Valuation of Two-Factor Interest Rate Contingent Claims using Green's Theorem

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Abstract

Over the years a number of two-factor interest rate models have been proposed that have formed the basis for the valuation of interest rate contingent claims. This valuation equation often takes the form of a partial differential equation, that is solved using the finite difference approach. In the case of two factor models this has resulted in solving two second order partial derivatives leading to boundary errors, as well as numerous first order derivatives. In this paper we demonstrate that using Green's theorem second order derivatives can be reduced to first order derivatives, that can be easily discretised; consequently two factor partial differential equations are easier to discretise than one factor partial differential equations. We illustrate our approach by applying it to value contingent claims based on the two factor CIR model. We provide numerical examples which illustrates that our approach shows excellent agreement with analytical prices and the popular Crank Nicolson method.

KEYWORDS: Box method, derivatives, Green's theorem

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Abstract

Over the years a number of two-factor interest rate models have been proposed that have formed the basis for the valuation of interest rate contingent claims. This valuation equation often takes the form of a partial differential equation, that is solved using the finite difference approach. In the case of two factor models this has resulted in solving two second order partial derivatives leading to boundary errors, as well as numerous first order derivatives. In this paper we demonstrate that using Green's theorem second order derivatives can be reduced to first order derivatives, that can be easily discretised; consequently two factor partial differential equations are easier to discretise than one factor partial differential equations. We illustrate our approach by applying it to value contingent claims based on the two factor CIR model. We provide numerical examples which illustrates that our approach shows excellent agreement with analytical prices and the popular Crank Nicolson method.

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1. Introduction

The fixed income market is one of the largest sectors of the financial markets where billions of dollars worth of assets are traded daily. Over the years a variety of interest rate models, both single-factor and multi-factors have been proposed that have formed the basis for the valuation of fixed income instruments. The most general of the single-factor interest rate models is the model proposed by Chan et-al (1992) (henceforth CKLS). The CKLS model encloses many of the earlier single-factor models such as that proposed by Vasicek (1977) and Cox et-al(1985b) (henceforth CIR). The main advantage of one-factor models is their simplicity as the entire yield curve is a function of a single state variable. However, there are several problems associated with single-factor models. First, single factor models assume that changes in the yield curve, and hence bond returns, are perfectly correlated. Secondly, the shape of the yield curve is severely restricted. To overcome these limitations a number of two factor models have been proposed, including those by Brennan and Schwartz (1979), Cox et-al (1985b) amongst others.

Complexity of the underlying stochastic processes used to model fixed income instruments means that except in few specific cases, numerical techniques are necessary. Widely used numerical techniques include the lattice approach of Cox et-al (1979), Monte Carlo Simulation of Boyle (1977) and the finite difference approach. The finite difference approach is widely used both in the financial markets and in academia. This approach involves transforming the valuation equation expressed as a partial differential equation into a set of finite difference equations. This set is then solved either iteratively or by elimination to obtain the value of the instruments. Number of researchers including Brennan and Schwartz (1979), Courtadon (1982), Hull and White (1990) have applied the finite difference approach to value fixed income instruments. Sorwar et-al (2007) introduced the Box method from the physical sciences and demonstrated by numerical experimentation that the Box method was superior to the finite difference approach for valuing fixed income instruments. In this paper we further develop the approach of Sorwar et-al (2007) to value two factor fixed income instruments. We demonstrate that by avoiding the traditional route of solving two factor valuation equation as suggested by Brennan and Schwartz (1979) and Hull and White (1990) and others, we in fact end up with a simpler valuation problem where only first order derivatives are present. Our approach involves using Green's theorem to convert a surface integral into a line integral, more specifically it allows us to convert a second order derivative into a first order derivative.

In Section 2, we provide a description of the general problem. We then state the valuation equation based on two-factor interest rate models. We then prove using Green's theorem how the valuation equation can be reduced to a form that involves first order derivatives only. We further demonstrate our approach in depth by applying the proposed technique to the two-factor CIR model. We develop the algorithm in depth and illustrate its accuracy by comparing calculated bond prices with analytical bond prices. In Section 3, we discuss the results. In the final Section we offer conclusions and discuss avenues for future research.

