

To determine $\lambda(\beta)$, we calculate

$$\begin{aligned}
-n \frac{\partial K_\psi(\lambda; \beta)}{\partial \lambda} &= - \sum_{i=1}^n \frac{\partial K_\psi^i(\lambda; \beta)}{\partial \lambda} \\
&= \sum_{i=1}^n \frac{\partial (\mu_i \lambda^T x_i + \frac{b(\theta_{0i}) - b(\theta_{0i} + \lambda^T x_i a(\phi))}{a(\phi)})}{\partial \lambda} \\
&= \sum_{i=1}^n [\mu_i x_i - b'(\theta_{0i} + \lambda^T x_i a(\phi)) \cdot x_i] \\
&= 0
\end{aligned}$$

Since $g(\cdot)$ is the canonical link, $\theta_i = x_i^T \beta$, and $-n \frac{\partial^2 K_\psi(\lambda; \beta)}{\partial \lambda \partial \lambda^T}$ is negative definite, this equation has a unique solution given by $\lambda(\beta) = \frac{\beta - \beta_0}{a(\phi)}$.

Then, by replacing this expression for λ in K_ψ and after simplification we obtain

$$-K_\psi^i(\lambda(\beta); \beta) = \frac{(\theta_i - \theta_{0i})\mu_i - (b(\theta_i) - b(\theta_{0i}))}{a(\phi)},$$

and

$$\begin{aligned}
h(\beta) &= -K_\psi(\lambda(\beta); \beta) \\
&= \frac{1}{n} \sum_{i=1}^n -K_\psi^i(\lambda(\beta); \beta) \\
&= \frac{1}{n} \sum_{i=1}^n \frac{(\theta_i - \theta_{0i})\mu_i - (b(\theta_i) - b(\theta_{0i}))}{a(\phi)} \\
&= \frac{1}{n} \sum_{i=1}^n \frac{b'(x_i^T \beta) x_i^T (\beta - \beta_0) - (b(x_i^T \beta) - b(x_i^T \beta_0))}{a(\phi)}.
\end{aligned}$$

APPENDIX B

Calculation of the integrals I_{i1} , I_{i2} , I_{i3}

(i)

$$\begin{aligned}
I_{i1} &= \int_{r_i < -c} e^{-\lambda^T c \frac{w(x_i)}{V^{1/2}(\mu_i)} \mu'_i - \lambda^T \tilde{a}(\beta)} \cdot e^{\frac{y\theta_{0i} - b(\theta_{0i})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= e^{-\lambda^T c \frac{w(x_i)}{V^{1/2}(\mu_i)} \mu'_i - \lambda^T \tilde{a}(\beta)} \cdot \int_{r_i < -c} e^{\frac{y\theta_{0i} - b(\theta_{0i})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= e^{-\lambda^T c \frac{w(x_i)}{V^{1/2}(\mu_i)} \mu'_i - \lambda^T \tilde{a}(\beta)} \cdot \int_{y < -cV^{1/2}(\mu_i) + \mu_i} e^{\frac{y\theta_{0i} - b(\theta_{0i})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= e^{-\lambda^T c \frac{w(x_i)}{V^{1/2}(\mu_i)} \mu'_i - \lambda^T \tilde{a}(\beta)} \cdot P(Z^i \leq -cV^{1/2}(\mu_i) + \mu_i)
\end{aligned}$$

where Z^i is a random variable distributed according to the exponential family

(2) with parameter θ_{0i} .

