

# SURE WINS, SEPARATING PROBABILITIES AND THE REPRESENTATION OF LINEAR FUNCTIONALS

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ABSTRACT. We discuss conditions under which a convex cone  $\mathcal{K} \subset \mathbb{R}^\Omega$  admits a finitely additive probability  $m$  such that  $\sup_{k \in \mathcal{K}} m(k) \leq 0$ . Based on these, we characterize those linear functionals that are representable as finitely additive expectations. A version of Riesz decomposition based on this property is obtained as well as a characterisation of positive functionals on the space of integrable functions.

## 1. INTRODUCTION

A long standing approach to probability, originating from the seminal work of de Finetti, views set functions  $P$  as maps which assign to each set (event)  $E$  in some class  $\mathcal{A}$  the price  $P(E)$  for betting 1 dollar on the occurrence of  $E$ . A set function generating a betting system which admits no sure wins was termed coherent by de Finetti who proved in [5] that a set function on a finite algebra  $\mathcal{A}$  is coherent if and only if it is a probability. Since then this result has been extended and generalized by various authors, among which Heath and Sudderth [11], Lane and Sudderth [12] and Regazzini [13], to name but a few; Borkar et al. [4] is a more recent example.

In this paper we examine the absence of sure wins for a convex cone  $\mathcal{K}$  of real valued functions on some arbitrary set  $\Omega$ , obtaining conditions for the existence of a finitely additive probability measure  $m$  such that  $\sup_{k \in \mathcal{K}} m(k) \leq 0$ , i.e. a *separating probability*. The special case in which  $\mathcal{K}$  is the kernel of some linear functional leads to the characterization of those functionals that admit the representation as finitely additive expectations, a topic addressed by Berti and Rigo in a highly influential paper [2]. A version of Riesz decomposition based on this representation property is obtained.

Throughout the paper  $\Omega$  will be a fixed set,  $2^\Omega$  its power set,  $\mathbb{R}^\Omega$  and  $\mathfrak{B}$  the classes of real-valued and of bounded functions on  $\Omega$  respectively (the latter endowed with the topology induced by the supremum norm). All spaces of real-valued functions on  $\Omega$  (e.g. bounded or integrable) will be considered as equipped with pointwise ordering, with no further mention.  $f^+$  and  $f^-$  will denote the positive and negative parts of  $f \in \mathbb{R}^\Omega$ . The term probability designates positive, finitely additive set functions  $m$  on  $2^\Omega$  (in symbols,  $m \in ba_+$ ) such that  $m(\Omega) = 1$ . The symbol  $\mathbb{P}_{ba}$  will be used to denote the family of all probability measures;  $\mathbb{P}$  the subfamily of all countably additive probability measures. If  $\mathcal{A} \subset 2^\Omega$  then by  $\mathcal{S}(\mathcal{A})$  and  $\mathfrak{B}(\mathcal{A})$  we denote the class of simple functions generated by  $\mathcal{A}$  and its closure in  $\mathfrak{B}$ . We adopt the useful convention of identifying single-valued functions with their range so that, for example, we may use 1 either to denote an element of  $\mathbb{R}$ , or a function  $f$  on  $\Omega$  such that  $f(\omega) = 1$  for all  $\omega \in \Omega$ . In the terminology adopted throughout the following sections a *sure win* is defined to be an element of  $\mathbb{R}^\Omega$  which exceeds 1.

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We recall that  $f \in \mathbb{R}_+^\Omega$  is integrable with respect to  $m \in ba_+$ , in symbols  $f \in L(m)$ , if and only if

$$(1.1) \quad \sup \{m(h) : h \in \mathfrak{B}, 0 \leq h \leq f\} < \infty$$

The integral  $m(f)$  coincides then with the left hand side of (1.1); moreover,  $f \wedge n$  converges to  $f$  in  $L(m)$  [9, theorem III.3.6]. A special notion of convergence in  $L(m)$  will be used in the following. A sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  is said to converge orderly in  $L(m)$  to  $f$  if  $f_n \in L(m)$  for all  $n$  and there exists a pointwise decreasing sequence  $\langle \bar{f}_n \rangle_{n \in \mathbb{N}}$  in  $L(m)_+$  which converges to 0 in  $L(m)$  and is such that  $|f_n - f| \leq \bar{f}_n$  for  $n \geq 1$ . It is easily seen that if a sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  converges to  $f$  orderly in  $L(m)$  then so does any of each subsequences; moreover, the space of sequences converging orderly in  $L(m)$  is a vector space.