2. Two Factor General Diffusion

2.1 Deriving simplified equation using Green's theorem

Assume two stochastic processes that dictate contingent claim prices

$$dx = a_x\left(\underline{\theta_x}, x, t\right)dt + \sigma_x(x,t)dz_x \qquad (1)$$

$$dy = a_y\left(\underline{\theta_y}, y, t\right)dt + \sigma_y(y,t)dz_y \qquad (2)$$

where x and y are model specific stochastic parameters; $a_x\left(\underline{\theta_x}, x, t\right)$, $a_y\left(\underline{\theta_y}, y, t\right)$ are the drift terms and $\sigma_x(x,t)$, $\sigma_y(y,t)$ are the diffusion terms and $\underline{\theta_x}$ and $\underline{\theta_y}$ are a vector of model specific parameters.

Using standard hedging arguments and taking instantaneous short rate as *r(x,y)*, the above two stochastic processes lead to the following partial differential equation for contingent claim $U(x,y,\tau)$ at time t , expiring at time *T*, and $\tau = T - t$; subject to the usual boundary conditions.

$$\frac{1}{2}\sigma_x^2\frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma_y^2\frac{\partial^2 U}{\partial y^2} + \rho\sigma_x\sigma_y\frac{\partial^2 U}{\partial x\partial y} + a_x\frac{\partial U}{\partial x} + a_y\frac{\partial U}{\partial y} - r(x,y)U = \frac{\partial U}{\partial \tau} \qquad (3)$$

The standard approach in finance literature is to discretise equation (3) using finite difference approximations both for the first order and second order derivatives as in Brennan and Schwartz (1979), Hull and White (1990) etc. Instead of following this traditional route, we generalise the approach of Sorwar et-al (2007) to simplify equation (3):

$$\frac{\partial}{\partial x}\left(S(x,y)\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(W(x,y)\frac{\partial U}{\partial y}\right)+\rho R(x,y)\sigma_x\sigma_y\frac{\partial^2 U}{\partial x\partial y}$$
$$-R(x,y)r(x,y)U-R(x,y)\frac{\partial U}{\partial \tau}=0 \qquad (4)$$

PROPOSITION 2.1: *Using Green's theorem we can reduce equation (2.4) to the following form:*

$$\int_{c_{ij}}\left(S\frac{\partial U}{\partial x}dy-W\frac{\partial U}{\partial y}dx\right)+\iint_{r_{ij}}\rho R(x,y)\sigma_x\sigma_y\frac{\partial^2 U}{\partial x\partial y}dxdy$$
$$-\iint_{r_{ij}}R(x,y)r(x,y)U\,dxdy-\iint_{r_{ij}}R(x,y)\frac{\partial U}{\partial \tau}dxdy=0 \qquad (5)$$

Proof: As in Sorwar et-al (2007) we integrate equation (4) before descretising. Unlike Sorwar et-all (2007) where only a single factor process is considered, consideration of two factors leads to an equation, which involves integrating over a surface r_{ij} :

$$\iint_{r_{ij}}\frac{\partial}{\partial x}\left(S(x,y)\frac{\partial U}{\partial x}\right)dxdy+\iint_{r_{ij}}\frac{\partial}{\partial y}\left(W(x,y)\frac{\partial U}{\partial y}\right)dxdy$$
$$-\iint_{r_{ij}}R(x,y)r(x,y)U\,dxdy-\iint_{r_{ij}}R(x,y)\frac{\partial U}{\partial \tau}dxdy=0 \qquad (6)$$

From Green's theorem, we know that for any two differentiable functions *A(x,y)* and *B(x,y)* defined in r_{ij} :

$$\iint_{r_{ij}}\left(\frac{\partial A}{\partial x}-\frac{\partial B}{\partial y}\right)dxdy=\oint_{c_{ij}}\left(Bdx+Ady\right) \qquad (7)$$

where c_{ij} is the boundary of r_{ij} and the line integral is taken in the positive sense.

Rewriting equation (7) by letting:

$$A=S\frac{\partial U}{\partial x} \qquad (8)$$

$$B=-W\frac{\partial U}{\partial y} \qquad (9)$$

$$\iint_{r_{ij}} \left(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \right) dxdy = \int_{c_{ij}} \left(S \frac{\partial U}{\partial x} dy - W \frac{\partial U}{\partial y} dx \right) \qquad (10)$$

Substituting the right hand side of equation (10) for the first two left hand side terms of equation (6) yields the required equation.

