Decomposition of Supermartingales Indexed by a Linearly Ordered Set

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Abstract We prove a version of the Doob Meyer decomposition for supermartingales with a linearly ordered index set.

Key words Doob Meyer decomposition, natural increasing processes, potentials, supermartingales.

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1 Introduction.

In this paper we prove a version of the Doob Meyer decomposition for supermartingales indexed by a linearly ordered set Δ. Given a standard probability space $(\Omega, \mathcal{F}, P)$ we consider a family $\tilde{\mathcal{F}} = (\mathcal{F}_\delta : \delta \in \Delta)$ of sub σ algebras of $\mathcal{F}$ with the property that $\delta, \varepsilon \in \Delta$ and $\delta \leq \varepsilon$ imply $\mathcal{F}_\delta \subset \mathcal{F}_\varepsilon$. $X = (X_\delta : \delta \in \Delta)$ is a supermartingale if $X_\delta \in L^1(\mathcal{F}_\delta)$ for each $\delta \in \Delta$ and $\delta, \varepsilon \in \Delta$ and $\delta \leq \varepsilon$ imply $X_\delta \geq P(X_\varepsilon | \mathcal{F}_\delta)$. $\mathcal{S}$ denotes the collection of all supermartingales X which are bounded with respect to the norm $\|X\|_{\mathcal{S}} = \sup_{\delta \in \Delta} \|X_\delta\|_{L^1}$.

As is well known, the Doob Meyer decomposition was initially established by Doob [2] for supermartingales indexed by $\mathbb{N}$ and later extended by Meyer [6] to the case of right continuous supermartingales indexed by $\mathbb{R}_+$ and under the "usual assumptions". Despite the key importance of this result in the theory of stochastic processes, several situations of interest lead outside of its range, particularly so in applications where it is often more useful to index processes by stopping times. The case treated in this work may be exemplified by taking Δ to be the collection of hitting times arising from a corresponding collection of pairwise nested subsets of $\mathbb{R}$.

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The main result of this paper, Proposition 1, provides a necessary and sufficient condition for $X \in \mathcal{S}$ to admit a version of the Doob Meyer decomposition in which the intervening increasing process is required to be natural (in the sense of definition 2). Although such decomposition may appear somehow "weaker" than the original one, it is established under significantly more general conditions than those usually considered.

Given its repeated use and despite being essentially known, we state here the following criterion for weak convergence in L^1, a slight generalization of [4, lemma 1, p. 441] (throughout the paper we identify sets with their indicators).

Lemma 1 *For each u in a directed set $\mathbf{U}$ let $\mathcal{H}_u$ be a sub σ algebra of $\mathcal{F}$ and $f_u \in L^1(\mathcal{H}_u)$. Set $\mathcal{H} = \bigcup_u \mathcal{H}_u$. $\langle f_u \rangle_{u \in \mathbf{U}}$ converges to a $\sigma\mathcal{H}$ measurable limit f weakly in L^1 if and only if*

(i). $\lim_u P(f_u H)$ *exists and is finite for each $H \in \mathcal{H}$ and*

(ii). $\langle f_u \rangle_{u \in \mathbf{U}}$ *is uniformly P integrable.*

Moreover, $\langle f_u \rangle_{u \in \mathbf{U}}$ converges in L^1 whenever (ii) is replaced by

(ii'). $\langle f_u \rangle_{u \in \mathbf{U}}$ *is positive and increasing.*

Proof If $\langle f_u \rangle_{u \in \mathbf{U}}$ converges weakly in L^1 then the set $\{f_u : u \in \mathbf{U}\}$ is compact in that same topology: (ii) follows from [3, IV.8.11, p. 294]; (i) is obvious. Assume that $\langle f_u \rangle_{u \in \mathbf{U}}$ satisfies (i). If (ii') holds then (ii) follows from (i) and the inequality

$$\sup_{u \in \mathbf{U}} P(f_u F) \leq P(f_{u'} F) + \lim_{u \in \mathbf{U}} P(f_u - f_{u'}), \quad F \in \mathcal{F}$$