## 2. SEPARATING PROBABILITIES

Fix a convex cone  $\mathcal{K} \subset \mathbb{R}^\Omega$  (that is  $f + g, \lambda f \in \mathcal{K}$  whenever  $f, g \in \mathcal{K}$  and  $\lambda \geq 0$ ) and let  $\mathcal{K}_b = \{k \in \mathcal{K} : k^- \in \mathfrak{B}\}$ . For each  $f \in \mathbb{R}^\Omega$  let  $\mathcal{U}(f) = \{\alpha \in \mathbb{R} : \alpha + k \geq f \text{ for some } k \in \mathcal{K}\}$  and define  $\pi_{\mathcal{K}} : \mathbb{R}^\Omega \rightarrow \overline{\mathbb{R}}$  as

$$(2.1) \quad \pi_{\mathcal{K}}(f) = \inf \{\alpha : \alpha \in \mathcal{U}(f)\}$$

The setting presented here, although inspired by de Finetti approach to probability as explained in the introduction, has a direct translation into the language of mathematical finance where the elements of  $\mathcal{K}$  represent net, discounted returns from available investment opportunities<sup>1</sup>. The functional  $\pi_{\mathcal{K}}$  is then well known under the name of *superhedging price*. The key mathematical property of models of financial markets is the absence of *arbitrage opportunities* that is the assumption that  $\mathcal{K}$  contains no strictly positive element (see [7, p. 31] and references therein). A sure win is in fact an arbitrage opportunity of a special type as it admits a positive, uniform lower bound.

From (2.1),  $\pi_{\mathcal{K}}$  is monotonic,  $\pi_{\mathcal{K}}(\lambda + f) = \lambda + \pi_{\mathcal{K}}(f)$  for each  $\lambda \in \mathbb{R}$  and  $f \in \mathbb{R}^\Omega$  and  $\pi_{\mathcal{K}}(f) \leq \sup_{\omega \in \Omega} f(\omega)$  (as  $0 \in \mathcal{K}$ ). Since  $\mathcal{K}$  is a convex cone,  $\mathcal{U}(f) + \mathcal{U}(g) \subset \mathcal{U}(f + g)$  and  $\mathcal{U}(\lambda f) = \lambda \mathcal{U}(f)$  for  $\lambda > 0$ :  $\pi_{\mathcal{K}}$  is thus subadditive and positively homogeneous; moreover,  $\pi_{\mathcal{K}}(k) \leq 0$  for all  $k \in \mathcal{K}$ .

Given that  $\pi_{\mathcal{K}}(0) = 2\pi_{\mathcal{K}}(0) \leq 0$  and  $\pi_{\mathcal{K}}(1) = \pi_{\mathcal{K}}(0) + 1$ , then  $\pi_{\mathcal{K}}(0) > -\infty$  implies  $\pi_{\mathcal{K}}(0) = 0$  and  $\pi_{\mathcal{K}}(1) = 1$ . Moreover there is  $k \in \mathcal{K}$  such that  $k \geq 1$  if and only if  $\pi_{\mathcal{K}}(1) \leq 0$ . Thus:

**Lemma 1.** *Let  $\mathcal{K} \subset \mathbb{R}^\Omega$  be a convex cone. Then the following are equivalent: (i)  $\pi_{\mathcal{K}}(0) > -\infty$ , (ii)  $\pi_{\mathcal{K}}(0) = 0$ , (iii)  $\pi_{\mathcal{K}}(1) = 1$ , (iv)  $\mathcal{K}$  contains no sure wins.*

Denote  $L(\pi_{\mathcal{K}}) = \{f \in \mathbb{R}^\Omega : \pi_{\mathcal{K}}(|f|) < \infty\}$ . It is clear that  $\mathfrak{B} \subset L(\pi_{\mathcal{K}})$ . Define also

$$(2.2) \quad \mathcal{M}(\mathcal{K}) = \left\{ m \in \mathbb{P}_{ba} : \mathcal{K} \subset L(m), \sup_{k \in \mathcal{K}} m(k) \leq 0 \right\}$$

and let  $\mathcal{M}(\mathcal{K}_b)$  be defined likewise. We shall refer to elements of  $\mathcal{M}(\mathcal{K})$  as *separating probabilities* for  $\mathcal{K}$ . It is clear that if  $m \in \mathcal{M}(\mathcal{K}_b)$  then  $L(\pi_{\mathcal{K}}) \subset L(m)$ .

**Proposition 1.** *Let  $\mathcal{K} \subset \mathbb{R}^\Omega$  be a convex cone. Then  $\mathcal{M}(\mathcal{K}_b)$  is non empty if and only if  $\mathcal{K}$  contains no sure wins.*

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<sup>1</sup>The relationship between the foundations of subjective probability and of asset pricing is, I believe, little known. A partial exception is [8].

