SUPERMARTINGALE DECOMPOSITION WITH A GENERAL INDEX SET

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This paper is a modest tribute to the memory of Lester Dubins.

Abstract. We prove results on the existence of Doléans-Dade measures and of the Doob-Meyer decomposition for supermartingales indexed by a general index set.

1. Introduction

By Doob's theorem, a supermartingale indexed by the natural numbers decomposes uniquely into the difference of a martingale and an increasing, predictable process; moreover, both such processes are uniformly integrable if the supermartingale is so. The relative ease of working with increasing processes explains the prominent role of this result in stochastic analysis and in the theory of stochastic integration and it motivates the interest for possible extensions to more general settings. Meyer [21] proved that, under the *usual conditions*, Doob's decomposition exists for right continuous supermartingales indexed by the positive reals if and only if the supermartingale is of class D. Doléans-Dade [10], later followed by Föllmer [13] and Metivier and Pellaumail [20], was the first to represent supermartingales as measures over predictable rectangles and to prove that a supermartingale is of class D if and only if its associated measure is countably additive. The first proof making no use of the usual conditions was obtained by Mertens [19].

In this paper we consider the case of processes indexed by a family of sets. We obtain results for supermartingales belonging to three different classes: D_0, D and D_*. In Theorem 1 we prove that the class D_0 property is necessary and sufficient for the existence of a (finitely additive) Doléans-Dade measure associated with a supermartingale. We then consider supermartingales of class D and show that this property is not enough to imply the existence of a suitable version of the Doob Meyer decomposition, save when the index set is linearly ordered. Looking for a more stringent condition, we prove in Theorem 3 that supermartingales of class D_* may be decomposed as required. In Corollary 1 we fully characterise supermartingales of uniformly integrable variation. Eventually, we show that the class D, the class D_* and the uniform integrable variation property are equivalent when the index set is linearly ordered.

Many papers over the years have treated the case of stochastic processes indexed by general sets, starting from the seminal work of Cairoli and Walsh [3], in which the index set consists of rectangles in $\mathbb{R}^2_+$ with one vertex in the origin. A rather complete list of references appears in the bibliography of the book by Ivanoff and Merzbach [16], where a general treatment of this topic is offered. The works more directly related to ours are those of Dozzi, Ivanoff and Merzbach [11] and of Slonowsky [24], who obtain a form of the Doob Meyer decomposition. Also relevant are the papers of Ivanoff and Merzbach [15], who extend such decomposition

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by a localization argument, and, to a much lesser extent, of De Giosa and Mininni [8], dealing with measures associated with supermartingales. A very recent but less relevant contribution is the paper by Yosef [25].

Part of the interest raised by this topic is for the additional mathematical machinery needed in order to obtain the results of Doob and Meyer in a more general setting. All of the aforementioned works, some of which draw in turn from an unpublished paper of Norberg [23], apply classical techniques, based on right continuity, separability and uniform integrability. In order to make this possible, a large number of set-theoretical as well as of topological restrictions on the index set has to be introduced. We take here a different approach in which the index set is given a minimal structure with no topological content. Likewise, the notion of predictability we adopt is elementary and, hopefully, intuitive. Our approach is based once more on the fundamental idea that supermartingale decompositions are related to a corresponding property of the Doléans-Dade measure associated with it. The mathematical novelty lies in the choice to work with finitely additive measures which has the advantage of making easier the proofs concerning existence although at the cost, as usual with finite additivity, of accepting non uniqueness.

2. Preliminaries and notation

We fix some general notation, mainly in accordance with [12]. When S is a set, 2^S denotes its power set and $\mathbf{1}_S$ its indicator function. If $\Sigma \subset 2^S$, typically an algebra, the symbols $ba(\Sigma)$ (resp. $ca(\Sigma)$) and $\mathfrak{B}(\Sigma)$ designate the spaces of bounded, finitely (resp. countably) additive set functions on Σ and the closure of the set of Σ simple functions with respect to the supremum norm, respectively. We prefer $ba(S)$ to $ba(2^S)$ and $\mathfrak{B}(S)$ to $\mathfrak{B}(2^S)$. The space of integrable functions with respect to some $m \in ba(\Sigma)$ is denoted $L(m)$. Finitely additive measures are identified with the linear functional arising from the corresponding expected value so writing $\mu(f)$ is preferred $\int f d\mu$. $\mathbb{P}(\Sigma)$ designates the collection of those elements P of $ba(S)_+$ whose restriction $P|\Sigma$ to Σ is a countably additive probability measure (thus $P \in \mathbb{P}(\Sigma)$ need not be itself countably additive). We recall two useful facts on finitely additive probabilities that we shall use repeatedly (see [2, Theorem 3.2.10, p. 70] and [5, Theorem 1, p. 588]):

Lemma 1. *Let $\Sigma_0 \subset 2^S$ be an algebra and $\Sigma \subset 2^S$ a σ algebra, $\mu \in ba(\Sigma_0)_+$ and $P \in \mathbb{P}(\Sigma)$. Then,*

(i) there is $\bar\mu \in ba(S)_+$ with $\bar\mu|\Sigma_0 = \mu$;

(ii) for each $f \in L(P)$ there exists a P a.s. unique element $P(f|\Sigma)$ of $L(P|\Sigma)$ such that

$$P(f\mathbf{1}_F) = P(P(f|\Sigma)\mathbf{1}_F) \qquad F \in \Sigma. \tag{1}$$

Thus, any (countably additive) probability measure on some algebra Σ may be extended to an element of $\mathbb{P}(\Sigma)$ while the conditional expectation relatively to Σ remains well defined.

We take two sets Ω and I as given, put $\bar\Omega \equiv \Omega \times I$ and, for $s, t \subset \bar\Omega$, we write $s \le t$ whenever $t \subset s$. $s < t$ means $s \le t$ and $s \cap t^c \neq \varnothing$. $t(i)$ denotes the i-section $\{\omega \in \Omega : (\omega, i) \in t\}$ of t, $\{s < t\} = \bigcup_{i\in I}(s \cap t^c)(i)$: thus $\{s < \varnothing\}$ is just the projection of s on Ω. The special case where $I = \mathbb{R}_+$ and some probability measure P on Ω is given will be referred to as the classical theory.