Assume (i) and (ii), let $H \in \sigma\mathcal{H}$, $H' \in \mathcal{H}$ and denote by $|H - H'|$ their symmetric difference. Then

$$\begin{aligned} 2 \sup_u P(|f_u| \, |H - H'|) &\geq \sup_{u', u'' \geq u} \{P(f_{u'}(H - H')) - P(f_{u''}(H - H'))\} \\ &\geq \limsup_{u'} P(f_{u'}(H - H')) - \liminf_{u''} P(f_{u''}(H - H')) \\ &= \limsup_{u'} P(f_{u'} H) - \liminf_{u''} P(f_{u''} H) \end{aligned}$$

Given that $P(|H - H'|)$ can be made arbitrarily small by an accurate choice of $H' \in \mathcal{H}$ and given (ii), $Q(h) = \lim_u P(f_u h)$ exists for any $h \in \sigma\mathcal{H}$ and, $\langle f_u \rangle_{u \in \mathbf{U}}$ being norm bounded, for each $h \in L^\infty(\mathcal{F})$. $Q \in ba(\sigma\mathcal{H})$ and, by (ii), $Q \ll P$. Let $f \in L^1(\sigma\mathcal{H})$ be the corresponding Radon Nikodym derivative. If $g \in L^\infty(\mathcal{F})$,

$$\lim_u P(f_u g) = \lim_u P(f_u P(g | \sigma\mathcal{H})) = P(f P(g | \sigma\mathcal{H})) = P(fg)$$

Under (ii'), $P(|f - f_u|) = P(f - f_u)$.

Denote by δ_∞ and δ_0 the indexes associated to the σ algebras $\bigvee_{\delta \in \Delta} \mathcal{F}_\delta$ and $\bigwedge_{\delta \in \Delta} \mathcal{F}_\delta$ respectively. By Lemma 1, whenever the supermartingale X is uniformly integrable we may and will consider the corresponding weak limits X_{δ_∞} and X_{δ_0} i.e. treat Δ as if it admitted a maximal as well as a minimal element.

2 Uniformly Integrable Potentials.

A stochastic process $A = (A_\delta : \delta \in \Delta)$ is increasing if $P(A_\varepsilon \geq A_\delta \geq A_{\delta_0} = 0) = 1$ for $\varepsilon \geq \delta$; it is integrable if $\sup_{\delta \in \Delta} P(A_\delta) < \infty$. Let

$$\mathcal{D} = \{\delta_1 \leq \ldots \leq \delta_N : \delta_1 = \delta_0, \delta_N = \delta_\infty, N \in \mathbb{N}\} \tag{1}$$

and for $d = \left\{\delta_1^d \leq \ldots \leq \delta_{N_d}^d\right\} \in \mathcal{D}$ and $]\delta, \varepsilon] = \left\{\delta' \in \Delta : \delta < \delta' \leq \varepsilon\right\}$ let

$$\mathcal{P}^d = \left\{ F_0 \{\delta_0\} \cup \bigcup_{n=1}^{N_d - 1} F_n \left]\delta_n^d, \delta_{n+1}^d\right] : F_0 \in \mathcal{F}_{\delta_0}, F_n \in \mathcal{F}_{\delta_n^d}, \right\} \tag{2}$$

Let moreover

$$\mathcal{P} = \left\{ F_0 \{\delta_0\} \cup \bigcup_{n=1}^{N} F_n \,]\delta_n, \varepsilon_n] : F_0 \in \mathcal{F}_{\delta_0}, F_n \in \mathcal{F}_{\delta_n^d}, \delta_n^d, \varepsilon_n \in \Delta, 1 \leq n \leq N, N \in \mathbb{N} \right\} \tag{3}$$

Definition 1 *Let $d \in \mathcal{D}$ and $A = (A_\delta : \delta \in \Delta)$. A is (i) d-predictable (ii) predictable or (iii) weakly predictable if (i) A is $\mathcal{P}^d$ measurable (ii) A is $\sigma\mathcal{P}$ measurable or (iii) for each $\delta \in \Delta$, A_δ is measurable with respect to the σ algebra $\mathcal{F}_{\delta-} = \bigvee_{\gamma<\delta} \mathcal{F}_\gamma$.*

To each and $U \in \underset{\delta \in \Delta}{\times} L^1$ we can associate the predictable process

$$\mathcal{P}^d(U) = P(U_{\delta_0} | \mathcal{F}_{\delta_0}) + \sum_{n=1}^{N_d - 1} P\left(U_{\delta_{n+1}^d} \middle| \mathcal{F}_{\delta_n^d}\right) \left]\delta_n^d, \delta_{n+1}^d\right] \tag{4}$$

Identify $f : \Omega \to \mathbb{R}$ with $f\Delta : \Omega \times \Delta \to \mathbb{R}$ and let $\mathcal{D}$ be directed by inclusion.