*Proof.* Assume that  $\mathcal{K}$  contains no sure wins. By Lemma 1 and the Hahn Banach Theorem, we may find a linear functional  $\phi$  on  $\mathfrak{B}$  such that  $\phi \leq \pi_{\mathcal{K}}$  on  $\mathfrak{B}$  and  $\phi(1) = 1$ . If  $f \in \mathfrak{B}_+$  then  $\phi(f) = -\phi(-f) \geq -\pi_{\mathcal{K}}(-f) \geq 0$ . Therefore  $\phi$  is positive and, since continuous [9, V.2.7], it may be represented as the expectation with respect to some  $m \in \mathbb{P}_{ba}$ . If  $f \in L(\pi_{\mathcal{K}})_+$ , the left hand side of (1.1) is bounded by  $\pi_{\mathcal{K}}(f)$  so that  $L(\pi_{\mathcal{K}}) \subset L(m)$ . Then  $\mathcal{K}_b \subset L(m)$  and

$$m(k) = \lim_n m(k \wedge n) \leq \pi_{\mathcal{K}}(k) \leq 0 \quad k \in \mathcal{K}_b$$

so that  $m \in \mathcal{M}(\mathcal{K}_b)$ . If  $m \in \mathcal{M}(\mathcal{K}_b)$  and  $k \in \mathcal{K}$  is a sure win, then  $k \in \mathcal{K}_b$  and  $m(k) \leq 0$ , a contradiction.  $\square$

A classical application of Proposition 1 considers the collection  $\mathcal{K}$  of all finite sums of the form  $\sum_n a_n(\mathbf{1}_{F_n} - \lambda(F_n))$  where  $a_1, \dots, a_N$  are real numbers,  $F_1, \dots, F_N$  are elements of some  $\mathcal{A} \subset 2^\Omega$  and  $\lambda : \mathcal{A} \rightarrow \mathbb{R}$ . It is then clear that  $\mathcal{K}$  admits no sure wins if and only if there is  $m \in \mathbb{P}_{ba}$  such that  $m|_{\mathcal{A}} = \lambda$ . If the sums in  $\mathcal{K}$  are allowed to admit countably many terms provided  $\sum_n |a_n \lambda(F_n)| < \infty$ , then  $m$  will possess the additional property that  $m(\bigcup_n F_n) = \sum_n m(F_n)$  when  $\langle F_n \rangle_{n \in \mathbb{N}}$  is a disjoint sequence in  $\mathcal{A}$ . This informal statement is essentially a reformulation of [11, theorems 5 and 6, p. 2074]<sup>2</sup>. It admits an interesting generalisation to the case of concave integrals, a special case of the monotone integral of Choquet treated, e.g., in [10].

**Definition 1.** An extended real-valued functional  $\gamma$  on a convex cone  $\mathcal{L} \subset \mathbb{R}^\Omega$  is a concave integral if it is positively homogeneous, monotone, superadditive and such that  $\gamma(c + f) = \gamma(c) + \gamma(f)$  when  $c, f \in \mathcal{L}$  and  $c$  is a constant. The symbol  $\mathcal{L}(\gamma)$  then designates the set  $\{f \in \mathcal{L} : |\gamma(f)| < \infty\}$ .

If  $\gamma$  is a concave integral its *core* is defined to be the set

$$(2.3) \quad \Gamma(\gamma) = \{\lambda \in ba_+ : \mathcal{L}(\gamma) \subset L(\lambda), \gamma(f) \leq \lambda(f), f \in \mathcal{L}\}$$

The following Lemma is essentially a restatement of a result of Shapley [14, theorem 2, p. 18]. It characterises the properties of a concave integral in terms of its core.

**Lemma 2.** Let  $\gamma$  be a concave integral on a convex cone  $\mathcal{L} \subset \mathbb{R}^\Omega$  containing the constants and such that  $f \in \mathcal{L}$  implies  $f^+ \in \mathfrak{B}$ . Then  $\gamma(1) < \infty$  if and only if for each convex set  $C \subset \mathcal{L}(\gamma) \cap \mathfrak{B}$  such that  $\gamma(C) \equiv \sup_{f \in C} \gamma(f) < \infty$  there exists  $\lambda_C \in \Gamma(\gamma)$  such that

$$(2.4) \quad \sup_{f \in C} \lambda_C(f) = \gamma(C)$$

*Proof.*  $\gamma(1) = 0$ ,  $f \in \mathcal{L}(\gamma)$  and  $g \in C$  imply  $\gamma(f) \leq \gamma(1) \sup_{\omega \in \Omega} f(\omega) \leq 0 = \gamma(g)$ . The claim follows upon choosing  $\lambda_C$  to be the null measure. Alternatively let, upon normalization,  $\gamma(1) = 1$  and suppose that

$$(2.5) \quad \alpha(k - \gamma(C)) \geq 1 + \sum_{n=1}^N (f_n - \gamma(f_n))$$

for some choice of  $\alpha \geq 0$ ,  $k \in C$  and  $f_n \in \mathcal{L}(\gamma)$ ,  $n = 1, \dots, N$ . The value under  $\gamma$  of the left hand side of (2.5) is less than 0 while that of the right hand side exceeds 1, contradicting monotonicity. Thus the collection  $\mathcal{K}_C$  of finite sums of the form  $\sum_{1 \leq n \leq N} (\gamma(f_n) - f_n) + \alpha(k - \gamma(C))$  for  $\alpha, k$  and  $f_n$ ,  $n = 1, \dots, N$  as above contains no sure win; moreover, it is a convex cone of uniformly lower bounded functions on  $\Omega$ . According to Proposition 1, there exists  $\lambda_C \in \mathcal{M}(\mathcal{K}_C)$ : thus,  $\lambda_C(f) \geq \gamma(f)$  for each  $f \in \mathcal{L}(\gamma)$  (i.e.  $\lambda_C \in \Gamma(\gamma)$ ) and  $\lambda_C(k) \leq \gamma(C)$  whenever  $k \in C$ , proving (2.4). The converse is obvious.  $\square$