Also given are a collection T of subsets of $\bar\Omega$ containing $\bar\Omega$ and $\varnothing$ and a filtration $\mathbb{A} = (\mathscr{A}_t : t \in T)$, that is an increasing collection of algebras of subsets of Ω satisfying:

$$F \cap (s \cap t^c)(i) \in \mathscr{A}_s \cap \mathscr{A}_t \quad \text{and} \quad F \cap \{s < t\} \in \mathscr{A}_s \cap \mathscr{A}_t \qquad s, t \in T,\ F \in \mathscr{A}_s,\ i \in I. \tag{2}$$

One should remark that in the present setting the second inclusion in (2) does not follow from the first one and must therefore be explicitly assumed. Define also $\mathscr{A} = \bigcup_{t\in T} \mathscr{A}_t$ and $\mathscr{F} = \sigma\mathscr{A}$. We denote by $\mathscr{D}$ the family of all finite, disjoint collections

$$d = \{s_n \cap t_n^c : n = 1, \ldots, N\} \quad \text{with} \quad s_n, t_n \in T,\ s_n \leq t_n \qquad n = 1, \ldots, N. \tag{3}$$

In the classical theory, T would typically be some family of stochastic intervals such as $]]\tau, \infty[[$ or $[[\tau, \infty[[$ with τ a stopping time (which explains our choice to define partial order by reverse inclusion). The literature has treated the case in which each $t \in T$ is deterministic, i.e. of the form $t = \Omega \times J$ with $J \subset I$, and may then be identified with a subset of I. Dozzi et al. [11] and Slonowsky [24], e.g., take T to be a collection of closed subsets of a (locally) compact topological space and assume, among other things, that T is closed with respect to countable intersections. The index set T is said to be *regular* if it is closed with respect to finite unions and intersections.

3. Finitely Additive Supermartingales

A finitely additive process (on $\mathbb{A}$) is an element $m = (m_t : t \in T)$ of the product space $\prod_{t \in T} ba(\mathscr{A}_t)$. A particular case of special importance is that of classical, integrable processes, $X \in \prod_{t \in T} L(P|\mathscr{A}_t)$, for some given $P \in \mathbb{P}(\mathscr{F})$ with which one associates the finitely additive process $(m_t : t \in T)$ defined by letting $dm_t = X_t dP$ for all $t \in T$. A finitely additive process m is bounded if $\|m\| \equiv \sup_{t \in T} \|m_t\| < \infty$.

We speak of the finitely additive process m as a finitely additive supermartingale if

$$m_t|\mathscr{A}_s \leq m_s \qquad s, t \in T,\ s \leq t. \tag{4}$$

As a consequence of the assumption $\bar{\Omega},\ \varnothing \in T$, all finitely additive supermartingales are actually bounded. Alternatively, when T is a directed set and m a bounded, finitely additive supermartingale, one may avoid assuming $\varnothing \in T$ by letting $m_\varnothing(F) = \lim_{\{t \in T : F \in \mathscr{A}_t\}} m_t(F)$ for all $F \in \mathscr{A}$.

A process $f : \bar{\Omega} \to \mathbb{R}$ is elementary, $f \in \mathscr{E}$, if it may be written in the form

$$f = \sum_{n=1}^N f_n \mathbf{1}_{s_n \cap t_n^c} \quad \text{with} \quad f_n \in \mathfrak{B}(\mathscr{A}_{s_n}),\ s_n, t_n \in T,\ s_n \leq t_n \qquad n = 1, \ldots, N, \tag{5}$$

while we write $f \in \mathscr{E}^*$ if (5) applies with $f_n \in \mathfrak{B}(\mathscr{A})$ for $n = 1, \ldots, N$. A subset of $\bar{\Omega}$ is predictable if it belongs to the algebra $\mathscr{P}$ generated by the elementary processes[1].

The next property is crucial in the following developments.

Definition 1. *A finitely additive supermartingale m is said to be of class D_0 if*

$$0 \geq \sum_{n=1}^N f_n \mathbf{1}_{s_n \cap t_n^c} \in \mathscr{E} \quad \text{imply} \quad \sum_{n=1}^N (m_{s_n} - m_{t_n})(f_n) \leq 0. \tag{6}$$

The following lemma shows that when the index set is sufficiently well behaved, then the class D_0 property simplifies to a condition which is entirely familiar in the classical framework.

Lemma 2. *Let T be regular. A finitely additive supermartingale m is of class D_0 if and only if it is regular, i.e. if it satisfies*

$$(m_{s \cup t} - m_t)(F) = (m_s - m_{s \cap t})(F \cap \{s < t\}) \qquad s, t \in T,\ F \in A_{s \cup t}. \tag{7}$$

[1] The terminology here is motivated by our focus on the Doob Meyer decomposition but, given that the collection T is entirely arbitrary, $\mathscr{P}$ may well be taken to be the family of optional stochastic intervals or of progressively measurable sets, in the classical approach.

Proof. Given that $F \in \mathscr{A}_{s\cup t}$ implies $0 = \mathbf{1}_F\mathbf{1}_{(s\cup t)\cap t^c} - \mathbf{1}_{F\cap\{s<t\}}\mathbf{1}_{s\cap(s\cap t)^c} \in \mathscr{E}$, a finitely additive supermartingale of class D_0 clearly meets (7). The converse is proved by induction. If $0 \geq f_1\mathbf{1}_{s_1\cap t_1^c} \in \mathscr{E}$ and m is regular, then $(m_{s_1} - m_{t_1})(f_1) = (m_{s_1} - m_{t_1})(f_1\mathbf{1}_{\{s_1<t_1\}}) \leq (m_{s_1} - m_{t_1})(\sup_i f\mathbf{1}_{s_1\cap t_1^c}(i)) \leq 0$. Assume that

$$\sum_{n=1}^{N-1}(m_{s_n} - m_{t_n})(f_n) \leq 0 \quad \text{when} \quad 0 \geq \sum_{n=1}^{N-1} f_n\mathbf{1}_{s_n\cap t_n^c} \in \mathscr{E} \tag{8}$$

and let $0 \geq f = \sum_{n=1}^N f_n\mathbf{1}_{s_n\cap t_n^c} \in \mathscr{E}$. To start a recursion, define $I_n^0 = s_n \cap t_n^c$,

$$I_n^k = I_n^{k-1} \cap I_k^{k-1} = I_n^0 \cap \bigcap_{j=1}^k I_j^{j-1} \quad \text{and} \quad f^k = \sum_{n=1}^N f_n\mathbf{1}_{I_n^k} = f\mathbf{1}_{\bigcap_{j=1}^k I_j^{j-1}} \leq 0. \tag{9}$$

It is easily seen that, letting $s_n^0 = s_n$ and $t_n^0 = t_n$, then $I_n^k = s_n^k \cap (t_n^k)^c$ where

$$s_n^k = (s_k^{k-1} \cap s_n^{k-1}) \cup t_n^{k-1} \quad \text{and} \quad t_n^k = (s_n^{k-1} \cap t_k^{k-1}) \cup t_n^{k-1}, \qquad k = 1, \ldots, N. \tag{10}$$