2.2 Application to two factor CIR processes

To illustrate our approach, we concentrate on two-factor model of the term structure, set within the CIR framework (1985a, 1985b). Two dependent state variables *x* and *y* determine the nominal instantaneous interest rate *r(x,y)*:

$$r(x,y) = x + y \qquad (11)$$

We assume the risk-adjusted factors are generated by independent square root processes as below:

$$dx = (\kappa_1\theta_1 - \kappa_1 x - \lambda_1 x)dt + \sigma_1\sqrt{x}dz_x \qquad (12)$$

$$dy = (\kappa_2\theta_2 - \kappa_2 y - \lambda_2 y)dt + \sigma_2\sqrt{y}dz_y \qquad (13)$$

The valuation equation for contingent claims $U(x, y, \tau)$ assuming no intermediate cash flows is:

$$\frac{1}{2}\sigma_1^2 x \frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma_2^2 y \frac{\partial^2 U}{\partial y^2} + \rho\sigma_1\sigma_2\sqrt{xy}\frac{\partial^2 U}{\partial x \partial y}$$
$$+ [\kappa_1\theta_1 - x(\kappa_1 - \lambda_1)]\frac{\partial U}{\partial x} + [\kappa_2\theta_2 - x(\kappa_2 - \lambda_2)]\frac{\partial U}{\partial y} - (x+y)U = \frac{\partial U}{\partial \tau}$$

$$(14)$$

Dividing equation (14) by *xy* gives:

$$\frac{1}{2}\frac{\sigma_1^2}{y}\frac{\partial^2 U}{\partial x^2}+\frac{1}{2}\frac{\sigma_2^2}{x}\frac{\partial^2 U}{\partial y^2}+\frac{\rho\sigma_1\sigma_2}{\sqrt{xy}}\frac{\partial^2 U}{\partial x\partial y}$$
$$+\left[\frac{\kappa_1\theta_1}{xy}-\frac{1}{y}(\kappa_1-\lambda_1)\right]\frac{\partial U}{\partial x}+\left[\frac{\kappa_2\theta_2}{xy}-\frac{1}{x}(\kappa_2-\lambda_2)\right]\frac{\partial U}{\partial y}-\left(\frac{1}{y}+\frac{1}{x}\right)U=\frac{1}{xy}\frac{\partial U}{\partial \tau}$$

(15)

Now consider *x* and *y* derivatives of equation (15) separately in terms of functions *P(x)* and *Q(y)* respectively:

$$\frac{1}{2}\frac{\sigma_1^2}{y}\frac{\partial^2 U}{\partial x^2}+\left[\frac{\kappa_1\theta_1}{xy}-\frac{1}{y}(\kappa_1-\lambda_1)\right]\frac{\partial U}{\partial x}=\frac{1}{P(x)}\frac{\partial}{\partial x}\left(P(x)\frac{\partial U}{\partial x}\right)$$

(16)

$$\frac{1}{2}\frac{\sigma_2^2}{x}\frac{\partial^2 U}{\partial y^2}+\left[\frac{\kappa_2\theta_2}{xy}-\frac{1}{x}(\kappa_2-\lambda_2)\right]\frac{\partial U}{\partial y}=\frac{1}{Q(y)}\frac{\partial}{\partial y}\left(P(y)\frac{\partial U}{\partial y}\right)$$

(17)

Expanding equations (16) and (17) and integrating gives:

$$P(x)=\exp\left\{\frac{2\kappa_1\theta_1}{\sigma_1^2}\ln(x)-\frac{2}{\sigma_1^2}(\kappa_1-\lambda_1)x\right\}=x^{\frac{2\kappa_1\theta_1}{\sigma_1^2}}\exp\left\{-\frac{2}{\sigma_1^2}(\kappa_1-\lambda_1)x\right\}$$

(18)

$$Q(y)=\exp\left\{\frac{2\kappa_2\theta_2}{\sigma_2^2}\ln(y)-\frac{2}{\sigma_2^2}(\kappa_2-\lambda_2)y\right\}=y^{\frac{2\kappa_2\theta_2}{\sigma_2^2}}\exp\left\{-\frac{2}{\sigma_2^2}(\kappa_2-\lambda_2)y\right\}$$

(19)

We now define a new function *R(x,y)* which is a product of *P(x)* and *Q(y)*:

$$R(x,y)=x^{\alpha_0}y^{\beta_0}\exp(-\alpha_1 x-\beta_1 y) \qquad (20)$$

where:

$$\alpha_0 = \frac{2\kappa_1\theta_1}{\sigma_1^2}, \beta_0 = \frac{2\kappa_2\theta_2}{\sigma_2^2}, \alpha_1 = \frac{2}{\sigma_1^2}(\kappa_1 - \lambda_1), \beta_1 = \frac{2}{\sigma_2^2}(\kappa_2 - \lambda_2)$$