<sup>2</sup>However we do not restrict  $\mathcal{A}$  nor  $\lambda$ . Heath and Sudderth seem to suggest that the existence of  $m$  need not exclude sure wins while it is clear that this cannot be the case. A less general version of this result was also proved, with different methods, in [4, theorem 2, p. 420]

Lemma 2 has an interesting implication.

**Corollary 1.** *Let  $\mathcal{T}$  be a collection of subsets of some set  $T$ , with  $\{T\} \in \mathcal{T}$ . For each  $\tau \in \mathcal{T}$ , let  $\mathcal{L}_\tau$  be a vector sublattice of  $\mathbb{R}^\Omega$  containing the constants and  $\phi_\tau$  a linear functional on  $\mathcal{L}_\tau$ . The following are equivalent:*

(i) *the collection  $(\phi_\tau : \tau \in \mathcal{T})$  is coherent in the sense that<sup>3</sup>*

$$(2.6) \quad \sup \left\{ \sum_{n=1}^N \phi_{\tau_n}(b_n) : b_n \in \mathcal{L}_{\tau_n} \ n=1, \dots, N, \sum_{n=1}^N b_n \mathbf{1}_{\tau_n} \leq 1, \ N \in \mathbb{N} \right\} < \infty$$

*and satisfies moreover  $\lim_k \phi_\tau(f \wedge k) = \phi_\tau(f)$  for all  $f \in \mathcal{L}_\tau$  and  $\tau \in \mathcal{T}$ .*

(ii) *there exists  $\lambda \in ba(\Omega \times T)_+$  such that  $\|\lambda\| = \phi_{\{T\}}(1)$  and  $\lambda(f \mathbf{1}_\tau) = \phi_\tau(f)$  for each  $f \in \mathcal{L}_\tau$  and  $\tau \in \mathcal{T}$ .*

*Proof.* Let  $\mathcal{L}$  denote the linear span of  $\{f \mathbf{1}_\tau : f \in \mathcal{L}_\tau, \tau \in \mathcal{T}\}$  and define  $\gamma : \mathfrak{B}(\Omega \times T) \rightarrow \mathbb{R}$  implicitly as

$$(2.7) \quad \gamma(b) = \sup \left\{ \sum_{n=1}^N \phi_{\tau_n}(b_n) : b_n \in \mathcal{L}_{\tau_n} \ n=1, \dots, N, \sum_{n=1}^N b_n \mathbf{1}_{\tau_n} \leq b, \ N \in \mathbb{N} \right\}$$

It is readily seen that  $\gamma$  is monotone, superadditive and positively homogeneous.  $\gamma(1) < \infty$  by (2.6) while  $1 \in \mathcal{L}_{\{T\}}$  implies that  $\gamma$  is additive relative to the constants. Lemma 2 guarantees the existence of  $\lambda \in \Gamma(\gamma)$ . Given that each  $\mathcal{L}_\tau$  is a linear space, it follows from (i) that  $\lim_n \lambda((f \wedge n) \mathbf{1}_\tau) = \lim_n \phi_\tau(f \wedge n) = \phi_\tau(f)$  for each  $f \in \mathcal{L}_\tau, f \geq 0$ . Thus  $\mathcal{L} \subset L(\lambda)$  and  $\lambda\left(\sum_{n=1}^N f_n \mathbf{1}_{\tau_n}\right) = \sum_{n=1}^N \phi_{\tau_n}(f_n)$  whenever  $\sum_{n=1}^N f_n \mathbf{1}_{\tau_n} \in \mathcal{L}$ .  $\|\lambda\| = \lambda(\Omega \times T) = \phi_{\{T\}}(1)$ . If  $\lambda$  is as in (ii) and  $1 \geq \sum_{n=1}^N f_n \mathbf{1}_{\tau_n} \in \mathcal{L}$  then  $\sum_{n=1}^N \phi_{\tau_n}(f_n) = \lambda\left(\sum_{n=1}^N f_n \mathbf{1}_{\tau_n}\right) \leq \|\lambda\|$ .  $\square$

A special case of this corollary is obtained by taking all  $\tau \in \mathcal{T}$  to be copies of  $T$ : the representing measure  $\lambda$  can then be taken to be an element of  $ba_+$ .