Let also $s = \bigcap_{n=1}^N s_n^N$ and $t = s \cap \bigcup_{n=1}^N t_n^N$. Observe that $s, s_n^k, t_n^k, t \in T$ and that $s_n^k \leq s_n^{k+1} \leq t_n^{k+1} \leq t_n^k$. Moreover, $I_n^N = \bigcap_{j=1}^N I_j^{j-1} = s \cap t^c$ for $n = 1, \ldots, N$ which implies

$$s_n^N = t_n^N \cup (s \cap t^c) \subset t_n^N \cup s = s_n^N \qquad \text{and} \quad s \cap t_n^N = s \cap \left((s_n^N)^c \cup t_n^N\right) = s \cap (s^c \cup t) = t \tag{11}$$

We then draw the following implications. First,

$$0 \geq f^{k-1}\mathbf{1}_{(s_k^{k-1})^c} = \sum_{n\neq k} f_n\mathbf{1}_{s_n^{k-1}\cap(t_n^{k-1}\cup s_k^{k-1})^c} = \sum_{n\neq k} f_n\mathbf{1}_{s_n^{k-1}\cap(s_n^k)^c}$$

and likewise $0 \geq \sum_{n\neq k} f_n\mathbf{1}_{t_n^k\cap(t_n^{k-1})^c}$ so that, by (8), $\sum_{n=1}^N(m_{s_n^{k-1}} - m_{s_n^k})(f_n) \leq 0$ and $\sum_{n=1}^N(m_{t_n^k} - m_{t_n^{k-1}})(f_n) \leq 0$, and thus

$$\sum_{n=1}^N(m_{s_n^{k-1}} - m_{t_n^{k-1}})(f_n) \leq \sum_{n=1}^N(m_{s_n^k} - m_{t_n^k})(f_n) \qquad 1 < k \leq N.$$

Second, from (11) and (7), $(m_{s_n^N} - m_{t_n^N})(f_n) = (m_{t_n^N\cup s} - m_{t_n^N})(f_n) = (m_s - m_{s\cap t_n^N})(f_n) = (m_s - m_t)(\mathbf{1}_{\{s<t\}}f_n)$. We conclude that

$$\sum_{n=1}^N(m_{s_n} - m_{t_n})(f_n) \leq \sum_{n=1}^N(m_{s_n^N} - m_{t_n^N})(f_n) = (m_s - m_t)\left(\mathbf{1}_{\{s<t\}}\sum_{n=1}^N f_n\right) \leq (m_s - m_t)\left(\sup_i f(i)\right) \leq 0.$$

and that thus (8) holds for any integer $N > 1$. □

All finitely additive supermartingales are regular when the index set is deterministic (or even linearly ordered). In the classical theory the optional sampling theorem extends this conclusion to *càdlàg* supermartingales indexed by bounded stopping times, provided the usual conditions apply. As is well known, this theorem is far from obvious with a general index set (see [14] and [18]) and it may actually fail even with $\mathbb{R}_+$ as the index set unless the usual conditions hold. All finitely additive supermartingales in this paper will be regular.

A regular index set has other two interesting properties.

Lemma 3. *Let the index set T be regular. Then,*

(i) each $f \in \mathscr{E}$ may be written in the form $\sum_{n=1}^N f_n\mathbf{1}_{s_n\cap t_n^c}$ where $f_n \in \mathfrak{B}(\mathscr{A}_{s_n})$ $n = 1, \ldots, N$ and the collection $d = \{s_n \cap t_n^c : n = 1, \ldots, N\}$ is disjoint, i.e. $d \in \mathscr{D}$;

(ii) writing $\delta' \geq \delta$ whenever $\delta, \delta' \in \mathscr{D}$ and each $s\cap t^c \in \delta$ may be written as $\bigcup_{n=1}^N s_n \cap t_n^c$ with $s_n\cap t_n^c \in \delta'$ and $s_n \subset s$ for $n = 1, \ldots, N$ makes $\mathscr{D}$ into a directed set.

Proof. Write $f \in \mathscr{E}$ in the form $\sum_{k=1}^K f'_k \mathbf{1}_{s'_k}$ and denote by $\{\pi_1, \ldots, \pi_N\}$ the collection of non empty subsets of $\{1, \ldots, K\}$. For $n = 1, \ldots, N$, define (with the convention $\bigcup \varnothing = \varnothing$)

$$s_n = \bigcap_{k \in \pi_n} s'_k, \quad t_n = s_n \cap \bigcup_{j \notin \pi_n} s'_j \quad \text{and} \quad f_n = \sum_{k \in \pi_n} f'_k. \tag{12}$$

Then clearly, $f = \sum_{n=1}^N f_n \mathbf{1}_{s_n \cap t_n^c}$. Given that all collections in $\mathscr{D}$ are disjoint, to prove the second claim it is enough to consider $\delta = \{s \cap t^c\}$ and $\delta' = \{u \cap v^c\}$ and to apply (12) to the collection $\{s, t, u, v\}$. □

The preceding results naturally raise interest for regular index sets and induce to consider whether the original model may be embedded into one which possesses this property or is endowed with additional mathematical structure. This point was first made quite clearly by Dozzi et al. [11] and Ivanoff and Merzbach [16] (but see the comments in Section 6). To fix terminology, consider a finitely additive process $\bar{m}$ defined on some filtration $\bar{\mathbb{A}} = \big(\bar{\mathscr{A}_u} : u \in U\big)$ where $T \subset U \subset 2^{\bar{\Omega}}$,

$$\mathscr{A}_t \subset \bar{\mathscr{A}_t} \quad \text{and} \quad \bar{m}_t | \mathscr{A}_t = m_t \qquad t \in T.$$

We then say that $\bar{\mathbb{A}}$ and $\bar{m}$ are extensions of $\mathbb{A}$ and m respectively. The existence of extensions of a given finitely additive supermartingale turns out to be related to the time honoured question of whether finitely additive supermartingales may be represented as measures on $\bar{\Omega}$, i.e. the existence of Doléans-Dade measures.