(ii)

$$\begin{aligned}
I_{i2} &= \int_{|r_i| < c} e^{\frac{y\lambda^T \mu'_i}{V^{1/2}(\mu_i)} \frac{w(x_i)}{V^{1/2}(\mu_i)}} \cdot e^{\frac{-\lambda^T \mu_i \mu'_i}{V^{1/2}(\mu_i)} \frac{w(x_i)}{V^{1/2}(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{y\theta_{0i} - b(\theta_{0i})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= \int_{|r_i| < c} e^{\frac{y\lambda^T \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{-b(\theta_{0i})}{a(\phi)}} \cdot e^{\frac{y\theta_{0i}}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= \int_{|r_i| < c} e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{-b(\theta_{0i})}{a(\phi)}} \cdot e^{\frac{y(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= \int_{|r_i| < c} e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\
&\quad \cdot e^{\frac{y(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\
&\quad \cdot \int_{|r_i| < c} e^{\frac{y(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)})}{a(\phi)}} \cdot e^{d(y;\phi)} \cdot dy \\
&= e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\
&\quad \cdot P(-cV^{1/2}(\mu_i) + \mu_i < Z_\lambda^i < cV^{1/2}(\mu_i) + \mu_i)
\end{aligned}$$

where Z_λ^i is a random variable distributed according to the exponential family

(2) with parameter $[\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}]$.

(iii) This result can be easily derived as in (i).

We obtain:

$$I_{i3} = e^{\lambda^T c \frac{w(x_i)}{V^{1/2}(\mu_i)} \mu'_i - \lambda^T \tilde{a}(\beta)} \cdot P(Z^i \geq cV^{1/2}(\mu_i) + \mu_i).$$

APPENDIX C

For $i = 1, \dots, n$, we have from Appendix B:

$$\begin{aligned} \frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} &= - \left[\frac{cw(x_i) \mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot I_{i1} \\ &- \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) - \frac{\mu'_i w(x_i)}{V(\mu_i)} b'(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) \right] \cdot I_{i2} \\ &+ e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \left[\frac{\partial}{\partial \lambda} P(|Z_\lambda^i| < c) \right] \\ &+ \left[\frac{cw(x_i) \mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot I_{i3} \\ &= - \left[\frac{cw(x_i) \mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot I_{i1} \\ &- \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) - \frac{\mu'_i w(x_i)}{V(\mu_i)} b'(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) \right] \cdot I_{i2} \\ &+ e^{\frac{-\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \cdot \frac{\mu'_i w(x_i)}{V(\mu_i)} E_{|r_i| < c}^{Z_\lambda^i} [Y] \\ &- \frac{\mu'_i w(x_i)}{V(\mu_i)} b'(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) \cdot I_{i2} \\ &+ \left[\frac{cw(x_i) \mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot I_{i3} \end{aligned}$$

$$\begin{aligned}
&= - \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot I_{i1} \\
&- \left[\frac{\mu_i\mu'_iw(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot I_{i2} \\
&+ e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \cdot \frac{\mu'_i w(x_i)}{V(\mu_i)} E_{|r_i| < c}^{Z_\lambda^i} [Y] \\
&+ \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot I_{i3}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial s(\lambda; \beta)}{\partial \lambda} &= \sum_{i=1}^n \frac{\partial \left[\frac{\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda}}{I_{i1} + I_{i2} + I_{i3}} \right]}{\partial \lambda} \\
&= \sum_{i=1}^n \frac{\frac{\partial^2(I_{i1} + I_{i2} + I_{i3})}{\partial \lambda \partial \lambda^T} \cdot (I_{i1} + I_{i2} + I_{i3}) - \left[\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} \right] \cdot \left[\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} \right]^T}{(I_{i1} + I_{i2} + I_{i3})^2}.
\end{aligned}$$

