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The Plant Size Problem and Monopoly Pricing*

Stefano Demichelis†and Ornella Tarola‡

Abstract

This paper is concerned with the so called "plant-size problem": a demand increasing over time has to be served by a firm when there are economies of scale in plant construction. Generally the operating firm is assumed to be a cost-minimizer. We depart from this by supposing that it is a profit-maximizer, namely can manipulate demand through its own price. For the properties of a simultaneous pricing and expansion capacity strategy to be exploited, we first fix the time points when the investment for adding capacity is undertaken and characterize the optimal policy in terms of the price and the size of investment. We show that the investment size is strictly smaller than the one realized if no price change is allowed. Then we proceed in the reverse way: we assume that the amount of investment is fixed and define the optimal policy in terms of the price regime and time points when the investment is realized and we prove that price manipulation induces the firm to postpone the increment of capacity. Indeed as for both the cases, prices are so charged to dampen demand and make it satified by the existing capacity at any time.

JEL classification: C61, L21, L23, L25.

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1 Introduction

This paper is concerned with the so called plant size problem. This problem can be phrased in terms of incrementing capacity to serve a demand increasing over

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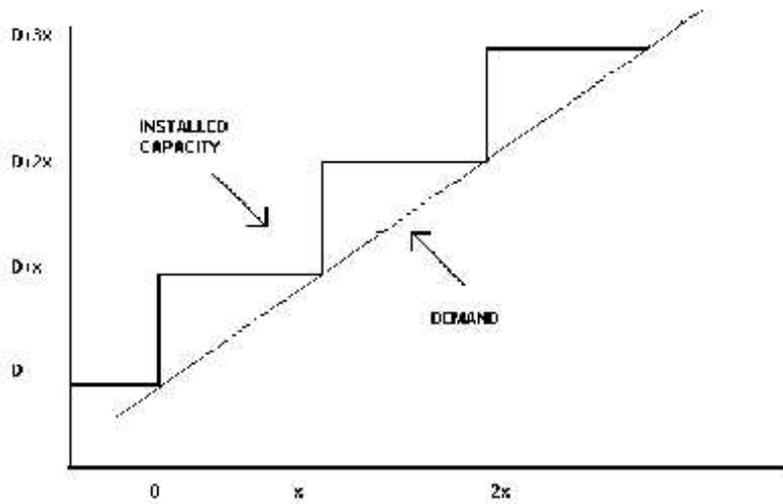
Figure 1: Time paths of demand and installed capacity

time when there are economies of scale in plant construction. Decisions about expanding communication network capacity, or manufacturing capacity to meet a growing demand are istances of a plant size problem.

Since 1960's several contributions have been devoted to analyze this problem[1]. Manne (1961) and (1967) faced the capacity expansion problem of several manufacturing facilities in India[2]. He assumed a demand growing linearly over time. The discount rate and the investment costs resulting from a capacity increase are stationary over time. Investment for adding capacity are subject to economies of scale. No undercapacity is admitted. Under these assumptions, the optimal policy consists of undertaking successive investment of the same size at equally spaced points of time (figure 1). The optimal policy is called by Manne (1961) *constant cycle time policy*[3]. Also, he extended his work to a probabilistic setting where demand follows a Brownian motion with deterministic drift. Uncertainty in demand growth induces to a larger size of capacity expansion.

Srinivasan (1967) formulated a model where the demand is assumed to grow

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[1] This topic has been introduced in the 1950s by Chanery (1952).

[2] Although his starting point was the Indian economy as it actually operated, yet the scope of his works did not confine itself to dealing with the evaluation of specific projects. Instead, the analysis was intended to serve as a general frame for defining when and how large a plant to built.

[3] The capacity expansion problem is formulated very similarly to inventory model. Indeed, a planning investment decision can be interpreted in terms of a (sS) policy in which the expansion point s and the expansion level S are identified with the investment for incrementing capacity and the plant size respectively.

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geometrically over an infinite horizon. In this setting it is still optimal to built a plant at each point of a sequence of equally spaced points of time. Yet contrary to what is proved by Manne (1961), the size of plant to be constructed at each such point of time grows exponentially, due to the geometric growth rate of the demand.

Gabszewicz and Vial (1972) extended Manne's work to exogenous technological progress, showing that if the time at which technological change occurs is known with certainty, depending on this time, the initial investment is of size larger or smaller than the capacity which would be optimal without technological progress. If the date at which technological improvement happens is random, then the optimal first investment is strictly smaller than the one realized without technological change.