Theorem 1. *Let m be a finitely additive supermartingale. The following are equivalent:*

(i) *m is of class D_0;*

(ii) *m admits a Doléans-Dade measure, that is an element of the set*

$$\mathscr{M}(m) = \big\{\lambda \in ba(\bar{\Omega})_+ : \lambda((F \times I) \cap t) = (m_t - m_\varnothing)(F),\ F \in \mathscr{A}_t,\ t \in T\big\};$$

(iii) *for any filtration $\bar{\mathbb{A}}$ which extends $\mathbb{A}$ there is a finitely additive supermartingale $\bar{m}$ of class D_0 on $\bar{\mathbb{A}}$ which extends m.*

If either one of the above conditions holds there exist a finitely additive martingale μ and a finitely additive increasing process α (defined as in [1, p. 287]*) such that*

$$m_t = \mu_t - \alpha_t \qquad t \in T. \tag{13}$$

Proof. For $t \in T$, let $\mathscr{L}_t = \{f \mathbf{1}_t : f \in \mathfrak{B}(\mathscr{A}_t)\}$ and define $\phi_t : \mathscr{L}_t \to \mathbb{R}$ implicitly as $\phi_t(f \mathbf{1}_t) = (m_t - m_\varnothing)(f)$. Then $\mathscr{L}_t$ is a linear subspace of $\mathfrak{B}(\bar{\Omega})$ and ϕ_t a linear functional on it. Given our assumption $\bar{\Omega} \in T$, (i) is easily seen to be equivalent to

$$\sup \left\{ \sum_{n=1}^N (m_{t_n} - m_\varnothing)(f_n) : 1 \geq \sum_{n=1}^N f_n \mathbf{1}_{t_n} \in \mathscr{E} \right\} < \infty \tag{14}$$

and corresponds to requiring that the collection $(\phi_t : t \in T)$ is coherent in the sense of [6, Corollary 1, p. 560]: thus (i) is equivalent to (ii). For $\lambda \in \mathscr{M}(m)$ and $H \subset \bar{\Omega}$, define $\lambda_H, m^\lambda_H \in ba(\Omega)$ by letting

$$\lambda_H(F) = \lambda((F \times I) \cap H) \quad \text{and} \quad m^\lambda_H = m^\lambda_\varnothing + \lambda_H \qquad F \subset \Omega, \tag{15}$$

where $m^\lambda_\varnothing \in ba(\Omega)$ is an extension of $m_\varnothing$ from $\mathscr{A}$ to 2^Ω. Thus, $(m^\lambda_H : H \subset \bar{\Omega})$ is an extension of m to $(2^\Omega : H \subset \bar{\Omega})$ and (iii) is proved by simply letting $\bar{m} = (m^\lambda_u | \mathscr{A}_u : u \in U)$. It is clear from (15) that $\bar{m}$ is a finitely additive supermartingale of class D_0 (since $\lambda \in \mathscr{M}(m^\lambda)$) and an extension of m. The implication $(iii) \to (i)$ is obvious. Assume (ii), choose $\lambda \in \mathscr{M}(m)$ and define

$$\mu_t = m^\lambda_{\bar{\Omega}} \big| \mathscr{A}_t \quad \text{and} \quad \alpha_t = \lambda_{t^c} \big| \mathscr{A}_t \qquad t \in T. \tag{16}$$

Given that α coincides with the restriction to $\mathbb{A}$ of the family $(\lambda_{t^c} : t \in T)$ in $ba(\Omega)$ which is increasing in t and such that $\lambda_{\bar{\Omega}^c} = 0$, α is indeed a finitely additive increasing process. $\square$

The choice of treating $\mathscr{M}(m)$ as a subset of $ba(\bar{\Omega})$ opens the door to the apparent arbitrariness implicit in the existence of a multiplicity of Doléans-Dade measures. This situation is almost unavoidable with finite additivity and will be a constant throughout the next sections. Let us recall that in the classical theory a Doléans-Dade measure is defined as an element of $ca(\mathscr{P})_+$ and its existence requires the supermartingale to be of class D. We believe that this additional property, briefly discussed in the following section, is not really essential to obtain several interesting results, such as the Doob Meyer decomposition. Moreover, our approach has the advantage of making clear that some important properties, such as the class D_0 and the ones to be introduced later, do not depend on the underlying filtration, that is on the actual unfolding of information as embodied in the filtration.

Implicit in Theorem 1 (and Lemma 4) is also the conclusion that a regular finitely additive supermartingale may be extended to a filtration endowed with a regular index set if and only if it is of class D_0. We shall return on this point in section 6.

Of course, (13) is only a rather primitive version of the Doob Meyer decomposition. Among other things its existence does not automatically imply the class D_0 property as finitely additive increasing process may in this setting fail to be of class D_0. (13) has been first obtained by Armstrong [1] (see also [5, Corollary 1, p. 591]). Our task is now to improve on it by requiring additional properties on Doléans-Dade measures.

4. Class D Supermartingales

A particularly interesting special case is that of classical supermartingales that we treat, in accordance with [5], without the assumption of a *given* probability measure. To avoid additional notation we assume in what follows (and without loss of generality) that $\mathscr{A}_t$ is a σ algebra for each $t \in T$.

Let m be a finitely additive supermartingale on $\mathbb{A}$ and define

$$\mathscr{M}^{uc} = \{\lambda \in ba(\bar{\Omega})_+ : \lambda_{\bar{\Omega}}|\mathscr{F} \in ca(\mathscr{F})\} \quad \text{and} \quad \mathscr{M}^{uc}(m) = \mathscr{M}^{uc} \cap \mathscr{M}(m). \tag{17}$$

Definition 2. *A finitely additive supermartingale m is of class D if $m_\varnothing \in ca(\mathscr{A})$ and $\mathscr{M}^{uc}(m) \neq \varnothing$.*

Of course a finitely additive supermartingale of class D is of class D_0 too. There are two immediate reasons of interest for this family of finitely additive supermartingales. First, if m is of class D and $\lambda \in \mathscr{M}^{uc}(m)$ there exists $P^\lambda \in \mathbb{P}(\mathscr{F})$ such that $m_\varnothing,\ \lambda_{\bar{\Omega}}|\mathscr{F} \ll P^\lambda|\mathscr{F}$. Exploiting (15), it follows that $m^\lambda_H\big|\mathscr{F} \ll P|\mathscr{F}$ for every $H \subset \bar{\Omega}$. But then, if $\mathscr{A}_H \subset \mathscr{F}$ is any σ algebra, one may define the following family of random quantities

$$X^\lambda_H = \frac{d\, m^\lambda_H\big|\mathscr{A}_H}{dP^\lambda|\mathscr{A}_H} \qquad H \subset \bar{\Omega}. \tag{18}$$

In other words, and given that $\lambda \in \mathscr{M}^{uc}(m^\lambda)$, a finitely additive supermartingale m of class D may be represented as a classical supermartingale of class D on *any* filtration of sub σ algebras of $\mathscr{F}$ indexed by some sub collection of $2^{\bar{\Omega}}$. This property, as defined above, is thus independent of the given filtration. Second, returning to (16), one easily realizes that the decomposition (13) admits a version in which μ and α may be represented as classical processes, but not necessarily adapted. In fact, letting A_t and $\bar{A}_t$ be the Radon Nikodym derivatives of α_t with respect to $P|\mathscr{A}_t$ and of $\lambda_{t^c}|\mathscr{F}$ with respect to $P^\lambda|\mathscr{F}$ respectively, the former process is adapted but not necessarily increasing while the latter is increasing but not adapted.