Further define:

$$S(x,y) = \frac{\sigma_1^2}{2y} R(x,y) \tag{21}$$

$$W(x,y) = \frac{\sigma_2^2}{2x} R(x,y) \tag{22}$$

Thus the original partial differential equation (14) becomes:

$$\frac{\partial}{\partial x}\left(S\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right) + \frac{\rho\sigma_1\sigma_2}{\sqrt{xy}} R\frac{\partial^2 U}{\partial x \partial y} - R\left(\frac{1}{x} + \frac{1}{y}\right)U = \frac{R}{xy}\frac{\partial U}{\partial \tau} \tag{23}$$

Taking the forward Euler-difference for the time derivative in equation (3.13) gives us:

$$\frac{\partial U}{\partial \tau} = \frac{U - U_0}{\Delta t} \tag{24}$$

Substituting the above approximation and re-arranging equation (23) yields:

$$-\left[\frac{\partial}{\partial x}\left(S\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right)\right] - \frac{\rho\sigma_1\sigma_2}{\sqrt{xy}} R\frac{\partial^2 U}{\partial x \partial y} - R\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy\Delta t}\right)U = \frac{RU_0}{xy\Delta t} \tag{25}$$

Using Proposition (2.1) allows us to transform equation (25) to the following form involving a line integral:

$$-\oint_{c_{ij}}\left(S\frac{\partial U}{\partial x}dy - W\frac{\partial U}{\partial y}dx\right) - \iint_{r_{ij}} R\frac{\rho\sigma_1\sigma_2}{\sqrt{xy}}\frac{\partial^2 U}{\partial x \partial y}dxdy + \iint_{r_{ij}} R\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy\Delta t}\right)Udxdy = \iint_{r_{ij}} \frac{R}{xy\Delta t}U_0 dxdy \tag{26}$$

For the line integral noting that *dy* is zero when moving along the *x* direction and *dx* is zero when moving along the *y* direction we have:

$$-\oint_{c_{ij}}\left(S\frac{\partial U}{\partial x}dy - W\frac{\partial U}{\partial y}dx\right) = \int_1^2 W\frac{\partial C}{\partial y}dx - \int_2^3 S\frac{\partial C}{\partial x}dy + \int_3^4 W\frac{\partial C}{\partial y}dx - \int_4^1 S\frac{\partial C}{\partial x}dy \tag{27}$$

We let *x*, *y* and τ take value on the interval $\Gamma_x = [0, X], \Gamma_y = [0, Y], \mathrm{T} = [0, T]$. To solve the above equation we need to fit the space $\Gamma_x \times \Gamma_y \times \mathrm{T}$. We let Δx, Δy and Δt represent the grid spacing the *x*, *y* and τ direction respectively, such that:

$x_i = i\Delta x \quad \text{for } 0 \le N_x \quad \text{such that } \Gamma_x = N_x \Delta x$

$y_j = j\Delta y \quad \text{for } 0 \le N_y \quad \text{such that } \Gamma_y = N_y \Delta y$

$\tau_m = m\Delta t \quad \text{for } 0 \le M \quad \text{such that } \mathrm{T} = M\Delta t$

Further we take:

$$x_{i+\frac{1}{2}} = X_U = \frac{x_{i+1} + x_i}{2}$$

$$x_{i-\frac{1}{2}} = X_L = \frac{x_i + x_{i-1}}{2}$$

$$y_{j+\frac{1}{2}} = Y_U = \frac{y_{j+1} + y_j}{2}$$

$$y_{j-\frac{1}{2}} = Y_L = \frac{y_j + y_{j-1}}{2}$$

Discretising each of the line integrals and simplifying gives:

$$\begin{aligned} -\oint_{c_{ij}}\left(S\frac{\partial U}{\partial x}dy - W\frac{\partial U}{\partial y}dx\right) &= \frac{\Delta x}{\Delta y}\left[W_{i,j+\frac{1}{2}}\left(U_{ij}^m - U_{i,j+1}^m\right) + W_{i,j-\frac{1}{2}}\left(U_{ij}^m - U_{i,j-1}^m\right)\right] \\ &\quad + \frac{\Delta y}{\Delta x}\left[S_{i+\frac{1}{2},j}\left(U_{ij}^m - U_{i+1,j}^m\right) + S_{i-\frac{1}{2},j}\left(U_{ij}^m - U_{i-1,j}^m\right)\right] \end{aligned} \tag{28}$$