Nickell (1977) modified these settings by including lead times for adding capacity. He showed that the presence of a fixed lead times for incrementing capacity would induce a firm to install new equipment earlier. Moreover, the longer the lead time, the earlier the advance of growth in demand. Davies et al. (1987) consider an expansion capacity model which incorporates both non-zero lead times and uncertain demand. A failure cost of meeting this demand is also introduced. Using stochastic calculus technique, they define the optimal policy in terms of numerical alghoritm. Chaouch and Buzacott (1994) analyze the effect of a fixed lead time on the timing of plant investment when demand grows at a constant rate in some period and stagnant growth in other ones. They show that it can be efficient to postpone investment for adding capacity behind the time when the existing capacity is completely absorbed. Ryan (2002) modeled a dynamic programming frame involving uncertain exponential demand[4] growth and deterministic lead times for adding capacity. When the expected lead time shortage is given, the discounted expansion costs are proved to be minimized by expanding capacity by a constant multiple of existing capacity.

Although these research lines represent a continuous advancement in deepening the plant size problem, typically they assume that demand for capacity is exogenously given and does not depend of the plant size plan[5]. Firms are required to define the timing and the increment of capacity needed to meet *that*

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[4]Other previous studies combined random demand with capacity expansion problem. In Freindfelds (1981) demand is modelled as a birth and death process for fixed investment for adding capacity: when the optimal plan has to be computed the randomness of demand can be meant as equivalent to a larger growth rate. Bean et al (1992) show that a capacity expansion problem with uncertain demand can be transformed into a deterministic problem by replacing the random demand with a deterministic one and discounting costs by a new smaller interest rate including an approximate proportion of uncertainty.

[5]Not even literature on pricing strategies is concerned with the simultaneous determination of pricing and investment planning. Pricing models assume that either investment process is prespecified at the beginning of the planning horizon (Bitrain and Mondschein 1995) or defined on a period by period basis (Balvers and Cosimano 1990).

stipulated demand[6]. This was a natural entry point since the past contributions were mainly devoted to solve planning problems in which the operating firm is not a profit maximizer, but only a cost minimizer. A different view point would be consider the firm as a profit maximizing entity. Then the plant size problem has to be reformulated taking into account how the price policy of the firm can affect the demand. In particular, pricing and investment strategies cannot be defined apart from each other, but have to be geared toward coordination of them. The aim of this paper is to relax the exogenous demand assumption. We do this by assuming that the operating firm is a monopoly. For the properties of a simultaneous pricing and expansion capacity strategy to be exploited, we first fix the time points when the investment for adding capacity is undertaken and characterize the optimal policy in terms of the monopoly's instruments: the price and the size of investment. When the monopolist can manipulate demand through price adjustments, the optimal plant size is proved to be strictly smaller than the one realized if no price change is allowed. Still, if demand grows linearly over time, the increment of capacity is constant, consistently with Manne (1961).

Then we proceed in the reverse way: we assume that the amount of investment is fixed and define the optimal policy in terms of the price regime and time points when the investment is realized. Once fixed the amount of investment, these time points are proved to be equally spaced, this result being in line with Manne (1961) again.

Under both the above mentioned assumptions, the optimal price policy displays two types of cycle, depending on the parameters. In the first type, corresponding for instance to a low value of the interest rate, the optimal stationary plan involves alternating monopolistic price regime and higher price regime. In the other case, corresponding for instance to a high value of interest rate, the optimal stationary plan involves only the higher price regime.

The model is described in Section 2. In Section 3 we present the general properties of the optimal plan, namely the ones which hold when both the time points are fixed and the optimal size is exogenously given. They refer to the optimal price policy. Then we conduct our analysis in the context of the two complementary cases: the optimal policy is fully characterized when the timing of investment is given (Section 3.1) and when the reverse case holds (Section 3.2), namely when the investment size is given. We summarize the main findings in the concluding Section 4. Details on several calculations are contained in the Appendix.

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[6] Admittedly, the coordination of pricing and inventory strategies has been faced in literature (Amihud and Mendelson 1983, Federgruen and Heching 1997). Yet, typically it is assumed that exceed demand can be backlogged. Thus, under the backlogged shortage assumption, the plant size problem calls for an extremely different analysis. A good review is provided by Eliashberg and Steinberg (1991).