Definition 2 *An increasing, integrable process A is natural if*

$$P\left(b \int f dA\right) = \lim_{d \in \mathcal{D}} P \int \mathcal{P}^d(b) f dA, \quad b \in L^\infty, f \in \mathcal{P} \tag{5}$$

Definition 3 *A supermartingale $X = (X_\delta : \delta \in \Delta)$ is Doob Meyer supermartingale, $X \in \mathcal{S}^m$, if for each $\delta \in \Delta$, $X_\delta = P(M | \mathcal{F}_\delta) - A_\delta$ where $M \in L^1$ and A is increasing, integrable and natural. X is weakly Doob Meyer, $X \in \mathcal{S}^w$, if A is weakly predictable rather than natural.*

Remark 1 Definition 2 differs from the traditional one (e.g. [7, VII, D18, p. 111]) inasmuch $\mathcal{P}^d(f)$ need not converge nor need its would be limit share the properties of the predictable projection. However, in the special case in which $\Delta = \mathbb{R}_+$, A is right continuous and the usual conditions hold, (5) implies that A is predictable. (This follows from [1, theorem VI.61, p. 126] upon selecting a sequence $\langle d_n \rangle_{n \in \mathbb{N}}$ such that each dyadic rational belongs to some d_n and $P(bA_{\delta_\infty}) = \lim_n P \int \mathcal{P}^{d_n}(b) \, dA$).

As sual, a potential is a positive supermartingale such that $\lim_\delta P(X_\delta) = 0$.

Lemma 2 *Let $k > 0$ and X be a potential such that $P(X_\delta \leq k) = 1$ for all $\delta \in \Delta$. Then there exists a sequence $\langle X^r \rangle_{r \in \mathbb{N}}$ in $\mathcal{S}^w$ such that $X \geq X^r \geq 0$ and $\lim_r \|X - X^r\|_{\mathcal{S}} = 0$.*

Proof Consider the case $k=1$, fix $d=\left\{\delta_1^d \leq \ldots \leq \delta_{N_d}^d\right\} \in \mathcal{D}$ and $\eta>1$ and adopt throughout the proof the conventions $0/0=0$ and $\prod\{\varnothing\}=1$. For $1 \leq n < N_d$ let

$$h_n^d=\prod_{i=1}^n \frac{\eta-X_{\delta_i^d}}{\eta-P\left(X_{\delta_{i+1}^d} \middle| \mathcal{F}_{\delta_i^d}\right)} \tag{6}$$

For each $1 \leq m \leq n < N_d$ and up to a P null set

$$1 \geq h_m^d \geq h_n^d \geq \frac{\eta-X_{\delta_0}}{\eta} \prod_{i=1}^{n-1} \frac{\eta-X_{\delta_{i+1}^d}}{\eta-P\left(X_{\delta_{i+1}^d} \middle| \mathcal{F}_{\delta_i^d}\right)} \geq 0 \tag{7}$$

Given that $X_{\delta_{N_d}^d}=0$,

$$\begin{aligned}
P\left(h_{N_d-1}^d \middle| \mathcal{F}_{\delta_n^d}\right) &= \eta^{-1} P\left(P\left(\eta-X_{\delta_{N_d-1}^d} \middle| \mathcal{F}_{\delta_{N_d-2}^d}\right) h_{N_d-2}^d \middle| \mathcal{F}_{\delta_n^d}\right) \\
&= \eta^{-1} P\left(\eta-X_{\delta_{n+1}^d} \middle| \mathcal{F}_{\delta_n^d}\right) h_n^d \\
&= \left(1-X_{\delta_n^d}\eta^{-1}\right) h_{n-1}^d
\end{aligned}$$

i.e.

$$h_{n-1}^d X_{\delta_n^d}=\eta P\left(h_{n-1}^d-h_{N_d-1}^d \middle| \mathcal{F}_{\delta_n^d}\right) \tag{8}$$