Let $S1_i$ and $S2_i$ such that:

$$\begin{aligned}
S1_i : &= \frac{\partial^2(I_{i1} + I_{i2} + I_{i3})}{\partial \lambda \partial \lambda^T} \cdot (I_{i1} + I_{i2} + I_{i3}) \\
&= \frac{\partial \left(\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} \right)}{\partial \lambda^T} \cdot (I_{i1} + I_{i2} + I_{i3}) \\
&= (I_{i1} + I_{i2} + I_{i3}) \cdot \left\{ \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right]^T \cdot I_{i1} \right. \\
&+ \left[\frac{\mu_i\mu'_iw(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu_i\mu'_iw(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right]^T \cdot I_{i2} \\
&- \left[\frac{\mu_i\mu'_iw(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu_i w(x_i)}{V(\mu_i)} \right]^T E_{|r_i| < c}^{Z_\lambda^i} [Y] \\
&\cdot e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\
&- \left[\frac{\mu_i\mu'_iw(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y] \\
&\cdot e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\
&+ e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y^2]
\end{aligned}$$

$$+ \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right]^T \cdot I_{i3} \Big\}$$

and

$$\begin{aligned} S_{2_i} : &= \left[\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} \right] \cdot \left[\frac{\partial I_{i1}}{\partial \lambda} + \frac{\partial I_{i2}}{\partial \lambda} + \frac{\partial I_{i3}}{\partial \lambda} \right]^T \\ &= \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right]^T \cdot I_{i1}^2 \\ &+ \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right]^T \cdot I_{i2}^2 \\ &+ e^{-2 \frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-2\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{2b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - 2b(\theta_{0i})}{a(\phi)}} \\ &\quad \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot [E_{|r_i| < c}^{Z_i} [Y]]^2 \\ &+ \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right]^T \cdot I_{i3}^2 \\ &+ 2 \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right]^T I_{i1} I_{i2} \\ &- 2 \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot E_{|r_i| < c}^{Z_i} [Y] \cdot I_{i1} \\ &\cdot e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\ &- 2 \left[\frac{cw(x_i)\mu'_i}{V^{1/2}} - \tilde{a}(\beta) \right] \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}} + \tilde{a}(\beta) \right]^T \cdot I_{i1} \cdot I_{i3} \\ &- 2 \cdot \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot E_{|r_i| < c}^{Z_i} [Y] \cdot I_{i2} \\ &\cdot e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \\ &- 2 \cdot \left[\frac{\mu_i \mu'_i w(x_i)}{V(\mu_i)} + \tilde{a}(\beta) \right] \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right]^T I_{i2} \cdot I_{i3} \\ &+ 2 \cdot \left[\frac{cw(x_i)\mu'_i}{V^{1/2}(\mu_i)} - \tilde{a}(\beta) \right] \cdot \left[\frac{\mu'_i w(x_i)}{V(\mu_i)} \right]^T \cdot E_{|r_i| < c}^{Z_i} [Y] \cdot I_{i3} \\ &\cdot e^{-\frac{\lambda^T \mu_i \mu'_i w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu'_i w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}}. \end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial s(\lambda; \beta)}{\partial \lambda} &= \sum_{i=1}^n \frac{[S1_i - S2_i]}{(I_{i1} + I_{i2} + I_{i3})^2} \\
&= \sum_{i=1}^n \frac{[\mu'_i \cdot \mu_i'^T]}{(I_{i1} + I_{i2} + I_{i3})^2} w^2(x_i) \left\{ \left[\frac{c}{V^{1/2}(\mu_i)} - \frac{\mu_i}{V(\mu_i)} \right]^2 \cdot I_{i1}I_{i2} + \left[2\frac{c}{V^{1/2}(\mu_i)} \right]^2 I_{i1}I_{i3} \right. \\
&\quad + \left. \left[\frac{c}{V^{1/2}(\mu_i)} + \frac{\mu_i}{V(\mu_i)} \right]^2 \cdot I_{i2}I_{i3} \right. \\
&\quad + 2 \cdot e^{-\frac{\lambda^T \mu_i \mu_i' w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu_i' w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \cdot \frac{1}{V(\mu_i)} \\
&\quad \cdot \left[\left(\frac{c}{V^{1/2}(\mu_i)} - \frac{\mu_i}{V(\mu_i)} \right) \cdot I_{i1} - \left(\frac{c}{V^{1/2}(\mu_i)} + \frac{\mu_i}{V(\mu_i)} \right) \cdot I_{i3} \right] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y] \\
&\quad + e^{-\frac{\lambda^T \mu_i \mu_i' w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu_i' w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \cdot \frac{1}{V^2(\mu_i)} \\
&\quad \cdot [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y^2] \\
&\quad \left. - \left[e^{-\frac{\lambda^T \mu_i \mu_i' w(x_i)}{V(\mu_i)}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot e^{\frac{b(\theta_{0i} + \frac{\lambda^T \mu_i' w(x_i) a(\phi)}{V(\mu_i)}) - b(\theta_{0i})}{a(\phi)}} \right]^2 \cdot \frac{1}{V^2(\mu_i)} \cdot [E_{|r_i| < c}^{Z_\lambda^i}[Y]]^2 \right\}.
\end{aligned}$$