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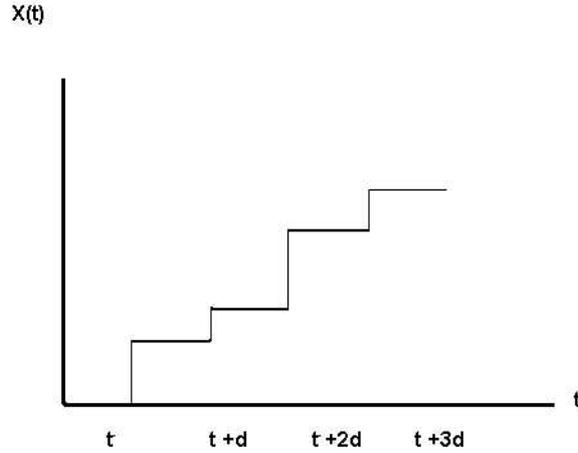
Figure 2: A constant cycle policy.

2   The model

Before presenting the model in detail we introduce several useful definitions [7].

A *decision point DP* is a point of time at which the investment can be realized. A *regeneration point RP* is a decision point x_{t_i} at which the current capacity is equal to demand. The distance between two regeneration points is called *cycle*.

A *policy* is a sequence of DP's for which at any time t the capacity is greater or equal to the demand at any t, $t\,[0,\infty[\,$.

An *optimal policy* — maximizing the discounted profits — is said to have the *RP-property* if the sequence of DC's involves only RP's.

A *constant cycle policy* (figure 2) is a policy which has the RP-property and whose both the investment size

decision for adding capacity and the pricing plan are the stationary, being fixed the regeneration points RPs.

Finally a *constant time cycle policy* (figure 3) is a policy which has the RP-property and whose both the distance between two regeneration points the pricing plan are the stationary, being fixed the investment for adding capacity.

A single firm is facing a demand increasing linearly and decreasing linearly in price, so up to rescaling price $D\left(t;p\right)=At-p$. Its productive capacity is bounded by amount of equipment capital $X\left(t\right)$. While the existing capacity can be larger than the current demand, no undercapacity is admitted, i.e. $D\left(t\right)\leq X\left(t\right)$.

At fixed times t_i $i=1,2...$ it decides the amount x_{t_i} that is invested or alternatively given a fixed amount of x_{t_i} it decides the times t_i when the investment

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[7]Some of them are borrowed from Manne (1961).

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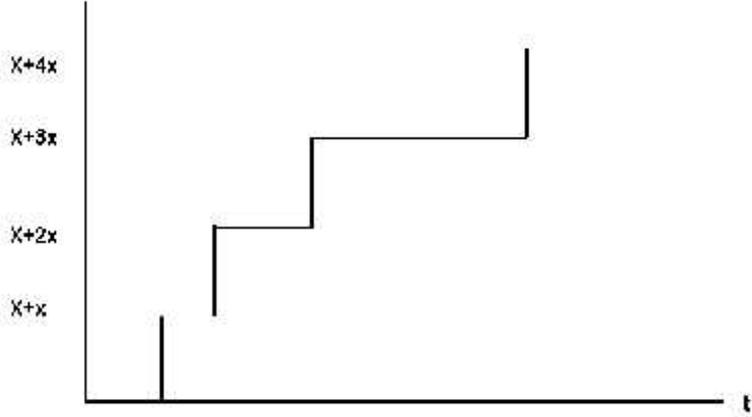
Figure 3: A constant time cycle policy.

is undertaken. We suppose that the equipment capital lasts forever, so that $X(t) = \sum_{i:t_i \leq t} x_{t_i}$. Note that $X(t)$ has a constant value, that will be called X_{t_i} on the interval $[t_i, t_{i+1}[$. The investment for adding capacity includes both fixed and variable costs, and the cost structure is assumed to hold forever (no technological progress). It can be expressed as follows: $f(x) = K + ax, K > 0$.

Time is discounted at a constant interest rate r. The time horizon is infinite.

The timing of actions in each period is as follows. The firm decides the amount x_{t_i} to be invested and the time t_i at which this investment is undertaken , for time $t \in [t_i, t_{i+1}[$, it sets the price $p(t)$ so as to maximize profits, given its plant size, then at time t_{i+1} a new investment $x_{t_{i+1}}$ is made and so on.

As stated in the introduction, we consider in the following two alternative assumptions. According to the first assumption, the decision points are fixed and we determine the optimal investment size and the optimal price policy. According to the second one, the size of investment is fixed and we determine the optimal timing together with the optimal policy[8].