Discretising the remaining components gives

$$\iint_{r_{ij}} \frac{R}{xy\Delta t} U_0 dxdy \approx \frac{R_{ij} U_{ij}^{m-1}}{\Delta t} \int_{X_L}^{X_U} \int_{Y_L}^{Y_U} \frac{1}{xy} dxdy = \frac{R_{ij}}{\Delta t} \ln\left(\frac{X_U Y_U}{X_L Y_L} \right) U_{ij}^{m-1} = H_{ij} U_{ij}^{m-1} \tag{29}$$

$$\begin{aligned} \iint_{r_{ij}} R\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy\Delta t} \right) U dxdy &= \int_{X_L}^{X_U} \int_{Y_L}^{Y_U} R\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy\Delta t} \right) U dxdy \\ &\approx R_{ij} \left[\ln\left(\frac{Y_U}{Y_L} \right) (X_U - X_L) + \ln\left(\frac{X_U}{X_L} \right) (Y_U - Y_L) + \frac{1}{\Delta t} \ln\left(\frac{X_U Y_U}{X_L Y_L} \right) \right] U_{ij}^m = G_{ij} U_{ij}^m \end{aligned} \tag{30}$$

$$\begin{aligned} -\iint_{r_{ij}} R \frac{\rho \sigma_1 \sigma_2}{\sqrt{xy}} \frac{\partial^2 U}{\partial x \partial y} dxdy = -\frac{\rho \sigma_1 \sigma_2}{16 \Delta x \Delta y} \left(\sqrt{X_U} - \sqrt{X_L} \right) \left(\sqrt{Y_U} - \sqrt{Y_L} \right) \times \\ \left(U_{i+1,j+1}^m - U_{i+1,j-1}^m - U_{i-1,j+1}^m + U_{i-1,j-1}^m \right) \end{aligned} \tag{31}$$

Collecting all the terms and re-arranging gives the final matrix equation as:

$$\begin{aligned} &A_{ij} U_{ij}^m + C_{ij} U_{i,j+1}^m + D_{ij} U_{i,j-1}^m + E_{ij} U_{i+1,j}^m + F_{ij} U_{i-1,j}^m \\ &+ I_{ij} \left(U_{i+1,j+1}^m - U_{i+1,j-1}^m - U_{i-1,j+1}^m + U_{i-1,j-1}^m \right) = H_{ij} U_{ij}^{m-1} \\ &\text{where} \\ &A_{ij} = \frac{\Delta x}{\Delta y} \left(\frac{W_{i,j+\frac{1}{2}}}{R_{ij}} + \frac{W_{i,j-\frac{1}{2}}}{R_{ij}} \right) + \frac{\Delta y}{\Delta x} \left(\frac{S_{i+\frac{1}{2},j}}{R_{ij}} + \frac{S_{i-\frac{1}{2},j}}{R_{ij}} \right) + G_{ij} \\ &C_{ij} = -\frac{\Delta x}{\Delta y} \frac{W_{i,j+\frac{1}{2}}}{R_{ij}}, D_{ij} = -\frac{\Delta x}{\Delta y} \frac{W_{i,j-\frac{1}{2}}}{R_{ij}} \\ &E_{ij} = -\frac{\Delta y}{\Delta x} \frac{S_{i+\frac{1}{2},j}}{R_{ij}}, F_{ij} = -\frac{\Delta y}{\Delta x} \frac{S_{i-\frac{1}{2},j}}{R_{ij}} \\ &I_{ij} = -\frac{\rho \sigma_1 \sigma_2}{16 \Delta x \Delta y} \left(\sqrt{X_U} - \sqrt{X_L} \right) \left(\sqrt{Y_U} - \sqrt{Y_L} \right) \end{aligned} \tag{32}$$