**Remark 1.** *Writing  $\tau \leq v$  when  $v \subset \tau$  makes of course  $\mathcal{T}$  into a partially ordered set. If  $(\phi_\tau : \tau \in \mathcal{T})$  meets property (i) of Corollary 1 and if  $(\mathcal{L}_\tau : \tau \in \mathcal{T})$  is increasing in  $\tau$  then necessarily  $\phi_v|_{\mathcal{L}_\tau} \geq \phi_\tau$  whenever  $\tau, v \in \mathcal{T}$  and  $\tau \leq v$ . This conclusion has a direct application to the theory of finitely additive supermartingales. Given a collection  $(\mathcal{A}_\tau : \tau \in \mathcal{T})$  of algebras of subsets of  $\Omega$  which increases with  $\tau$ , a finitely additive supermartingale is an element  $(m_\tau : \tau \in \mathcal{T})$  of  $\prod_{\tau \in \mathcal{T}} ba(\mathcal{A}_\tau)$  such that  $m_\tau \geq m_v|_{\mathcal{A}_\tau}$  whenever  $\tau \leq v$ . Letting  $\mathcal{L}_\tau = \mathfrak{B}(\mathcal{A}_\tau)$  and identifying  $\phi_\tau$  with the expected value with respect to  $m_\tau$ , the second half of condition (i) is necessarily satisfied so that (2.6) is equivalent to the existence of a representing measure or, in the terminology of classical stochastic processes, a Doléans-Dade measure. A more systematic statement of this result is in [6, theorem 1].*

Much of this section rests on the conclusion, established in Proposition 1, that  $\mathcal{K}_b$  admits a separating probability in the absence of sure wins. This result, however, does not have an extension to  $\mathcal{K}$  of a corresponding simplicity. To this end we shall need some results on the representation of linear functionals, to be developed in the next section.

<sup>3</sup>The inequality that follows is meant to hold pointwise in  $\Omega \times T$

## 3. THE REPRESENTATION OF LINEAR FUNCTIONALS

It is the purpose of this section to establish conditions for a linear functional  $\phi$  on some linear subspace  $\mathcal{L}$  of  $\mathbb{R}^\Omega$  with  $1 \in \mathcal{L}$  to admit the representation

$$(3.1) \quad \phi(f) = \phi(1)m(f) \quad f \in \mathcal{L}$$

for some  $m \in ba$  such that  $\mathcal{L} \subset L(m)$ , referred to as a *representing measure* for  $\phi$ . We use the symbols  $\mathcal{K}^\phi$  and  $\mathcal{K}_b^\phi$  to denote the sets  $\{f \in \mathcal{L} : \phi(f) = 0\}$  and  $\{f \in \mathcal{K}^\phi : f^- \in \mathfrak{B}\}$ , respectively. If  $\phi(1) \neq 0$ , then  $\mathcal{K}_b^\phi = \{f - \phi(1)^{-1}\phi(f) : f \in \mathcal{L}, f^- \in \mathfrak{B}\}$ . Thus if  $\mathcal{L}$  is a vector sublattice of  $\mathbb{R}^\Omega$  then  $m \in \mathcal{M}(\mathcal{K}_b^\phi)$  implies  $\mathcal{L} \subset L(m)$  and  $\phi(f) = \phi(1)m(f)$  for every  $f \in \mathcal{L} \cap \mathfrak{B}$  (which clarifies the connection between separating probabilities and representing measures).

The content of this section, as will soon become clear, owes much to the work of Berti and Rigo [2].

**Theorem 1.** *Let  $\mathcal{A} \subset 2^\Omega$  be an algebra,  $\mu \in ba(\mathcal{A})$ ,  $\mathcal{L}$  a vector sublattice of  $L(\mu)$  with  $1 \in \mathcal{L}$  and  $\phi$  a positive linear functional on  $\mathcal{L}$ . Denote by  $\mathcal{L}^*$  the set of limit points of sequences from  $\mathcal{L}$  which converge orderly in  $L(\mu)$ . The following are equivalent.*

- (i)  $\phi$  extends to a monotone function  $\phi^* : \mathcal{L}^* \rightarrow \mathbb{R}$ ;
- (ii)  $\lim_n \phi(h_n) = 0$  whenever  $\langle h_n \rangle_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{L}$  which converges to 0 orderly in  $L(\mu)$ ;
- (iii)  $-\infty < \lim_n \phi(g_n) \leq \lim_n \phi(f_n) < \infty$  whenever  $\langle f_n \rangle_{n \in \mathbb{N}}$  and  $\langle g_n \rangle_{n \in \mathbb{N}}$  are sequences in  $\mathcal{L}$  which converge orderly in  $L(\mu)$  to  $f$  and  $g$  respectively, with  $f \geq g$ ;
- (iv)  $\phi$  admits a positive representing measure  $m$  such that  $m^*(h) \equiv \lim_n m(h_n)$  exists in  $\mathbb{R}$  and is unique for every sequence  $\langle h_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{L}$  which converges to  $h$  orderly in  $L(\mu)$ .