Let

$$A_{\delta_{n+1}^d}^d=\eta\left(1-h_n^d\right), \ 1 \leq n < N_d \tag{9}$$

Define also $A^d=\left(A_\delta^d : \delta \in \Delta\right)$ and $X^d=\left(X_\delta^d : \delta \in \Delta\right)$ by

$$A_\delta^d=\sum_{n=1}^{N_d-1} A_{\delta_{n+1}^d}^d\left\{\delta_n^d < \delta \leq \delta_{n+1}^d\right\} \quad \text{and} \quad X_\delta^d=P\left(A_{\delta_{N_d}^d}^d \middle| \mathcal{F}_\delta\right)-A_\delta^d \tag{10}$$

By (7) A^d is increasing and bounded by η; by (6) A^d is d-predictable; by (8) $X_{\delta_0}^d=X_{\delta_0}$ and

$$X_\delta^d=\eta P\left(h_n^d-h_{N_d-1}^d \middle| \mathcal{F}_\delta\right)=h_n^d P\left(X_{\delta_{n+1}^d} \middle| \mathcal{F}_\delta\right), \quad 1 \leq n < N_d, \ \delta_n^d < \delta \leq \delta_{n+1}^d \tag{11}$$

We thus conclude from (11) that $X \geq X^d$ and

$$\left\|X_\delta-X_\delta^d\right\|=\inf_{n:\delta_{n+1}^d \geq \delta} P\left(X_\delta-h_n^d X_{\delta_{n+1}^d}\right) \leq \eta^{-1}+\inf_{n:\delta_{n+1}^d \geq \delta} P\left(X_\delta-X_{\delta_{n+1}^d}\right) \tag{12}$$

Endowing L^1 with the weak topology and $\times_{\delta \in \Delta} L^1$ with the corresponding product topology, the set $\left\{Z \in \times_{\delta \in \Delta} L^1 : \eta \geq Z \geq 0\right\}$ is compact and contains A^d. Moving to a subnet if necessary (still indexed by $\mathcal{D}$), $\left\langle A^d\right\rangle_{d \in \mathcal{D}}$ converges to a limit A^η. Let $X_\delta^\eta=P\left(A_{\delta_\infty}^\eta \middle| \mathcal{F}_\delta\right)-A_\delta^\eta$ and $X^\eta=(X_\delta^\eta : \delta \in \Delta)$. For each $\delta \in \Delta$, A_δ^η is $\mathcal{F}_{\delta-}$ measurable, by Lemma 1, and $0 \leq X_\delta^\eta \leq X_\delta$; moreover, by (12)

$$\|X^\eta-X\|_{\mathcal{S}}=\sup_{\delta \in \Delta} P\left(X_\delta-X_\delta^\eta\right)=\sup_{\delta \in \Delta}\lim_{d \in \mathcal{D}} P\left(X_\delta-X_\delta^d\right) \leq \eta^{-1}$$

If $k>0$ is arbitrary and $X^\eta \in \mathcal{S}^w$ is such that $0 \leq X^\eta \leq k^{-1}X$ and $\left\|X^\eta-k^{-1}X\right\|_{\mathcal{S}} \leq \eta^{-1}$ then $kX^\eta \in \mathcal{S}^w$, $0 \leq kX^\eta \leq X$ and $\|kX^\eta-X\|_{\mathcal{S}} \leq k\eta^{-1}$: the claim follows from η being arbitrary.

We obtain next the following characterization where upper bar denotes the closure in the norm topology of $\mathcal{S}$ and $\mathcal{S}^u$ denotes uniformly integrable supermartingales.

Theorem 1 $\overline{\mathcal{S}^w} = \mathcal{S}^u$.

Proof $\overline{\mathcal{S}^m} \subset \mathcal{S}^u$ as $\mathcal{S}^w \subset \mathcal{S}^u$ and $\mathcal{S}^u$ is closed. Let $Y \in \mathcal{S}^u$ and $X + M$ its Riesz decomposition as the sum of a martingale $M = \left(P\left(Y_{\delta_\infty} \middle| \mathcal{F}_\delta\right) : \delta \in \Delta\right)$ and a potential X, both uniformly integrable. For each $k > 0$ the process $X^k = \left(X^k_\delta : \delta \in \Delta\right)$, with $X^k_\delta = X_\delta \wedge k$, is a potential bounded by k. Then $Y^k = M + X^k \in \overline{\mathcal{S}^w}$ by Lemma 2 and $\lim_k \left\|Y - Y^k\right\|_{\mathcal{S}} = \lim_k \sup_\delta P\left(X_\delta - X^k_\delta\right) \leq \lim_k \sup_\delta P\left(Y_\delta \left\{Y_\delta > k\right\}\right) = 0$.