3   The optimal policy: several general properties

Formally the problem is to find $\mathbf{x} = (x_1; x_2; ....x_{t_i}...) \in R_+^\infty$ and $\mathbf{t} = (t_1; t_2; ....t_i...) \in R_+^\infty$ increasing and $p(t)$, measurable function of $t \in R_+$ so that the objective function is maximized:

$$V(\mathbf{x}, p(t)) = \int_0^\infty p(t) D(t, p(t)) e^{-rt} dt - \sum_{i=1}^\infty (K + ax_{t_i}) e^{-rt_i}$$

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[8]A more general cases where firm can define both the investment for adding capacity and the timing will be dealt within a future paper.

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s.t. the no undercapacity constraint holds

$$D\left(t, p\left(t\right)\right) \le X(t)$$

In order to find the optimal solution we first remark that the cost function $\sum_{i=1}^{\infty}\left(k + ax_{t_i}\right)$ does not depend on the choice of $p\left(t\right)$ so a sufficient condition for the optimality of $p\left(t\right)$ is that it maximizes the integrand $p\left(t\right)D\left(t; p\left(t\right)\right)$ at any point t, given the no undercapacity constraint.

The next two propositions are proved under the first assumption. It is left to the reader to prove that they also hold under the second one.

Proposition 1 *If* $\mathbf{x}$ *is kept fixed* $V\left(\mathbf{x}, p\left(t\right)\right)$ *achieves its maximum for* $p\left(t\right)$ *given by*

$$p(t) = \max(At/2, At - X_{t_i})$$

for $i = \max[n \mid t_n < t]$ the integer part of t.

Proof. This follows simply from the maximization problem

$$\max_{p(t)} = p(t)D\left(t; p\left(t\right)\right)$$
$$\text{s.t. } D\left(t; p\left(t\right)\right) \le X_{t_i}$$

$t_i = [t]$. **Q.E.D**.

So during the period $[t_i, t_{i+1}[$ we have two possible price regimes: at some t the constraint is not binding and $p(t)$ is set equal to the price $p^M\left(t\right) = \frac{At}{2}$, otherwise we have $D\left(t; p\left(t\right)\right) = X_{t_i}$ i.e. the firm chooses price $p^C\left(t\right) = At - X_{t_i}$ so as to contract the demand $D\left(t; p\left(t\right)\right)$ within the limits imposed by its plant size.

Let us call $t_i^* = \frac{2X_{t_i}}{A}$ the switching point.

Suppose first that $t_i < t_i^* < t_{i+1}$ so that the interval $[t_i; t_{i+1}[$ is split into two subintervals $[t_i; t_i^*]$ and $]t_i^*; t_{i+1}[$. From proposition 1 it follows that in the first subinterval $p(t) = p^M(t)$, in the second one $p(t) = p^C(t)$. We shall call them the *monopoly price regime* and the *constrained price regime* respectively. The resulting demands will be denoted respectively $D^M\left(t\right)$ and $D^C\left(t\right)$. At the switching point t_i^*, $p^M(t) = p^C(t)$.

Then assume that the decision point t_i exceeds the switching point t_i^*, so the plant size X_{t_i} is low and the firm may have to begin immediately with $p^C(t)$. Finally assume that the switching point t_i^* exceeds the regeneration point t_{i+1}, then X_{t_i} is high and only the monopoly regime holds.

Once the investment x_{t_i} has been made at time t_i the firm tries to maximize profits by setting the instantaneous monopoly price. When demand expands, the firm may be compelled to impose a higher price $p^C(t)$ to be able to satisfy it.

Thus, if, and how soon, the monopoly price regimes gets into the higher price regime will in general depend on the size of investment. If the optimal investment size for adding capacity is so big as to meet the demand $D^M(t)$ at all time points in $[t_i, t_{i+1}[$, the time of the switch can possibly be delayed up to the end of the cycle so that the price policy only consists of the monopoly price regime. However, the latter situation never arises for optimal investment plan. Indeed, we show now that for optimal investment plans in any cycle $[t_i; t_{i+1}[$, there is a nonempty subinterval in which constrained price policy applies.