Moreover, if  $\phi$  is a positive linear functional on a vector sublattice  $\mathcal{L}$  of  $\mathbb{R}^\Omega$  with  $1 \in \mathcal{L}$  then there exists a unique positive linear functional  $\phi^\perp$  on  $\mathcal{L}$  such that  $\phi^\perp(1) = 0$  and that

$$(3.2) \quad \phi(f) = \phi(1)m(f) + \phi^\perp(f) \quad f \in \mathcal{L}$$

for some  $m \in ba_+$  satisfying  $\mathcal{L} \subset L(m)$ .

*Proof.* Let  $\langle h_n \rangle_{n \in \mathbb{N}}$  be as in (ii) and  $\langle \bar{h}_n \rangle_{n \in \mathbb{N}}$  be a decreasing sequence in  $L(m)$  converging to 0 in  $L(m)$  and such that  $\bar{h}_n \geq |h_n|$ ,  $n = 1, 2, \dots$ . Fix a sequence  $\langle \alpha_n \rangle_{n \in \mathbb{N}}$  in  $\mathbb{R}_+$  such that  $\lim_n \alpha_n = \infty$ . Any subsequence of  $\langle h_n \rangle_{n \in \mathbb{N}}$  admits a further subsequence (still denoted by  $\langle h_n \rangle_{n \in \mathbb{N}}$  for convenience) such that  $\sum_n \alpha_n \|h_n\| < \infty$ . Fix  $\eta > 0$  arbitrarily and set

$$(3.3) \quad h_n^\eta = (|h_n| - \eta)^+, \quad g_k^\eta = \sum_{n \leq k} \alpha_n h_n^\eta \quad \text{and} \quad g^\eta = \sum_n \alpha_n h_n^\eta$$

Then,  $\{\sum_{n > k} \alpha_n h_n^\eta > \epsilon\} \subset \{\bar{h}_k \geq \eta\}$  and  $\left\| \sum_{k < n \leq k+p} \alpha_n h_n^\eta \right\| \leq \sum_{n > k} \alpha_n \|h_n^\eta\| \leq \sum_{n > k} \alpha_n \|h_n\|$ . Thus,  $\langle g_k^\eta \rangle_{k \in \mathbb{N}}$  is an increasing sequence in  $\mathcal{L}$  which converges orderly in  $L(\mu)$  to  $g^\eta \in \mathcal{L}^*$  [9, theorem III.3.6]. If (i) holds then  $\alpha_n^{-1} \phi^*(g^\eta) \geq \phi(h_n^\eta) \geq \phi(|h_n|) - \eta \phi(1)$  so that  $\lim_n \phi(h_n) = 0$ , i.e. (ii) holds as well. Let  $\langle g_n \rangle_{n \in \mathbb{N}}$  and  $\langle f_n \rangle_{n \in \mathbb{N}}$  be as in (iii). The inequality  $f_n - g_n \geq (f_n - f) + (g - g_n)$  together with (ii) induces the conclusion that  $(f_n - g_n)^-$  converges to 0 orderly in  $L(\mu)$  and thus that  $\liminf_n \phi(f_n) = \liminf_n \{\phi(g_n) + \phi((f_n - g_n)^+)\} \geq \liminf_n \phi(g_n)$ . The case in which  $\langle g_n \rangle_{n \in \mathbb{N}}$  is a subsequence of  $\langle f_n \rangle_{n \in \mathbb{N}}$  suggests that  $\liminf_n \phi(f_n) = \limsup_n \phi(f_n)$ . If  $\lim_n \phi(f_n) = \infty$  then one may select a subsequence  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$  such that, letting  $h_k = f_{n_{k+1}} - f_{n_k}$ ,  $\lim_k \phi(h_k) = \infty$ . However this contrasts with (ii) since the sequence

$\langle h_k \rangle_{k \in \mathbb{N}}$  converges to 0 orderly in  $L(\mu)$ . This proves (iii). In the general case in which  $\mathcal{L}$  is a vector sublattice of  $\mathbb{R}^\Omega$ , fix  $f \in \mathcal{L}_+$  and choose  $m \in \mathcal{M}(\mathcal{K}_b^\phi)$  if  $\phi(1) > 0$ , or  $m = 0$  otherwise. Then,

$$(3.4) \quad \phi(f) = \lim_n \phi(f \wedge n) + \lim_n \phi(f - (f \wedge n)) = \phi(1)m(f) + \phi^\perp(f)$$

a conclusion which extends to general  $f \in \mathcal{L}$  by considering  $f^+$  and  $f^-$  separately. The functional  $\phi^\perp$ , as defined implicitly in (3.4), is clearly positive, linear and such that  $\phi^\perp(1) = 0$ . Decomposition (3.2) thus exists. If  $\phi(f) = \phi(1)v(f) + \psi^\perp(f)$  were another decomposition such as (3.2), with  $v \in ba_+$ ,  $\mathcal{L} \subset L(v)$  and  $\psi^\perp$  a positive, linear functional on  $\mathcal{L}$  with  $\psi^\perp(1) = 0$ , then  $f \in \mathcal{L}_+$  would imply

$$(\phi^\perp - \psi^\perp)(f) = \lim_n (\phi^\perp - \psi^\perp)(f - (f \wedge n)) = \phi(1) \lim_n (m + v)(f - (f \wedge n)) = 0$$

which proves uniqueness of (3.2). Returning to the case  $\mathcal{L} \subset L(\mu)$ , if (iii) holds, then it is obvious from (3.4) that  $\phi^\perp = 0$ ; in addition the limit  $\lim_n m(h_n)$  exists in  $\mathbb{R}$  for each sequence  $\langle h_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{L}$  which converges orderly in  $L(\mu)$  and does not depend but on the limit point  $h$ .  $\square$

One noteworthy implication of Theorem 1 is obtained by replacing  $\mathcal{L}$  with  $L(\mu)$ .