A more classical version of the class D property would be the following:

Definition 3. *A finitely additive supermartingale is said to be of class D_α if m is regular and there exists $P \in \mathbb{P}(\mathscr{F})$ such that for any $\eta > 0$ and some $\delta > 0$, $P\left(\bigcup_{n=1}^N F_n\right) < \delta$ implies $\sum_{n=1}^N |m_{t_n}|(F_n) < \eta$ whenever $\{F_n : F_n \in \mathscr{A}_{t_n},\ n = 1, \ldots, N\}$ is a disjoint collection.*

For each disjoint collection $\{F_n : F_n \in \mathscr{A}_{t_n},\ n = 1, \ldots, N\}$ it would be tempting to consider the set $\tau = \bigcup_{n=1}^N F_n \times t_n$ as a stopping time. Likewise, if X is a classical supermartingale one may interpret the random quantity $\sum_{n=1}^N X_{t_n} \mathbf{1}_{F_n}$ as the value X_τ of X at the stopping time τ. The class D_α property amounts thus to a uniform integrability property across stopping times with finitely many values and is thus comparable (but more general) to the one originally considered by Meyer [21]. In the next section it will be shown (see Corollary 2 and Theorem 3) that the class D_α and D properties are equivalent and imply the Doob Meyer decomposition, if T is linearly ordered (a conclusion that generalises [4, theorem 4, p. 799]).

Lemma 4. *Let m be a finitely additive supermartingale and consider the following properties: (i) m is of class D, (ii) there exist $\lambda \in \mathscr{M}(m)$ and $P \in \mathbb{P}(\mathscr{F})$ such that $\lim_k \lambda(f^k) = 0$ whenever $\langle f^k \rangle_{k\in\mathbb{N}}$ is a sequence in $\mathscr{E}$ and $\lim_k P(\sup_{i\in I} |f^k(i)| > \eta) = 0$ and (iii) m is of class D_α. (i) and (ii) imply (iii); if T is linearly ordered, (iii) implies (ii).*

Proof. Let m be a finitely additive supermartingale and $\lambda \in \mathscr{M}^{uc}(m)$. Then, $F \in \mathscr{A}_t$ implies

$$|m_t|(F) \le (m_t - m_\varnothing)(F) + |m_\varnothing|(F) = \lambda_t(F) + |m_\varnothing|(F) \le \lambda_{\bar\Omega}(F) + |m_\varnothing|(F),$$

so that if $\{F_n : F_n \in \mathscr{A}_{t_n},\ n = 1, \ldots, N\}$ is a disjoint collection we have $\sum_{n=1}^N |m_{t_n}|(F_n) \le (\lambda_{\bar\Omega} + |m_\varnothing|)\left(\bigcup_{n=1}^N F_n\right)$. The proof of the implication $(i)\to(iii)$ is completed by choosing $P \in \mathbb{P}(\mathscr{F})$ such that $m_\varnothing, \lambda_{\bar\Omega}|\mathscr{F} \ll P|\mathscr{F}$. $(ii)\to(iii)$ is clear. For the converse, observe that when T is linearly ordered, and m is of class D_α then necessarily m is of class D_0, by Lemma 2: choose $\lambda \in \mathscr{M}(m)$. If $f = \sum_{n=1}^N f_n \mathbf{1}_{t_n \cap t_{n+1}^c} \in \mathscr{E}$, let $f_0^* = 0$ and $f_n^* = \sup_{k\le n} |f_k|$. Then the inequality $f \le c + \|f\|_{\mathfrak{B}} \sum_{n=1}^N \mathbf{1}_{\{f_n^* > c \ge f_{n-1}^*\}} \mathbf{1}_{t_n}$ implies

$$\lambda(f) \le c\|\lambda\| + \|f\|_{\mathfrak{B}} \left(|m_\varnothing|(f^* > c) + \sum_{n=1}^N |m_{t_n}|(f_n^* > c \ge f_{n-1}^*) \right). \tag{19}$$

We conclude that if $P \in \mathbb{P}(\mathscr{F})$ is as in Definition 3 and $\langle f^k \rangle_{k\in\mathbb{N}}$ is a sequence in $\mathscr{E}$ such that $\sup_i |f^k(i)|$ converges to 0 in P probability, then $\lim_k \lambda(f^k) = 0$. ☐

The implication $(iii)\to(i)$ for the case of a linearly ordered index set will given in the next section.

5. The Doob Meyer Decomposition

In the classical theory, the existence of a predictable increasing process associated with class D supermartingales rests on the existence of a predictable compensator of each element of $ca(\mathscr{P})$ and the fact that this may be represented as an increasing process. We encounter two difficulties in adapting this approach to our setting. First, the elements of $\mathscr{M}^{uc}(m)$ need not be countably additive in restriction to $\mathscr{P}$. To prove this implication, Dellacherie and Meyer exploit a Dini/Daniell argument [9, lemma p. 185] which requires (local) compactness of the index set. Second, a suitable notion of predictable compensator and of predictable projection is not available here[2]. In fact, compactness and separability of the index set imply that the class D property defined above is equivalent to countable additivity of Doléans-Dade measures relatively to $\mathscr{P}$.

[2]In Dozzi et al. [11, A3, p. 516] a related operator is introduced by assumption.

For the rest of this section we shall assume, without further mention, that T is a regular index set. In view of Theorem 1 this may be done with no loss of generality when the finitely additive supermartingales considered are of class D_0.

Fix $P \in \mathbb{P}(\mathscr{F})$. For given $d \in \mathscr{D}$ define the following elementary process

$$\mathscr{P}_P^d(b) = \sum_{s\cap t^c \in d} P\left(\inf_{i\in s\cap t^c} b(i)\middle| \mathscr{A}_s\right)\mathbf{1}_{s\cap t^c} \qquad b \in \mathfrak{B}(\bar\Omega), \tag{20}$$

where the conditional expectation is defined as in Lemma 1. We shall use the following fact:

Lemma 5. *Let $P \in \mathbb{P}(\mathscr{F})$ and define the mapping $\mathscr{P}_P^d : \mathfrak{B}(\bar\Omega) \to \mathscr{E}$ implicitly via (20). For given $f_0 \in \mathscr{E}$ and $f_1, f_2 \in \mathscr{E}^*$, there is $d_0 \in \mathscr{D}$ such that*

$$\mathscr{P}_P^d(f_0) = f_0 \quad \textit{and} \quad \mathscr{P}^d(f_1+f_2) = \mathscr{P}^d(f_1) + \mathscr{P}^d(f_2) \qquad \textit{for all} \qquad d \geq d_0.$$