APPENDIX D

Special cases

(i) $Y_i \sim \mathbf{N}(\mu_i, \sigma^2)$

$$b(\theta_i) = \frac{\theta_i^2}{2} \quad a(\phi) = \sigma^2$$

and in this case $\tilde{a}(\beta) = 0$. Then, we have :

$$\begin{aligned}
\frac{\partial s(\lambda; \beta)}{\partial \lambda} &= \sum_{i=1}^n x_i x_i^T \cdot \frac{w^2(x_i)}{(I_{i1} + I_{i2} + I_{i3})^2} \left\{ (c - x_i^T \beta)^2 \cdot I_{i1}I_{i2} \right. \\
&\quad + (2c)^2 \cdot I_{i1}I_{i3} + (c + x_i^T \beta)^2 \cdot I_{i2}I_{i3} \\
&\quad + 2 \cdot e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \\
&\quad \cdot [(c - x_i^T \beta) I_{i1} - (c + x_i^T \beta) \cdot I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y] \\
&\quad + e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \cdot [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y^2] \\
&\quad \left. - \left[e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \right]^2 \cdot \frac{1}{V^2(\mu_i)} \cdot [E_{|r_i| < c}^{Z_\lambda^i}[Y]]^2 \right\}.
\end{aligned}$$

$$\begin{aligned}
& - \left[e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \right]^2 \cdot \left[E_{|r_i| < c}^{Z_\lambda^i} [Y] \right]^2 \Big\} \\
& = \sum_{i=1}^n x_i x_i^T \cdot A_i(\lambda),
\end{aligned}$$

where $A_i(\lambda)$ is scalar function defined by

$$\begin{aligned}
A_i(\lambda) &= \frac{w(x_i)}{(I_{i1} + I_{i2} + I_{i3})^2} \cdot \\
& \Big\{ (c - x_i^T \beta)^2 \cdot I_{i1} I_{i2} + (2c)^2 \cdot I_{i1} I_{i3} + (c + x_i^T \beta)^2 \cdot I_{i2} I_{i3} \\
& + 2 \cdot e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \\
& \cdot [(c - x_i^T \beta) I_{i1} - (c + x_i^T \beta) \cdot I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y] \\
& + e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \cdot [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y^2] \\
& - \left[e^{x_i^T \lambda w(x_i) x_i^T (2\beta_0 - \beta) + (x_i^T \lambda w(x_i) \sigma)^2} \right]^2 \cdot \left[E_{|r_i| < c}^{Z_\lambda^i} [Y] \right]^2 \Big\}.
\end{aligned}$$