**Theorem 2.** *Let  $\mathcal{A} \subset 2^\Omega$  be an algebra and  $\mu \in ba(\mathcal{A})$ . Every positive linear functional  $\phi$  on  $L(\mu)$  admits a positive representing measure  $m$  such that  $\lim_n m(h_n) = 0$  for every sequence  $\langle h_n \rangle_{n \in \mathbb{N}}$  in  $L(\mu)$  which converges to 0 orderly in  $L(\mu)$ .*

Given that  $L(\mu)$  is a normed Riesz space, its dual space is a vector lattice [1, theorem 12.1, p. 175]. Thus Theorem 2 also implies that continuous linear functionals on  $L(\mu)$ , decomposing as the difference of two positive linear functionals, admit a representing measure [2, theorem 7, p. 3255].

Another application concerns more general functionals. In fact it is clear that the implication (i)  $\rightarrow$  (ii) in Theorem 2 does not require  $\phi$  to be linear.

**Theorem 3.** *Let  $\mathcal{L} \subset \mathbb{R}^\Omega$  be either (i) a Banach lattice (see e.g. [1, p. 174]) containing the constants or (ii)  $\mathcal{L} = L(\mu)$  for some  $\mu \in ba(\mathcal{A})$  and some algebra  $\mathcal{A} \subset 2^\Omega$ . Assume that  $\phi : \mathcal{L} \rightarrow \mathbb{R}$  is a monotone functional such that*

$$(3.5) \quad \lim_n \inf_{\{f \in \mathcal{L} : \phi(f) > \eta\}} \phi(nf) = \infty \quad \eta > 0$$

and, under (ii),

$$(3.6) \quad \lim_{k \downarrow 0} \sup_{f \in \mathcal{L}} \{\phi(f) - \phi(f - k)\} = 0$$

Then,  $\limsup_n \phi(h_n) \leq 0$  whenever  $\langle h_n \rangle_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{L}$  that converges to 0 in norm or, under (ii), orderly in  $L(\mu)$ . In particular monotone, positively homogeneous and subadditive functionals on Banach lattices are continuous.

*Proof.* Each subsequence of  $\langle h_n \rangle_{n \in \mathbb{N}}$  contains a further subsequence for which it is possible to define  $g_k^\eta$  and  $g^\eta$  as in (3.3). Under (i),  $\langle g^\eta \rangle_{k \in \mathbb{N}}$  converges to  $g^\eta$  in norm for all  $\eta \geq 0$ ; under (ii) only for  $\eta > 0$ . In either case we conclude that  $\phi(g^\eta) \geq \phi(\alpha_n h_n^\eta) \geq \phi(\alpha_n (h_n - \eta))$  and, given (3.5),  $\liminf_n \phi(h_n - \eta) \leq 0$ . Choosing  $\eta = 0$  under (i) or exploiting (3.6) under (ii) and recalling that the initial choice of the subsequence was arbitrary, we conclude that  $\limsup_n \phi(h_n) \leq 0$ . It is clear that a positively homogeneous, subadditive functional  $\phi$  meets (3.5), (3.6) and, if monotone,  $|\phi(h) - \phi(h_n)| \leq \phi(|h_n - h|)$ .  $\square$

Given the preceding results, it is now easy to extend Proposition 1 to  $\mathcal{K}$ .

**Corollary 2.** *Let  $\mathcal{K} \subset \mathbb{R}^\Omega$  be a convex cone. Then  $\mathcal{M}(\mathcal{K})$  is non empty if and only if there exist an algebra  $\mathcal{A} \subset 2^\Omega$  and  $\mu \in \mathbb{P}_{ba}(\mathcal{A})$  such that  $\mathcal{K} \subset L(\mu)$  and that the closure  $\overline{C}^\mu$  of  $C = \mathcal{K} - \mathcal{S}(\mathcal{A})_+$  in the norm topology of  $L(\mu)$  admits no sure wins.*