Proof. Given Lemma 3, it is enough to suppose f_i to be of the form $f_i = h_i\mathbf{1}_{s_i\cap t_i^c}$ with $s_i \cap t_i^c \in \mathscr{D}$ $f_i \in \mathfrak{B}(\mathscr{A})$ for $i = 1, 2$ and $f_0 \in \mathfrak{B}(\mathscr{A}_{s_0})$. Let $d_0 \in \mathscr{D}$ be such that $d_0 \geq \{s_i \cap t_i^c : i = 0, 1, 2\}$. Then $d \in \mathscr{D}$ and $d \geq d_0$ imply

$$\mathscr{P}_P^d(f_i) = \sum_{\{s\cap t^c\in d: s\cap t^c\subset s_i\cap t_i^c,\ s\subset s_i\}} P(h_i|\mathscr{A}_s)\mathbf{1}_{s\cap t^c},$$

from which the claim follows straightforwardly. □

Theorem 2. *Let $\lambda \in \mathscr{M}^{uc}$, $P \in \mathbb{P}(\mathscr{F})$ and $\lambda_{\bar\Omega}|\mathscr{F} \ll P|\mathscr{F}$. There is $\lambda^P \in ba(\bar\Omega)_+$ such that $\lambda^P_{\bar\Omega}\big|\mathscr{F}$ vanishes on $P|\mathscr{F}$ null sets and*[3]

$$\lambda^P(fg) = \operatorname*{LIM}_{d\in\mathscr{D}} \lambda\left(\mathscr{P}_P^d(f)g\right) \qquad f \in \mathscr{E}^*,\ g \in \mathscr{E}. \tag{21}$$

If $\lambda^P|\sigma\mathscr{P}$ and $\lambda|\sigma\mathscr{P}$ are countably additive then

$$\lambda^P(fh) = \lambda\left(\lambda^P(f|\sigma\mathscr{P})h\right) \qquad f \in \mathscr{E}^*,\ h \in \mathfrak{B}(\sigma\mathscr{P}). \tag{22}$$

Proof. Consider the functional $\gamma : \mathfrak{B}(\bar\Omega) \to \mathbb{R}$ defined as $\gamma(f) = \mathrm{LIM}_{d\in\mathscr{D}}\, \lambda\left(\mathscr{P}_P^d(f)\right)$ for any $f \in \mathfrak{B}(\bar\Omega)$. Then, γ is a concave integral in the sense of [6, Definition 1, p. 560], it is linear on $\mathscr{E}^*$ by Lemma 5 and such that $\gamma = \lambda$ in restriction to $\mathscr{E}$; moreover, $\gamma(b) = 0$ for all b in $\mathscr{L} = \{g \in \mathfrak{B}(\bar\Omega) : P(\sup_{i\in I} |g(i)| > \eta) = 0 \text{ for all } \eta > 0\}$. Given that $\mathscr{L}$ is a linear space, then [6, Lemma 2, p. 560] implies that there exists $\lambda^P \in ba(\bar\Omega)_+$ such that

$$\lambda^P(g) = 0 \quad \text{and} \quad \lambda^P(f) \geq \gamma(f) \qquad g \in \mathscr{L},\ f \in \mathfrak{B}(\bar\Omega).$$

If $g \in \mathscr{E}$ and $f \in \mathscr{E}^*$, then

$$\lambda^P(fg) = \gamma(fg) = \operatorname*{LIM}_{d\in\mathscr{D}} \lambda(\mathscr{P}_P^d(fg)) = \operatorname*{LIM}_{d\in\mathscr{D}} \lambda(\mathscr{P}_P^d(f)g).$$

The last claim is obvious. □

Theorem 2 establishes the existence of a P predictable compensator, λ^P, associated with any $\lambda \in \mathscr{M}^{uc}$ and $P \in \mathbb{P}(\mathscr{F})$ such that $\lambda_{\bar\Omega}|\mathscr{F} \ll P|\mathscr{F}$. In the classical theory this concept interplays with the notion of predictable projection and requires countable additivity on $\mathscr{P}$. The failure of this latter property is overcome by means of the approximation procedure adopted in (21).

[3]LIM denotes the Banach limit.

Definition 4. *Let $\lambda \in \mathscr{M}^{uc}$ and $P \in \mathbb{P}(\mathscr{F})$ be such that $\lambda_{\bar{\Omega}}|\mathscr{F} \ll P$. Any $\lambda^P \in ba(\bar{\Omega})_+$ meeting (21) will be referred to as a P predictable compensator of λ.*

Remark that if $\lambda \in \mathscr{M}^{uc}(m)$ for some finitely additive supermartingale m, then its P compensator λ^P is itself a Doléans-Dade measure for m, i.e. $\lambda^P \in \mathscr{M}(m)$. It is, however, not possible to conclude that $\lambda^P \in \mathscr{M}^{uc}(m)$ in the general case: under finite additivity $\lambda^P_{\bar{\Omega}}$ may well vanish on P null while failing to be absolutely continuous. The class D property has then to be reinforced into the following.

Definition 5. *A finitely additive supermartingale m is said to be of class D_* if it is of class D and if there is $\lambda \in \mathscr{M}^{uc}(m)$ which admits as its P predictable compensator an element λ^P of $\mathscr{M}^{uc}(m)$.*

When $P \in \mathbb{P}(\mathscr{F})$, $(A_t : t \in T)$ is a P increasing process if $P(0 = A_{\bar{\Omega}} \le A_s \le A_t) = 1$ for all $s, t \in T$ with $s \le t$. $(B_t : t \in T)$ is then a modification of A if $P(A_t = B_t) = 1$ for all $t \in T$.

Theorem 3. *Let m be a finitely additive supermartingale of class D_*. Then for some $P \in \mathbb{P}(\mathscr{F})$ there exists one and only one (up to modification) way of writing*

$$m_t(F) = P((M - A_t)\mathbf{1}_F) \qquad t \in T,\ F \in \mathscr{A}_t, \tag{23}$$

where $M \in L(P)$ and A is an increasing process, adapted to $\mathbb{A}$ and such that

$$P\int f dA = \operatorname*{LIM}_{d\in\mathscr{D}} P\int \mathscr{P}^d_P(f)dA \qquad f \in \mathscr{E}^*. \tag{24}$$

Proof. Let $\lambda \in \mathscr{M}^{uc}(m)$ and fix $P \in \mathbb{P}(\mathscr{F})$ such that $m_\varnothing, \lambda|\mathscr{F} \ll P|\mathscr{F}$. Let also $\lambda^P \in \mathscr{M}^{uc}(m)$ be the P compensator of λ. Define m^{λ^P} as in (15) and let M and A'_t to be the Radon Nikodym derivatives of $m^{\lambda^P}_{\bar{\Omega}}\Big|\mathscr{F}$ and $\lambda^P_{t^c}\Big|\mathscr{F}$ with respect to $P|\mathscr{F}$. (23) is thus a version of (13). Clearly,

$$P\int f dA' = P\sum_{s\cap t^c\in d} f_s(A'_t - A'_s) = \lambda^P(f) \quad \text{for all} \quad f = \sum_{s\cap t^c\in d} f_s\mathbf{1}_{s\cap t^c} \in \mathscr{E}^*. \tag{25}$$