(ii) $Y_i \sim \mathcal{P}(\mu_i)$

$$b(\theta) = e^\theta, \quad a(\phi) = 1$$

Then, we have :

$$\begin{aligned}
\frac{\partial s(\lambda; \beta)}{\partial \lambda} &= \sum_{i=1}^n x_i x_i^T \cdot \frac{w^2(x_i) \cdot e^{2x_i^T \beta}}{(I_{i1} + I_{i2} + I_{i3})^2} \Big\{ (ce^{-\frac{1}{2}x_i^T \beta} - 1)^2 \cdot I_{i1} I_{i2} \\
& + (2ce^{-\frac{1}{2}x_i^T \beta})^2 \cdot I_{i1} I_{i3} + (ce^{-\frac{1}{2}x_i^T \beta} + 1)^2 \cdot I_{i2} I_{i3} \\
& + 2 \cdot e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i) \lambda)} - e^{x_i^T \beta_0}]} \cdot e^{-x_i^T \beta} \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y] \\
& \cdot [(ce^{-\frac{1}{2}x_i^T \beta} - 1) \cdot I_{i1} - (ce^{-\frac{1}{2}x_i^T \beta} + 1) \cdot I_{i3}] \\
& + e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i) \lambda)} - e^{x_i^T \beta_0}]} \cdot e^{-2x_i^T \beta} [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i} [Y^2] \\
& - \left[e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i) \lambda)} - e^{x_i^T \beta_0}]} \right]^2 \cdot e^{-2x_i^T \beta} \cdot \left(E_{|r_i| < c}^{Z_\lambda^i} [Y] \right)^2 \Big\} \\
& = \sum_{i=1}^n x_i x_i^T \cdot A_i(\lambda),
\end{aligned}$$

where $A_i(\lambda)$ is scalar function defined by

$$\begin{aligned}
A_i(\lambda) &= \frac{w^2(x_i) \cdot e^{2x_i^T \beta}}{(I_{i1} + I_{i2} + I_{i3})^2} \\
&\left\{ \begin{aligned} &(ce^{-\frac{1}{2}x_i^T \beta} - 1)^2 \cdot I_{i1}I_{i2} + (2ce^{-\frac{1}{2}x_i^T \beta})^2 \cdot I_{i1}I_{i3} + (ce^{-\frac{1}{2}x_i^T \beta} + 1)^2 \cdot I_{i2}I_{i3} \\ &+ 2 \cdot e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i)\lambda)} - e^{x_i^T \beta_0}]} \cdot e^{-x_i^T \beta} \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y] \\ &\cdot [(ce^{-\frac{1}{2}x_i^T \beta} - 1) \cdot I_{i1} - (ce^{-\frac{1}{2}x_i^T \beta} + 1) \cdot I_{i3}] \\ &+ e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i)\lambda)} - e^{x_i^T \beta_0}]} \cdot e^{-2x_i^T \beta} [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y^2] \\ &- [e^{-x_i^T \lambda w e^{x_i^T \beta} - \lambda^T \tilde{a}(\beta)} \cdot e^{[e^{x_i^T (\beta_0 + w(x_i)\lambda)} - e^{x_i^T \beta_0}]}]^2 \cdot e^{-2x_i^T \beta} \cdot \left(E_{|r_i| < c}^{Z_\lambda^i}[Y]\right)^2 \end{aligned} \right\}.
\end{aligned}$$

(iii) $Y_i \sim \text{Bin}(m, \pi_i)$

$$b(\theta) = m \cdot \log(1 + e^\theta), \quad a(\phi) = 1$$

Then, we have :