To prove this, it suffices to show that the optimal existing capacity $X_{t_i}^*$ at any i must be strictly less than the capacity X_{t_i} satisfying the monopoly price demand $D^M(t)$ (figure 4). Suppose that $\mathbf{x}^* = (x_1^*; ...x_{t_i}^*; ...x_{t_{i+1}}^*; ...)$ is an optimal plan, then $V(x^*) \geq V(x)$ for any other x, and by contradiction suppose that $t^* = t_{i+1}$ so $X_{t_i} \geq D_{t_{i+1}}^M$. Let's choose $\mathbf{x} = (x_1^*; ...x_{t_i}^* - \varepsilon; ...x_{t_{i+1}}^* + \varepsilon; x_{t_{i+2}}^*...)$, i.e. we alter the plan by transferring ε of the investment from t_i to t_{i+1}.

The cost function is reduced by $\varepsilon a \left(e^{-rt_i} - e^{-r(t_{i+1})} \right)$, a linear function of ε. To see the change in the revenue function we remark that $X_k = X_k^*$, $k \leq t_{i-1}$; $X_{t_i} = X_{t_i}^* - \varepsilon$ and $X_k = X_k^*$, $k \geq t_{i+1}$, so the only change in the integral is from t_i to t_{i+1}. In this period there will be a possible loss of revenue due to the fact that the plant size is reduced and so the demand may have to be lower.

However, if X_{t_i} is strictly higher than $\frac{A}{2}(t_{i+1})$, it is seen that there is no change for small ε because the monopoly demand is strictly lower than the plant size. In case of equality, the next proposition shows that the loss in revenue is of the order of ε^3 so for small ε is negligible compared to the loss in cost.

So in both cases X_{t_i} cannot be optimal.

Proposition 2 *During any cycle $[t_i, t_{i+1}[$ the switch of price between the two optimal regimes $p_{t_i}^M$ and $p_{t_i}^C$ happens at $t_i^* < t_{i+1}$.*

Proof. *(i) By contradiction,* let X_{t_i}' be the existing optimal capacity at any t_i such that $X_{t_i}' > D_{t_{i+1}}^M$. The associated optimal policy consists then only of the monopoly price regime $p_{t_i}^M$. Since the pair $\left\{ \left[X_{t_i}', p_{t_i}^M \right] \right\}_{i=0}^{\infty}$ is optimal, it should be profit-maximizing. Now assume that the investment size drops by a small quantity ε, where $X_{t_i}' - D_{t_{i+1}}^M > \varepsilon$ and $0 < \varepsilon < 1$. The firm has no loss since the demand is still met. Yet the monopolist gains the discounted cost saved by reducing the investment by ε, namely $a\varepsilon \left(e^{-r(t_{i+1})} - e^{-rt_i} \right)$. Hence if the investment were to reduce, the corresponding profit would increase contradicting the hypothesis that $\left\{ \left[X_{t_i}', p_{t_i}^M \right] \right\}_{i=0}^{\infty}$ was the optimal choice.

(ii) By contradiction, let X_{t_i}'' be the existing optimal capacity at any t_i such that $X_{t_i}'' = D_{t_{i+1}}^M$. As before, the pair $\left\{ \left[X_{t_i}'', p_{t_i}^M \right] \right\}_{i=0}^{\infty}$ is the optimal policy. The present value of the discounted flow of revenues during the cycle t_i is $R_{t_i} = \int_{t_i}^{t_{i+1}} \left(\frac{At}{2} \right)^2 e^{-rt} dt$. Now assume that the investment size drops by a small quantity

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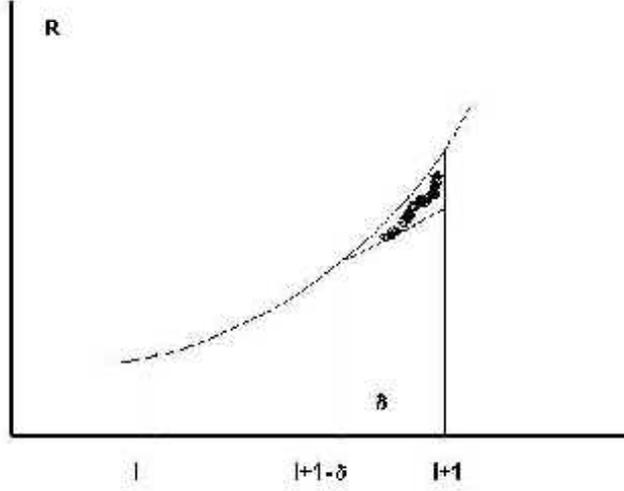
Figure 4: Loss of revenue due to a small reduction of investment.