To determine contingent claim prices we use the following SOR iteration subject to appropriate boundary conditions:

$$
\begin{aligned}
Z_{ij}^{m} &= \frac{1}{A_{ij}}\begin{pmatrix} G_{ij}U_{ij}^{m-1} - C_{ij}U_{i,j+1}^{m} - D_{ij}U_{i,j+1}^{m} - E_{ij}U_{i+1,j}^{m} - F_{ij}U_{i-1,j}^{m} \\ -I_{ij}\left(U_{i+1,j+1}^{m} - U_{i+1,j-1}^{m} - U_{i-1,j+1}^{m} + U_{i-1,j-1}^{m}\right) \end{pmatrix} \\
U_{ij}^{m} &= \omega Z_{ij}^{m} + (1-\omega)U_{ij}^{m-1} \\
&\text{for } i = 1,\ldots\ldots, N_x - 1 \quad j = 1,\ldots\ldots, N_y - 1 \text{ and } \omega \in (1,2]
\end{aligned}
$$

(33)

3. Discussion of Results

In Table 1 and Table 2, we compare numerically evaluated bond prices with analytical bond prices. The maturities of the bonds range from 1 year to 15 years. The face value of the zero coupon bonds are \$100. Interest rates ranging from 5% to 9% are considered. For direct comparison with analytical bond prices the correlation coefficient is zero. Further we alter the annual number of time steps from 20 to 50. This variation serves as a check on the stability of the numerical scheme.

In Table 3 to Table 6 we compare American call and put prices calculated using the Box Method and the Crank-Nicolson method based on time step of 1/50 per year. In Table 3 to Table 4 we focus on short expiry options, whereas in Table 5 – Table 6 we focus on long dated options.

Examining Table 1 and Table 2, we can conclude the following. First, it is possible to obtain accurate bond prices using the Box Method and Crank Nicolson, with as little as 20 annual time steps per year. For example, for a five year bond, with x = 5%, y = 5% the analytical bond price is \$59.6215 compared with the Box price of \$59.5986 and Crank Nicolson price of \$59.3703. Secondly, increasing the annual number of time steps leads to more accurate bond prices. Again, considering the same five year bond at the same interest rates, we find the Box price with 50 annual time steps is \$59.5396 and Crank Nicolson price is \$59.3710. Thirdly, we find that Crank Nicolson leads to more accurate bond prices for short term bonds, whereas the Box Method leads to more accurate bond prices at longer maturities. Finally, bond prices show expected trends. Thus, as the short-term interest rates and the term to maturity of the bonds increase, bond prices decline. From Table 3 – Table 6 we find that Box call options are always higher than Crank Nicoslon call prices and Box put are always lower than Crank Nicolson puts. This discrepancy is larger for longer maturity options. Given the evidence in Table – Table 2, it is likely that the Crank-Nicolson method is marginally less accurate than the Box method for longer expiry options.

4. Conclusion

This paper focuses on the valuation of two factor interest rate, contingent claims. By expanding the earlier work of Sorwar et-al (2007), this paper proposes a more elegant technique based on Green's theorem to solve the valuation equation.

Further this paper illustrates how the proposed technique can be applied in the case of a specific two-factor model. Concentrating on the two factor CIR model, this paper illustrates the steps necessary to develop the system of equations needed to value the contingent claims. Numerical experimentation shows excellent agreement between analytical CIR bond prices and computed CIR bond prices.

Over the years a large number of two factor interest rate models have been proposed, many of which do not lead to analytical contingent claim prices. The proposed Box method offers an easy and quick route to examine the pricing implications of these models. These implications remain the avenue of future research.

Table 1. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$\kappa_1 = 0.5,\ \sigma_1 = 0.15,\ \theta_1 = 0.06,\ \lambda_1 = 0$