The classical Doob Meyer decomposition (see [1], [6] and [7]) does not follow from uniform integrability but requires the class D property [7, VII.T29]. Meyer himself considered this condition as “not very easy to handle” and its relationship with uniform integrability as yet unclear [6, p. 195][1]. Theorem 1 contributes (if at all) to this discussion by clarifying the role of uniform integrability for the Doob Meyer decomposition.

3 The Class D_σ Property.

Proposition 1 justifies the following terminology.

Definition 4 *A stochastic process X is of class D_σ if the collection*

$$D_\sigma(X) = \left\{\sum_{n\in\mathbb{N}} X_{\delta^d_n} F_n : \delta^d_n \in \Delta, \delta^d_n \leq \delta^d_{n+1} < \delta_\infty;\ F_n \in \mathcal{F}_{\delta^d_n};\ F_n F_{n'} = \varnothing;\ n, n' \in \mathbb{N}\right\} \tag{13}$$

is uniformly integrable.

If X is a uniformly integrable potential, fix an increasing sequence $d^X = \left\langle \delta^X_n \right\rangle_{n\in\mathbb{N}}$ such that $\delta^X_1 = \delta_0$, $\delta^X_n < \delta_\infty$ and $\lim_n P\left(X_{\delta^X_n}\right) = 0$ and denote by $\mathcal{D}_X$ the collection of all increasing sequences in Δ formed by adding to d^X a finite subset of Δ. Let $\mathcal{D}_X$ be directed by inclusion.

Proposition 1 *Let $X \in \mathcal{S}$. X is Doob Meyer if and only if X is of class D_σ.*

Proof Let $X_\delta = P\left(M \middle| \mathcal{F}_\delta\right) - A_\delta$ be the Doob Meyer decomposition of $X \in \mathcal{S}^m$ and A_{δ_∞} be the L^1 limit of A. If $h = \sum_{n\in\mathbb{N}} X_{\delta_n} F_n \in D_\sigma(X)$ then $P(|h|) \leq P\left(|M| + A_{\delta_\infty}\right)$ and

$$\begin{aligned} P\left(|h|\left\{|h| > c\right\}\right) &= P\sum_{n\in\mathbb{N}} \left|X_{\delta_n}\right| \left\{F_n \left|X_{\delta^d_n}\right| > c\right\} \\ &\leq P\sum_{n\in\mathbb{N}} \left(|M| + A_{\delta_\infty}\right)\left\{F_n \left|X_{\delta_n}\right| > c\right\} \\ &= P\left(\left(|M| + A_{\delta_\infty}\right)\left\{|h| > c\right\}\right) \end{aligned}$$

[1] An attempt to investigate the relationship between uniform integrability and the class D property was done in [6, proposition 1, p. 195]. A uniformly integrable supermartingale not of class D was later constructed in [5, pp. 61-2] where also a characterization for right continuous supermartingales was obtained (see. p. 59).

X is then of class D_σ since $P(|h|>c) \le c^{-1}P(|M|+A_{\delta_\infty})$.

Assume now that X is of class D_σ and, *a fortiori*, uniformly integrable. By Riesz decomposition we can actually focus on the case in which X is a potential of class D_σ. For each $d = \left\langle \delta_n^d \right\rangle_{n\in\mathbb{N}} \in \mathcal{D}_X$ let $\hat{\mathcal{F}}^d = \left(\mathcal{F}_{\delta_n^d} : n \in \mathbb{N}\right)$ and denote by $\mathcal{S}(d)$ the class of L^1 bounded supermartingales on $\hat{\mathcal{F}}^d$ and by δ_∞^d the index assigned to the σ algebra $\bigvee_{n\in\mathbb{N}} \mathcal{F}_{\delta_n^d}$. The potential $\hat{X}^d = \left(X_{\delta_n^d} : n \in \mathbb{N}\right)$ is uniformly integrable: for each $r \in \mathbb{N}$ there exists $\hat{X}^{d,r} \in \mathcal{S}^m(d)$ such that $0 \le \hat{X}^{d,r} \le \hat{X}^d$ and $\left\|\hat{X}^{d,r} - \hat{X}^d\right\|_{\mathcal{S}(d)} \le 2^{-r}$, by Lemma 2. Let $\hat{A}^{d,r}$ be the increasing, integrable and weakly predictable process associated with $\hat{X}^{d,r}$ and $\hat{A}^{d,r}_{\delta_\infty^d}$ its L^1 limit. The processes $\hat{X}^d$, $\hat{X}^{d,r}$ and $\hat{A}^{d,r}$ defined on $\hat{\mathcal{F}}^d$ extend to processes X^d, $X^{d,r}$ and $A^{d,r}$ on $\tilde{\mathcal{F}}$ by letting