Therefore, (21) implies that (24) holds for A' and its modifications, among which, we claim, there is one which is adapted. In fact, if $b \in \mathfrak{B}(\mathscr{F})$, $s_0 \in T$ and $P(b|\mathscr{A}_{s_0}) = 0$, then, choosing $d \in \mathscr{D}$ such that $d \ge \{s_0^c\}$

$$\mathscr{P}^d_P(b)\mathbf{1}_{s_0^c} = \sum_{\{s\cap t^c\in d: s\cap t^c\subset s_0^c\}} P\left(b\middle|\mathscr{A}_s\right)\mathbf{1}_{s\cap t^c} = \sum_{\{s\cap t^c\in d: s\cap t^c\subset s_0^c\}} P\left(b\mathbf{1}_{\{s<s_0\}}\middle|\mathscr{A}_s\right)\mathbf{1}_{s\cap t^c} = 0,$$

a conclusion following from (2) and the fact that $P\left(b\mathbf{1}_{\{s<s_0\}}\middle|\mathscr{A}_s\right) = P\left(P(b|\mathscr{A}_{s_0})\mathbf{1}_{\{s<s_0\}}\middle|\mathscr{A}_s\right)$. Thus $\lambda^P_{s_0^c}(b) = \lambda^P(b\mathbf{1}_{s_0^c}) = 0$ and, letting $A_s = P(A'_s|\mathscr{A}_s)$ and $F \in \mathscr{F}$,

$$P(A'_s\mathbf{1}_F) = \lambda^P_{s^c}(F) = \lambda^P_{s^c}\left(P(F|\mathscr{A}_s)\right) = P(A'_sP(F|\mathscr{A}_s)) = P(A_s\mathbf{1}_F). \tag{26}$$

A is thus an adapted modification of A' and therefore itself an increasing process meeting (24). Suppose that $P(N|\mathscr{A}_t) - B_t$ is another decomposition such as (23). Then if $F \in \mathscr{A}_t$ and $d \in \mathscr{D}$

$$P\int \mathscr{P}^d_P(\mathbf{1}_F\mathbf{1}_t)dA = -P\int \mathscr{P}^d_P(\mathbf{1}_F\mathbf{1}_t)dX = P\int \mathscr{P}^d_P(\mathbf{1}_F\mathbf{1}_t)dB$$

and, if both A and B meet (24), $P(A_t\mathbf{1}_F) = P(B_t\mathbf{1}_F)$. □

Remark that Theorem 3 is actually weaker than the classical Doob Meyer decomposition first of all because the class D_* property is only a sufficient condition. Indeed the predictable increasing process A generates a measure on $\bar{\Omega}$ which satisfies (21) by construction, but it is hard to prove that its $\mathscr{F}$ marginal is countably additive. This difficulty is due to the lack of an appropriate topology on the underlying space. Second, we established uniqueness only up to a modification rather than indistinguishability, a circumstance which is almost unavoidable in the absence of separability of the index set and of right continuity of the process.

It should also be remarked that it may not be possible to establish the above decomposition if the index set T is not regular. Of course, a finitely additive supermartingale of class D_* may always be extended to a class D_* finitely additive supermartingale on a filtration endowed with a regular index set and thus admitting a Doob Meyer decomposition. However, the intervening increasing process A may not be adapted to the *original* filtration while its projection on it may not be increasing. On the other hand the class D_* property is a global property and is thus preserved under any enlargement of the filtration. In fact, if m is a finitely additive supermartingale of class with $\lambda^P \in \mathscr{M}^*(m)$ and if $\bar{\mathbb{A}} = (\bar{\mathscr{A}_u} : u \in U)$ is an extension of $\mathbb{A}$ with $\bigcup_{u \in U} \bar{\mathscr{A}_u} \subset \mathscr{F}$, then, using (15), $\bar{m} = (m_u^{\lambda^P} | \bar{\mathscr{A}_u} : u \in U)$ is clearly an extension of m to $\bar{\mathbb{A}}$ such that $\lambda^P \in \mathscr{M}^*(\bar{m})$. It appears therefore that the decomposition of Doob and Meyer depends more on the structure of the index set than on the filtration.

A less general decomposition is based on a further uniform integrability condition for processes.

Corollary 1. *Let m be a finitely additive supermartingale. Then the following are equivalent:*

(i) there exists $\lambda \in \mathscr{M}^{uc}(m)$ and $P \in \mathbb{P}(\mathscr{F})$ such that $m_\varnothing$, $\lambda_{\bar{\Omega}}|\mathscr{F} \ll P|\mathscr{F}$ and

$$\lim_{P(F)\to 0} \sup_{d \in \mathscr{D}} \lambda(\mathscr{P}^d(\mathbf{1}_{F\times I})) = 0; \tag{27}$$

(ii) m admits a Doob Meyer decomposition (23) where the increasing process A satisfies (24) and is of uniformly integrable variation, i.e. such that

$$\lim_{P(F)\to 0} \sup_{d \in \mathscr{D}} P\left(\mathbf{1}_F \sum_{s \cap t^c \in d} (P(A_t|\mathscr{A}_s) - A_s)\right) = 0. \tag{28}$$

Proof. It is clear that if a Doob Meyer decomposition exists with A as the P increasing process and $P \in \mathbb{P}(\mathscr{F})$ then (27) and (28) are equivalent. Thus, in view of Theorem 3, we only need to prove that (27) implies that m is of class D_*. But this follows from the inequality $\mathrm{LIM}_{d\in\mathscr{D}} \lambda(\mathscr{P}^d(\mathbf{1}_{F\times I})) \le \sup_{d\in\mathscr{D}} \lambda(\mathscr{P}^d(\mathbf{1}_{F\times I}))$ characterising Banach limits. □

The characterisation provided in Corollary 1 is less satisfactory than it may appear at first sight. In fact the property involved is significantly stronger than what is considered in the classical setting. Even increasing processes may fail to be of uniformly integrable variation.

The special case of a linearly ordered index set is eventually considered, with the aim of showing that the aforementioned properties generalize more classical ones. A natural example of this special case is easily obtained by extracting from any partially ordered index set a maximal linearly ordered subset. A more explicit example may be given by taking $U = \mathbb{R}_+ \times \mathbb{R}_+$ to be endowed with lexicographic order in terms of which $(x_1, y_1) \ge (x_2, y_2)$ if and only if either (i) $x_1 > x_2$ or (ii) $x_1 = x_2$ and $y_1 \ge y_2$.