$\kappa_2 = 0.005,\ \sigma_2 = 0.07,\ \theta_2 = 0.03,\ \lambda_2 = 0$

$\rho = 0,\ \Delta r = 0.5\%,\ \Delta t = 1/20$ years

Maturity	Method	*x*=5,y=5	*x*=5,y=7	*x*=5,y=9	*x*=7,y=5	*x*=7,y=7	*x*=7,y=9	*x*=9,y=5	*x*=9,y=7	*x*=9,y=9
1 year	Anal.	90.3114	88.5290	86.7817	88.9052	87.1506	85.4305	87.5209	85.7936	84.1003
	Box	90.3401	88.5663	86.8295	88.9503	87.2048	85.4958	87.5837	85.8662	84.1844
	CN	90.3115	88.5291	86.7818	88.9052	87.1505	85.4304	87.5208	85.7934	84.1001
5 years	Anal.	59.4534	53.9762	48.9872	57.3677	52.0739	47.2687	55.3551	50.2471	45.6105
	Box	59.5986	54.1285	49.1643	57.5013	52.2256	47.4366	55.4824	50.3937	45.7734
	CN	59.3703	53.8936	48.9214	57.3190	52.0316	47.2310	55.3232	50.2200	45.5865
10 years	Anal.	35.9733	30.0203	25.0524	34.6260	28.8960	24.1142	33.3292	27.8138	23.2110
	Box	36.0449	30.1646	25.2158	34.6289	29.0271	24.2657	33.3754	27.9348	23.3533
	CN	35.5625	29.6589	24.7454	34.3248	28.6302	23.8882	33.0924	27.6044	23.0330
15years	Anal.	22.6975	17.7325	13.8535	21.8439	17.0656	13.3325	21.0224	16.4238	12.8311
	Box	22.4823	17.7154	13.9088	21.6825	17.0444	13.3826	20.8088	16.4000	12.8772
	CN	22.0139	17.1259	13.3483	21.3032	16.5823	12.9285	20.5708	16.0177	12.4908

Table 2. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$$\kappa_1 = 0.5,\ \sigma_1 = 0.15,\ \theta_1 = 0.06,\ \lambda_1 = 0$$

$$\kappa_2 = 0.005,\ \sigma_2 = 0.07,\ \theta_2 = 0.03,\ \lambda_2 = 0$$

$$\rho = 0,\ \Delta r = 0.5\%,\ \Delta t = 1/50 \text{ years}$$

Maturity	**Method**	***x*=5,y=5**	***x*=5,y=7**	***x*=5,y=9**	***x*=7,y=5**	***x*=7,y=7**	***x*=7,y=9**	***x*=9,y=5**	***x*=9,y=7**	***x*=9,y=9**
1 year	Anal.	90.3114	88.5290	86.7817	88.9052	87.1506	85.4305	87.5209	85.7936	84.1003
	Box	90.3284	88.5485	86.8050	88.9272	87.1756	85.4595	87.5493	85.8254	84.1364
	CN	90.3114	88.5290	86.7817	88.9052	87.1505	85.4305	87.5209	85.7936	84.1003
5 years	Anal.	59.4534	53.9762	48.9872	57.3677	52.0739	47.2687	55.3551	50.2471	45.6105
	Box	59.5396	54.0588	49.0817	57.4402	52.1545	47.3534	55.4192	50.3212	45.6895
	CN	59.3710	53.8926	48.9199	57.3201	52.0311	47.2301	55.3246	50.2198	45.5859
10 years	Anal.	35.9733	30.0203	25.0524	34.6260	28.8960	24.1142	33.3292	27.8138	23.2110
	Box	35.9581	30.0844	25.1361	34.5994	28.9500	24.1891	33.2950	27.8606	23.2796
	CN	35.5671	29.6598	24.7447	34.3294	28.6314	23.8878	33.0970	27.6057	23.0328
15 years	Anal.	22.6975	17.7325	13.8535	21.8439	17.0656	13.3325	21.0224	16.4238	12.8311
	Box	22.3896	17.6430	13.8477	21.5396	16.9749	13.3240	20.7235	16.3333	12.8210
	CN	22.0197	17.1284	13.3489	21.3090	16.5848	12.9293	20.5765	16.0203	12.4917

Table 3. Option prices calculated using the Box and the Crank Nicholson methods.