ε, where $X''_{t_i} - D^M_{t_{i+1}} < \varepsilon$ and $0 < \varepsilon < 1$. Again the monopolist gains the discounted cost saved by reducing the investment by ε. Yet, contrary to what happened in the previous case, the demand is not completely met. This induces to switch from the monopoly price regime $p^M_{t_i}$ to the higher price one $p^C_{t_i}$. Thus, the present value of the discounted flow of revenues during the cycle t_i turns into $R_{t_i} = \int_{t_i}^{t_{i+1-\delta}} \left(\frac{At}{2}\right)^2 e^{-rt}dt + \int_{t_{i+1-\delta}}^{t_{i+1}} \left(\frac{A(t_{i+1-\delta})}{2}\right)^2 + \frac{A}{2}^2(t_{i+1-\delta})(t - t_{i+1-\delta}))e^{-rt}dt$, where $\delta = \frac{A\varepsilon}{2}$, The loss induced by the switch between the two price regime is then $R_{t_i} = \int_{t_{i+1-\delta}}^{t_{i+1}} \left[\left(\frac{At}{2}\right)^2 - \left(\frac{A(t_{i+1-\delta})}{2}\right)^2 - \frac{A}{2}^2(t_{i+1-\delta})(t - t_{i+1-\delta})\right] e^{-rt}dt$.

Since the gain and the loss are respectively of first and third order in ε, this is the integral of a function of the order of δ^2 over an interval of size δ so its order of magnitude δ^3, for ε small enough the net loss is negative (figure 4). Then if the investment were to reduce, the corresponding discounted profit would increase contradicting the hypothesis that $\left\{\left[X''_{t_i}, p^M_{t_i}\right]\right\}_{i=0}^{\infty}$ was the profit-maximizing choice. **Q.E.D.**

Hence, the optimal policy prescribes an investment size at any time t_i that is strictly lower than the one guaranteeing a monopoly price till the end of the *cycle* (figure 5).

On the other side, for special values of the parameters, it is possible that the cycle consists of the constrained price regime only[9].

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[9]An in-depth analysis of both the case when the optimal policy consists of the constrained price regime only and the one when it involves alternating monopoly price regime and constrained price regime will be provide in the next two sections.
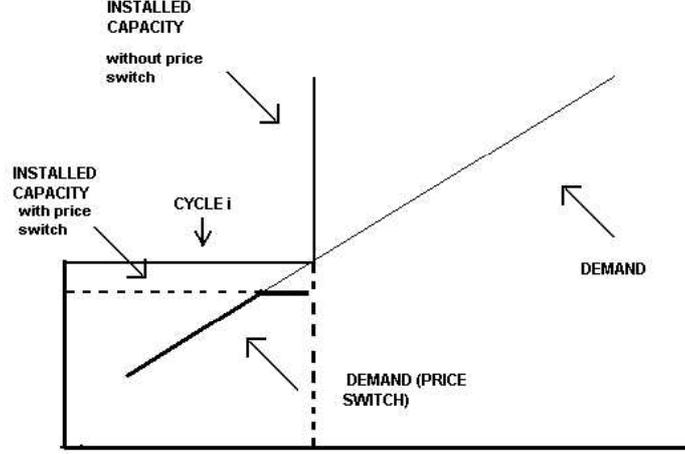
Figure 5: Optimal price policy.

3.1 Optimal plan with exogenously given decision points

Let us first assume that $t_i s$ are exogenously given and equally spaced, so that up to a change of units, we can write $t_i = i$. The distance between two regeneration points is constant and without loss of generality normalized to 1. Then, given a sequence of RP's $[i_1, i_2...i_n,]$, it follows that $i_{i+1} = i + 1$.

We now have to determine the size of investment $\mathbf{x}^*$ which maximizes

$$V\left(\mathbf{x}\right) = \sum_{t_i=0}^{\infty} \left(\int_i^{i+1} e^{-rt} \pi^T(t) dt \right) - e^{-ri}(K + ax_i) \tag{1}$$

or, by substituting the optimal price values,

$$\int_{t_i}^{t_{i+1}} e^{-rt} \pi^T(t) dt = \int_i^{t^*} e^{-rt} \frac{(At)^2}{4} dt + \int_{t^*}^{i+1} e^{-rt} X_i (At - X_i) dt$$