Corollary 2. *Let m be a finitely additive supermartingale and let T be linearly ordered. Then the following are equivalent: (i) m is of class D, (ii) m is of class D_α, (iii) m satisfies (27), (iv) m is of class D_*.*

Proof. $(i)\to(ii)$ was proved in Lemma 4. If T is linearly ordered then each $d \in \mathscr{D}$ may be taken to be of the form $\{s_n \cap s_{n+1}^c : n = 1, \ldots, N-1\}$. Assume (ii) and choose $\lambda \in \mathscr{M}(m)$ and $P \in \mathbb{P}(\mathscr{F})$ as in Definition 3.

If $F \in \mathscr{F}$, $\eta > 0$ and $d \in \mathscr{D}$ then, letting $M_n^*(F) = \sup_{\{k \leq n\}} P(F|\mathscr{A}_{s_k})$ we have, exactly as in (19),

$$\lambda(\mathscr{P}_P^d(F)) \leq \eta\|\lambda\| + \|f\|_{\mathfrak{B}} \left(|m_\varnothing|(M_N^*(F) > \eta) + \sum_{n=1}^{N} |m_{t_n}|(M_n^*(F) > c \geq M_{n-1}^*(F)) \right). \tag{29}$$

Given that, by Doob maximal inequality, $\lim_{P(F)\to 0} P(M_N^*(F) > \eta) = 0$ uniformly in N, we conclude that (27) holds. The implication $(iii)\to(iv)$ was obtained in the proof of Corollary 1; $(iv)\to(i)$ is obvious. □

6. Some Remarks on the Literature

In the preceding sections we considered the possibility of extending finitely additive supermartingales to settings possessing more structure. This extension was achieved in (15) and was based on the class D_0 property; the possibility of extending classical supermartingales was considered for class D processes for which we established (18). Most papers in the literature, including Dozzi, Ivanoff and Merzbach [11, Proposition 2.1], Ivanoff and Merzbach [15, p. 85], Ivanoff and Sawyer [17, Proposition 6, p. 3] and De Giosa and Mininni [8, p. 74], obtain an additive extension based on a lemma by Norberg [23, Proposition 2.3, p. 9] concerning functions defined on lattices. In this section we shall briefly examine this approach, that does not make use of any other assumption, and show that this is troublesome.

Fix $P \in \mathbb{P}(\mathscr{F})$ and consider the semi-algebra

$$T(d) = \{s \cap t^c : s, t \in T,\ s \leq t\}. \tag{30}$$

According to the aforementioned references, any process $X = (X_t : t \in T)$ admits an additive extension to $T(d)$ defined by letting

$$X_{s\cap t^c} = X_s - X_t \qquad s \cap t^c \in T(d). \tag{31}$$

This should be compared to the extensions m_H and X_H defined in (15) and in (18) respectively. The former is indeed additive but requires the class D_0 property; the second is not even additive because of the measurability requirements. We provide two examples in which the lack of the class D_0 property and of measurability hinder the validity of the extension defined in (31).

Example 1. *Let T consist of finite unions of rectangles in $\mathbb{R}^2$ with one vertex in the origin, as in Cairoli and Walsh [3]. Let $\mathscr{A}_t = \mathscr{B}(\mathbb{R}^2)$ be the Borel σ algebra of $\mathbb{R}^2$, for each $t \in T$, and $l \in ba(\mathscr{B}(\mathbb{R}^2))$ be the product Lebesgue measure. Define the finitely additive supermartingale $(m_t : t \in T)$ implicitly by letting*

$$m_t(F) = l(t)l(t \cap F) \qquad F \in \mathscr{B}(\mathbb{R}^2),\ t \in T.$$

Let $s_1, t_1, u \in T$ be rectangles with $t_1 \subset s_1 \subset u^c$. Let $s_2 = s_1 \cup u$ and $t_2 = t_1 \cup u$: $s_2, t_2 \in T$. Of course, $(m_{s_i} - m_{t_i})(\Omega) = l(s_i)^2 - l(t_i)^2$ for $i = 1, 2$ so that

$$(m_{s_2} - m_{t_2})(\Omega) = (m_{s_1} - m_{t_1})(\Omega) + 2l(u)(l(s_1) - l(t_1)).$$

If $l(u) > 0$ and $l(s_1) > l(t_1)$ this result contradicts (31) since $s_1 \cap t_1^c = s_2 \cap t_2^c$.

In the preceding example it is clear that the finitely additive supermartingale considered is not of class D_0 and that (31) fails exactly for this reason. This difficulty does not arise in the framework adopted by Norberg [23], where the index set consists of all lower sets $\downarrow f \equiv \{g \in \mathscr{L} : g \leq f\}$ of elements f of some lattice $\mathscr{L}$. In terms of Example 1, we would have $s_i = \downarrow f_i$ and $t_i = \downarrow g_i$ for some $f_i, g_i \in \mathscr{L}$. Then $s_i \leq t_i$

is equivalent to $f_i \geq g_i$ and $s_1 \cap t_1^c = s_2 \cap t_2^c$ holds if and only if either $f_i = g_i$ – and thus $s_i \cap t_i^c = \varnothing$ – for $i = 1, 2$ or $f_1 = f_2$ and $g_1 = g_2$: in either case the extension (31) is indeed well defined.

The extension (31) may however be inconsistent even in the case of finitely additive supermartingales of class D.

Example 2. *Let $\mathscr{A}_t$ be a σ algebra for each $t \in T$. Suppose that $s_i, t_i \in T$ for $i = 1, 2$ with $s_i \leq t_i$, $t_1 \leq t_2$ and $s_i = t_i \cup t_j^c$ for $i \neq j$, fix $P \in \mathbb{P}(\mathscr{F})$ and consider the process $X = (P(F|\mathscr{A}_t) : t \in T)c$ for some $F \in \mathscr{F}$. Then (31) implies $P(F|\mathscr{A}_{t_2}) = P(F|\mathscr{A}_{s_2}) - P(F|\mathscr{A}_{s_1}) + P(F|\mathscr{A}_{t_1})$. However it is easily seen that, letting F vary in $\mathscr{A}_{t_2}$, this is contradictory unless $\mathscr{A}_{t_2} = \mathscr{A}_{t_1} \vee \mathscr{A}_{s_2}$, a restriction that has therefore to be explicitly assumed.*

It is significant that the more recent contributions to this literature, [16, Definition 3.1., p. 54], [25, Definition 5, p. 1084], treat the additive extension (31) as part of the definition of a set-indexed process.

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