$\kappa_1 = 0.5$, $\sigma_1 = 0.15$, $\theta_1 = 0.06$, $\lambda_1 = 0$

$\kappa_2 = 0.005$, $\sigma_2 = 0.07$, $\theta_2 = 0.03$, $\lambda_2 = 0$

$\rho = 0$, $\Delta r = 0.5\%$, $\Delta t = 1/50$ years

$x=y=5\%$

	1year maturity bond			5 year maturity bond		
	6 month expiry option			1 year expiry option		
Method	K	Call	Put	K	Call	Put
Box	85	9.5109	0.0000	55	9.8923	0.0998
CN		9.5004	0.0000		9.7339	0.0973
Box	86	8.5601	0.0000	56	9.0093	0.1669
CN		8.5497	0.0000		8.8520	0.1655
Box	87	7.6093	0.0002	57	8.1367	0.2738
CN		7.5990	0.0000		7.9810	0.2756
Box	88	6.6585	0.0013	58	7.2786	0.4405
CN		6.6483	0.0004		7.1253	0.4495
Box	89	5.7077	0.0107	59	6.4404	0.6953
CN		5.6796	0.0055		6.2904	0.7185
Box	90	4.7569	0.0885	60	5.6284	1.0769
CN		4.7469	0.0702		5.4828	1.1259
Box	91	3.8065	0.6716	61	4.8500	1.6376
CN		3.7964	0.6886		4.7100	1.7284
Box	92	2.8582	1.6716	62	4.1133	2.4604
CN		2.8482	1.6886		3.9802	2.6290
Box	93	1.9236	2.6716	63	3.4266	3.4604
CN		1.9142	2.6886		3.3019	3.6290
Box	94	1.0537	3.8716	64	2.7980	4.4604
CN		1.0478	3.6886		2.6826	4.6290
Box	95	0.3868	4.6716	65	2.2342	5.4604
CN		0.3863	4.6886		2.1291	5.6290

Table 4. Option prices calculated using the Box and the Crank Nicholson methods.

$\kappa_1 = 0.5$, $\sigma_1 = 0.15$, $\theta_1 = 0.06$, $\lambda_1 = 0$

$\kappa_2 = 0.005$, $\sigma_2 = 0.07$, $\theta_2 = 0.03$, $\lambda_2 = 0$

$\rho = 0$, $\Delta r = 0.5\%$, $\Delta t = 1/50$ years

$x=y=5\%$

	10year maturity bond			15 year maturity bond		
	5 year expiry option			10 year expiry option		
Method	K	Call	Put	K	Call	Put
Box	30	18.0992	0.1724	17	16.2769	0.2008
CN		17.7206	0.1871		15.7250	0.2376
Box	31	17.5051	0.2495	18	15.9174	0.2940
CN		17.1276	0.2720		15.3603	0.3479
Box	32	16.9114	0.3560	19	15.5580	0.4222
CN		16.5351	0.3900		14.9968	0.4989
Box	33	16.3184	0.5015	20	15.1986	0.5961
CN		15.9434	0.5512		14.6345	0.7019
Box	34	15.7261	0.6978	21	14.8393	0.8287
CN		15.3525	0.7694		14.2733	0.9708
Box	35	15.1348	0.9596	22	14.4800	1.1366
CN		14.7626	1.0610		13.9131	1.3216
Box	36	14.5447	1.3052	23	14.1209	1.5400
CN		14.1741	1.4461		13.5538	1.7734
Box	37	13.9561	1.7564	24	13.7619	2.0636
CN		13.5872	1.9481		13.1954	2.3470
Box	38	13.3694	2.3407	25	13.4031	2.7386
CN		13.0023	2.5972		12.8379	3.0679
Box	39	12.7849	3.0824	26	13.0446	3.6104
CN		12.4198	3.4329		12.4811	3.9803
Box	40	12.2031	4.0419	27	12.6863	4.6104
CN		11.8402	4.4329		12.1250	4.9803

Reference

Sorwar, G., Barone-Adesi, G., and W. Allegretto (1997): "Valuation of derivatives based on single-factor interest rate models," *Global Finance Journal.* 18, 251-269.

Boyle, P. P., (1977): "Options: A Monte Carlo Approach," *Journal Financial Economics.* 4, 323-338.

Brennan, M., and E. Schwartz (1979): "A Continuous-Time Approach to the Pricing of Bonds," *Journal of Banking Finance* 3, 133-155.

Chan, K. C., Karolyi, G. A., Longstaff, F. A., and A.B. Sanders, (1992). "An empirical comparison of the short-term interest rate," *Journal of Finance,* 1(7), 1209−1227.

Chen, R.R., and L. Scott (1992): "Pricing Interest Rate Option in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure," *Review of. Financial Stud*ies, 5, 613-636.

Courtadon, G., (1982): "The Pricing of Options on Default-Free Bonds," *Journal of Financial and Quantitative Analysis,* 17, 75-100.

Cox, J.C., S. A. Ross, and M. Rubinstein (1979): "Option Pricing: A Simple Approach," *Journal of Financial Econ*omics, 7, 229-263.

Cox, J. C, J. E. Ingersoll, and S. A. Ross (1985a): "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363-384.

Cox, J. C, J. E. Ingersoll, and S. A. Ross (1985b): "A Theory of the Term Structure of Interest Rate ," *Econometrica*, 53, 385-407.

Hull, J., and A. White (1990): "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis* 25, 87-99.

Vasicek, O. A., (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics.* 5, 177-188.