If there is no monopoly regime, only the second integral in the sum is taken and t^* is equal to i. The monopoly price regime takes turn with the constrained price regime only if the investment size for incrementing capacity is large enough to meet a demand increasing over time, otherwise constrained price regime only applies. Namely, when the investment cost is high in comparison with the growth of demand, then the revenue resulting from a non-negative investment size does not countervail the associated investment cost: the firm refrains from increasing capacity and the constrained price regime is only adopted in order to dampen the demand. Once the investment cost becomes low with respect to the growth of demand, then a non-negative investment is undertaken: the price regime tends to

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decrease as the investment increases. When the investment size is so large as to satisfy the increasing demand, the constrained price regime meant as unique price regime is replaced by a price policy alternating monopoly prices and constrained prices[10]. So we say that for $\breve{x} \leq x$ both the monopoly price regime and the constrained one apply, while for $\breve{x} > x$, the constrained price regime only arises, where $\breve{x}$ is a threshold value depending on the investment cost (a and r) and the growth of demand (A)[11].

We discuss now the properties of the optimal investment plans both in the case when the investment cost is low in comparison with the growth of demand and in the reverse one. Under the first assumption, namely when $\breve{x} \leq x$, the switching point t_i^* is greater than the decision point i and both the price regimes apply. Under the second assumption, namely when $x < \breve{x}$, the switching point t_i^* is strictly smaller than the decision point i and the constrained price regime only applies.

Let us first remark the following, which will be used in the proof of *Proposition 3* and *Proposition 4*. As in the proof of the preceding proposition, x^* optimal implies that $V(x^*) > V(x)_{i,h}$ for $\mathbf{x}_{i,h} = (x_1, ...x_i - h, x_i + h, x_{i+1}...)$ for any i and $h \leq x_i$. So, it follows that:

$$\frac{d}{dh} V(x_{i,h}) = 0 \ , \ i = 1, 2...$$

Proposition 3 *When the investment cost is low with respect to the growth of demand, the optimal investment policy is unique and stationary and it involves alternating monopoly regime and constrained regime.*

Proof.

Given $i < t_i^*$ (figure 6), the first order conditions become

$$A e^{-ri} \int_{t_i^* - i}^{1} e^{-rs} [s - (t_i^* - i)] ds = a e^{-ri} \left(1 - e^{-r} \right)$$

or

$$f(t_i^* - i) = a$$

where $f(l) = A \int_l^1 e^{-rs} (s - l) ds / (1 - e^{-r})$ is a function that does not depend on i (figure 7) and $s = t - i$.

It is immediate to see that condition *(i)* implies that $t_i^* = i + \bar{l}$ where $\bar{l}$ is the unique solution to $f(l) = a$. So the solution is stationary, as claimed. **Q.E.D.**

Proposition 4 *When the investment cost is high with respect to the growth of demand, the optimal investment policy is unique and stationary and it involves the constrained price regime meant as unique price regime.*