*Proof.* If  $\mu \in \mathcal{M}(\mathcal{K})$  then  $\mu$  is a separating measure for  $\overline{C}^\mu$  which rules out sure wins. As for sufficiency, observe that ordinary separation theorems imply the existence of a continuous linear functional  $\phi : L(\mu) \rightarrow \mathbb{R}$  such that  $\sup_{f \in \overline{C}^\mu} \phi(f) \leq 0$  and  $1 = \phi(1)$ . Given that  $\mathcal{K}$  contains the origin,  $-\mathcal{S}(\mathcal{A})_+ \subset C$  so that  $\phi$  is positive on  $\mathcal{S}(\mathcal{A})$  and, since  $\mathcal{S}(\mathcal{A})_+$  is dense in  $L(\mu)_+$  and  $\phi$  is  $L(\mu)$  continuous, it is positive over the whole of  $L(\mu)$ . The claim follows from Theorem 2.  $\square$

Corollary 2 is related to a result of Yan [15], where  $\mathcal{K} \subset L(P)$  and  $P$  is countably additive. Yan theorem has been widely used in mathematical finance. In fact extending the absence of sure wins from  $\mathcal{K}$  to  $C$  as in Corollary 2 has a direct analogy in the extension of the no arbitrage principle into that of absence of *free lunches* in mathematical finance.

The representation (3.1) extends beyond  $L(\mu)$ .

**Corollary 3.** *Let  $\mathcal{L} \subset \mathbb{R}^\Omega$  be a linear space. A linear functional  $\phi$  on  $\mathcal{L}$  admits a representing measure if and only if there exists  $\mu \in ba$  such that  $\mathcal{L} \subset L(\mu)$  and  $\phi$  is continuous with respect to the norm topology of  $L(\mu)$ . If, in addition,  $\phi$  is positive and  $\mathcal{L}$  a vector sublattice of  $\mathbb{R}^\Omega$ , there exists a positive representing measure.*

*Proof.* The direct implication is obvious. For the converse, let  $\mu \in ba$  be as in the statement and denote by  $\bar{\phi}$  the continuous, linear extension of  $\phi$  to  $L(\mu)$ . If  $\mathcal{L}$  is a vector lattice and  $\phi$  is positive, the inequality  $\phi(f) \leq \bar{\phi}(f^+)$  implies that such extension may be chosen to be positive and continuous. In either case the claim follows from Theorem 2.  $\square$

Daniell theorem also follows easily.

**Corollary 4.** *Let  $\mathcal{L}$  be a vector sublattice of  $\mathbb{R}^\Omega$  containing 1 and  $\phi$  a positive linear functional on  $\mathcal{L}$ . Then  $\lim_n \phi(f_n) = 0$  for every sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{L}$  which decreases to 0 pointwise if and only if  $\phi$  admits a representing measure  $m$  which is countably additive in restriction to the  $\sigma$  algebra generated by  $\mathcal{L}$ .*

*Proof.* Consider the case  $\phi \neq 0$ , the claim being otherwise trivial. Then, by (3.2),  $\phi(1) > 0$  and  $\phi$  admits a representing probability  $m$ . Let  $\mathcal{A} = \left\{ E \subset \Omega : \inf_{\{g \in \mathcal{L} : g \geq \mathbf{1}_E\}} m(g) = \sup_{\{f \in \mathcal{L} : f \leq \mathbf{1}_E\}} m(f) \right\}$  and consider a decreasing sequence  $\langle E_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{A}$  with  $\bigcap_n E_n = \emptyset$ . For each  $\eta > 0$  there are sequences  $\langle f_n \rangle_{n \in \mathbb{N}}$  and  $\langle g_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{L}_+$  with  $g_n \geq \mathbf{1}_{E_n} \geq f_n$  and  $m(f_n) \geq m(g_n) - \eta 2^{-n}$ . Let  $h_n = \inf_{\{k \leq n\}} f_k$ .  $m(h_1) \geq m(g_1) - \eta 2^{-1}$ ; if  $m(h_{n-1}) \geq m(g_{n-1}) - \eta \sum_{k=1}^{n-1} 2^{-k}$  for some  $n$  then,  $h_{n-1} + f_n = h_n + (h_{n-1} \vee f_n) \leq h_n + g_{n-1}$  implies

$$m(h_n) \geq m(f_n) + m(h_{n-1}) - m(g_{n-1}) \geq m(f_n) - \eta \sum_{k=1}^{n-1} 2^{-k} \geq m(g_n) - \eta \sum_{k=1}^n 2^{-k}$$

Thus the sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  may be chosen to be decreasing to 0 and such that  $m(f_n) \geq m(g_n) - \eta$  for each  $n$ . Then,  $0 = \lim_n m(f_n) \geq \lim_n m(E_n) - \eta$ . It is well known that  $\mathcal{A}$  is an algebra and that  $\mathcal{L} \cap \mathfrak{B} \subset \mathfrak{B}(\mathcal{A})$ , see e.g. [3, p. 774]. Thus,  $m|_{\mathcal{A}}$  admits a countably additive extension to  $\sigma\mathcal{A}$  and this, in turn, an extension  $\mu$  to  $2^\Omega$ . Since  $\mu$  and  $m$  coincide on  $\mathcal{A}$ ,  $\mu$  is another representing measure for  $\phi$ . The converse is a straightforward implication of monotone convergence.  $\square$

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