$$X_\delta^d = X_{\delta_0}\{\delta_0\} + \sum_{n\in\mathbb{N}} P\left(X_{\delta_{n+1}^d} \middle| \mathcal{F}_\delta\right)\left\{\delta_n^d < \delta \le \delta_{n+1}^d\right\} \tag{14}$$

$$X_\delta^{d,r} = \hat{X}_{\delta_0}^{d,r}\{\delta_0\} + \sum_{n\in\mathbb{N}} P\left(\hat{X}^{d,r}_{\delta_{n+1}^d} \middle| \mathcal{F}_\delta\right)\left\{\delta_n^d < \delta \le \delta_{n+1}^d\right\} \tag{15}$$

and

$$A_\delta^{d,r} = \sum_{n\in\mathbb{N}} \hat{A}^{d,r}_{\delta_{n+1}^d}\left\{\delta_n^d < \delta \le \delta_{n+1}^d\right\} + \hat{A}^{d,r}_{\delta_\infty^d}\bigcap_n\left\{\delta > \delta_n^d\right\} \tag{16}$$

Then $X^{d,r}$, $X^d \in \mathcal{S}$ are such that

$$X_\delta \ge X_\delta^d \ge X_\delta^{d,r} = P\left(A^{d,r}_{\delta_\infty^d} \middle| \mathcal{F}_\delta\right) - A_\delta^{d,r}, \quad \delta \in \Delta \tag{17}$$

$$\left\|X^d - X^{d,r}\right\|_{\mathcal{S}} = \left\|\hat{X}^{d,r} - \hat{X}^d\right\|_{\mathcal{S}(d)} \le 2^{-r} \quad \text{and} \quad \left\|X_\delta^d - X_\delta\right\| = 0, \;\; \delta \in d \tag{18}$$

$A^{d,r}$ is increasing, integrable and d-predictable: then $h^q = \sum_{n\in\mathbb{N}} X_{\delta_n^d}\left\{A^{d,r}_{\delta_{n+1}^d} > q \ge A^{d,r}_{\delta_n^d}\right\} \in D_\sigma(X)$.

$$\begin{aligned}
P\left(A^{d,r}_{\delta_\infty}\left\{A^{d,r}_{\delta_\infty} > q\right\}\right) &= \sum_{n\in\mathbb{N}} P\left(A^{d,r}_{\delta_\infty}\left\{A^{d,r}_{\delta_{n+1}^d} > q \ge A^{d,r}_{\delta_n^d}\right\}\right) \\
&\le \sum_{n\in\mathbb{N}} P\left(\left(X_{\delta_n^d} + A^{d,r}_{\delta_n^d}\right)\left\{A^{d,r}_{\delta_{n+1}^d} > q \ge A^{d,r}_{\delta_n^d}\right\}\right) \\
&\le \sum_{n\in\mathbb{N}} P\left(\left(X_{\delta_n^d} + q\right)\left\{A^{d,r}_{\delta_{n+1}^d} > q \ge A^{d,r}_{\delta_n^d}\right\}\right) \\
&= P\sum_{n\in\mathbb{N}} X_{\delta_n^d}\left\{A^{d,r}_{\delta_{n+1}^d} > q \ge A^{d,r}_{\delta_n^d}\right\} + qP\left(A^{d,r}_{\delta_\infty} > q\right) \\
&= P\left(h^q\left\{A^{d,r}_{\delta_\infty} > q\right\}\right) + qP\left(A^{d,r}_{\delta_\infty} > q\right)
\end{aligned} \tag{19}$$