$$\begin{aligned}
\frac{\partial s(\lambda; \beta)}{\partial \lambda} &= \sum_{i=1}^n x_i x_i^T \frac{w^2(x_i) e^{2x_i^T \beta}}{(I_{i1} + I_{i2} + I_{i3})^2} \left\{ \begin{aligned} &\left(\frac{c - \sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta} (1 + e^{x_i^T \beta})}\right)^2 I_{i1}I_{i2} \\ &+ \left(\frac{2c}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta} (1 + e^{x_i^T \beta})}\right)^2 I_{i1}I_{i3} + \left(\frac{c + \sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta} (1 + e^{x_i^T \beta})}\right)^2 I_{i2}I_{i3} \\ &+ 2 \cdot \left(\frac{1 + x_i^T \beta_0 + x_i^T \lambda w(x_i)}{1 + \beta_0^T x_i}\right)^m \cdot e^{\frac{-mx_i^T \lambda w(x_i) e^{x_i^T \beta}}{1 + e^{x_i^T \beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{m e^{x_i^T \beta}} \\ &\cdot \left[\left(\frac{c - \sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta} (1 + e^{x_i^T \beta})}\right) I_{i1} - \left(\frac{c + \sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T \beta} (1 + e^{x_i^T \beta})}\right) I_{i3}\right] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y] \\ &+ \left(\frac{1 + x_i^T \beta_0 + x_i^T \lambda w(x_i)}{1 + x_i^T \beta_0}\right)^m \cdot e^{\frac{-mx_i^T \lambda w(x_i) e^{x_i^T \beta}}{1 + e^{x_i^T \beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{m^2 e^{2x_i^T \beta}} \\ &\cdot [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y^2] \\ &- \left(\frac{1 + x_i^T \beta_0 + x_i^T \lambda w(x_i)}{1 + x_i^T \beta_0}\right)^m \cdot e^{\frac{-mx_i^T \lambda w(x_i) e^{x_i^T \beta}}{1 + e^{x_i^T \beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{m^2 e^{2x_i^T \beta}} [E_{|r_i| < c}^{Z_\lambda^i}[Y]]^2 \end{aligned} \right\} \\
&= \sum_{i=1}^n x_i x_i^T \cdot A_i(\lambda),
\end{aligned}$$

where $A_i(\lambda)$ is scalar function defined by

$$\begin{aligned}
A_i(\lambda) &= \frac{w^2(x_i)e^{2x_i^T\beta}}{(I_{i1} + I_{i2} + I_{i3})^2} \left\{ \left(\frac{c - \sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}(1 + e^{x_i^T\beta})} \right)^2 I_{i1}I_{i2} \right. \\
&+ \left(\frac{2c}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}(1 + e^{x_i^T\beta})} \right)^2 I_{i1}I_{i3} + \left(\frac{c + \sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}(1 + e^{x_i^T\beta})} \right)^2 I_{i2}I_{i3} \\
&+ 2 \cdot \left(\frac{1 + x_i^T\beta_0 + x_i^T\lambda w(x_i)}{1 + x_i^T\beta_0} \right)^m \cdot e^{\frac{-mx_i^T\lambda w(x_i)e^{x_i^T\beta}}{1 + e^{x_i^T\beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{me^{x_i^T\beta}} \\
&\cdot \left[\left(\frac{c - \sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}(1 + e^{x_i^T\beta})} \right) I_{i1} - \left(\frac{c + \sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}}{\sqrt{m} \cdot e^{\frac{1}{2}x_i^T\beta}(1 + e^{x_i^T\beta})} \right) I_{i3} \right] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y] \\
&+ \left(\frac{1 + x_i^T\beta_0 + x_i^T\lambda w(x_i)}{1 + x_i^T\beta_0} \right)^m \cdot e^{\frac{-mx_i^T\lambda w(x_i)e^{x_i^T\beta}}{1 + e^{x_i^T\beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{m^2 e^{2x_i^T\beta}} \\
&\cdot [I_{i1} + I_{i2} + I_{i3}] \cdot E_{|r_i| < c}^{Z_\lambda^i}[Y^2] \\
&- \left(\frac{1 + x_i^T\beta_0 + x_i^T\lambda w(x_i)}{1 + x_i^T\beta_0} \right)^m \cdot e^{\frac{-mx_i^T\lambda w(x_i)e^{x_i^T\beta}}{1 + e^{x_i^T\beta}}} \cdot e^{-\lambda^T \tilde{a}(\beta)} \cdot \frac{1}{m^2 e^{2x_i^T\beta}} [E_{|r_i| < c}^{Z_\lambda^i}[Y]]^2 \Big\}.
\end{aligned}$$