Following [9, lemma 2] rather strictly, we deduce from (19)

$$P\left(h^q\left\{A^{d,r}_{\delta_\infty} > q\right\}\right) \ge P\left(\left(A^{d,r}_{\delta_\infty} - q\right)\left\{A^{d,r}_{\delta_\infty} > q\right\}\right) \ge P\left(\left(A^{d,r}_{\delta_\infty} - q\right)\left\{A^{d,r}_{\delta_\infty} > 2q\right\}\right) \ge qP\left(A^{d,r}_{\delta_\infty} > 2q\right) \tag{20}$$

which, combined again with (19), delivers

$$\begin{aligned} P\left(A^{d,r}_{\delta_\infty}\left\{A^{d,r}_{\delta_\infty}>2q\right\}\right) &\leq P\left(h^{2q}\left\{A^{d,r}_{\delta_\infty}>2q\right\}\right)+2qP\left(A^{d,r}_{\delta_\infty}>2q\right) \\ &\leq P\left(h^{2q}\left\{A^{d,r}_{\delta_\infty}>q\right\}\right)+2P\left(h^{q}\left\{A^{d,r}_{\delta_\infty}>q\right\}\right) \\ &\leq 3\sup_{h\in D_\sigma(X)} P\left(h\left\{A^{d,r}_{\delta_\infty}>q\right\}\right) \end{aligned}$$

$P\left(A^{d,r}_{\delta_\infty}>q\right)\leq q^{-1}P\left(A^{d,r}_{\delta_\infty}\right)\leq q^{-1}P\left(X_{\delta_0}\right)$ and the class D_σ property imply that for each $\epsilon>0$ and q sufficiently high $\sup_{d\in\mathcal{D}_X,r\in\mathbb{N}}P\left(A^{d,r}_{\delta_\infty}\left\{A^{d,r}_{\delta_\infty}>2q\right\}\right)\leq\epsilon$.

The collection $\left\{A^{d,r}_{\delta_\infty}:\ d\in\mathcal{D}_X,r\in\mathbb{N}\right\}$ is thus uniformly integrable. For fixed $d\in\mathcal{D}_X$ a subsequence of $\left\langle A^{d,r}_{\delta_\infty}\right\rangle_{r\in\mathbb{N}}$ (still indexed by r) admits a limit $A^d_{\delta_\infty}$ in the weak topology of L^1 [7, II.T23], [3, I.7.9]. Define A^d by letting

$$A^d_\delta=P\left(A^d_{\delta_\infty}\middle|\mathcal{F}_\delta\right)-X^d_\delta,\quad\delta\in\Delta \tag{21}$$

For each $F\in\mathcal{F}_\delta$ we get from (17) and (18)

$$P\left(A^d_\delta F\right)=\lim_r P\left(\left(A^{d,r}_{\delta_\infty}-X^{d,r}_\delta\right)F\right)=\lim_r P\left(A^{d,r}_\delta F\right)$$

By Lemma 1 A^d is increasing, integrable and d-predictable; moreover, A^d_δ converges in L^1 to $A^d_{\delta_\infty}$ as X^d is a potential. This same argument applied to the net, $\left\langle A^d_{\delta_\infty}\right\rangle_{d\in\mathcal{D}_X}$ delivers an increasing, integrable and weakly predictable process A such that

$$A_\delta=P\left(A_{\delta_\infty}\middle|\mathcal{F}_\delta\right)-X_\delta,\quad\delta\in\Delta \tag{22}$$

If $h\in L^\infty\left(\mathcal{F}_{\delta^d_n}\right)$, then (21), (18) and (22) imply

$$P\left(h\left(A^d_{\delta^d_{n+1}}-A^d_{\delta^d_n}\right)\right)=-P\left(h\left(X^d_{\delta^d_{n+1}}-X^d_{\delta^d_n}\right)\right)=-P\left(h\left(X_{\delta^d_{n+1}}-X_{\delta^d_n}\right)\right)=P\left(h\left(A_{\delta^d_{n+1}}-A_{\delta^d_n}\right)\right)$$

so that for $b\in L^\infty_+$ and $f\in\mathcal{P}$

$$P\left(b\int fdA\right)=\lim_d P\left(b\int fdA^d\right)=\lim_d P\int\mathcal{P}^d(b)\,fdA^d=\lim_d P\int\mathcal{P}^d(b)\,fdA$$

After the work of Mertens [8, T2] the Doob Meyer decomposition of a supermartingale indexed by $\mathbb{R}_+$ is known to exist even in the absence of the usual assumptions on the filtration and the right continuity of trajectories, provided X is an optional, strong supermartingale of class D [1, theorem 20, p. 414]. Although our supermartingale decomposition does not compare exactly to that of Doob and Meyer, its existence does not require but the class D_σ property, a condition considerably less restrictive than what usually assumed.

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