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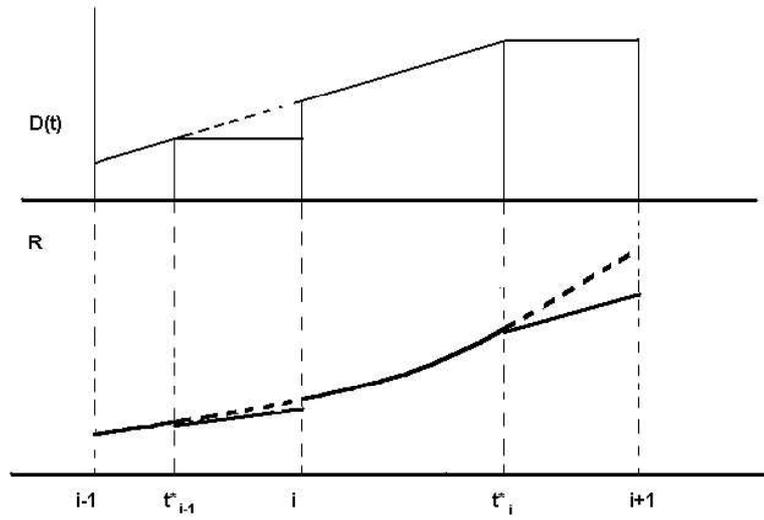
Figure 6: Monopoly/constrained regime.
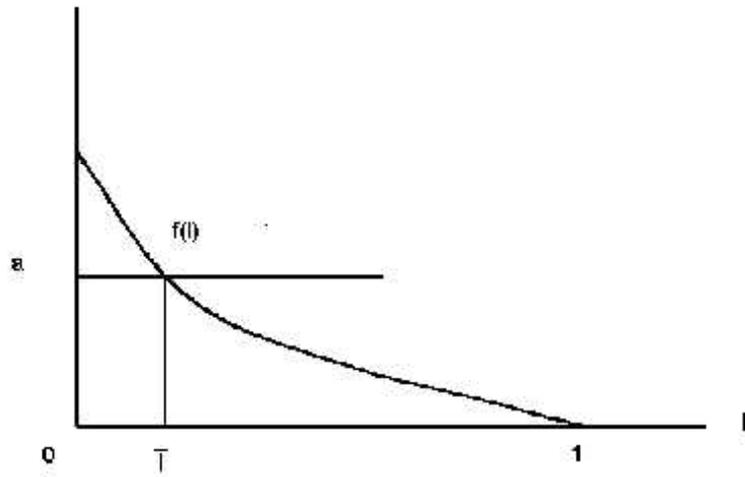
Figure 7: $f(l)$ does not depend on i.
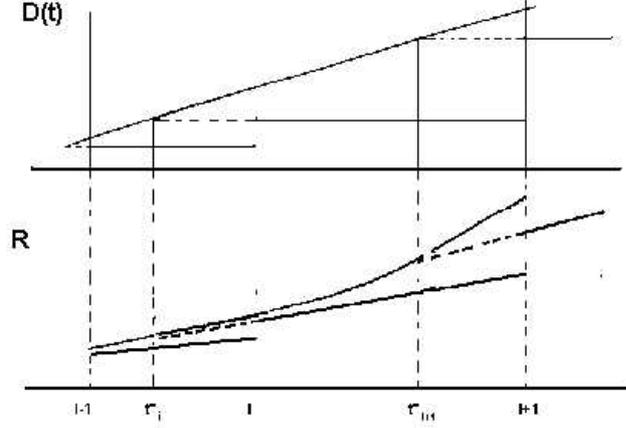
Figure 8: Only constrained price regime.

Proof.

Given $i > t_i^*$ (figure 8) the first order condition can be written

$$\int_i^{i+1} (At - 2X_i) e^{-rt} dt = ae^{-ri} \left(1 - e^{-r}\right)$$

where $X_i = X_{i-1} + \lambda_i$ and $t_i^* = \frac{2X_i}{A}$ or

$$\int_0^1 (At - 2X_{i-1}) e^{-rs} ds - a \left(1 - e^{-r}\right) = 2\lambda_i \int_0^1 e^{-rs} ds \qquad (2)$$

where λ_i is the solution of (2).

Then

$$x_i = \max[0, \lambda_i]$$

Indeed for

$$i < i_0, x_i = 0$$

$$i \geq i_0, x_i = z$$

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[10] The price policy involving the monopoly price regime meant as unique price regime is never optimal, as it has been proved in Proposition 2.

[11] The relationship between the investment cost and the growth of demand behind the optimal investment plan is discussed in detail in Appendix A.

where i_0 is the first decision point i s.t. the revenue $\int_i^{i+1} Ate^{r(s-i)}ds$ and the cost $a(1-e^{-r})$ resulting from a non-negative investment size counterbalance each other and z is an investment size strictly greater than 0, and independent of the time i.

So, due to $x_i = z$ where $z > 0$ the price regime is the highest one till i_0. Then $x_i > 0$ and the price regime tends to decrease as the investment for adding capacity increases. At the time point when the plant size is so big as to satisfy the demand growing linearly over time, the constrained price regime is replaced by the monopolistic price regime and the case when there are both the price regimes applies. **Q.E.D**.

3.2 Optimal plan with exogenously given investment size

Let us proceed now in the reverse way assuming, instead, that the amount of investment for adding capacity is fixed. Under the assumption that the investment size is exogenously given, there are two different types of cycle depending on the size of the fixed investment x_i.

- If $x_i \geq \bar{x}$ where $\bar{x}$ is a threshold value[12], then price policy involves alternating between monopoly price and constrained price regime

- If $x_i < \bar{x}$, then price policy involves only constrained prices, periodically adjusted[13].

We discuss now the properties of the optimal plan in both these cases. We show that the optimal investment time policy is unique and stationary regardless of the value of x, which solely affects the price policy.

So we have to determine **t** that maximizes

$$V(\mathbf{t}) = \int_{t_i}^{t_{i+1}} e^{-rt}\pi^T(t)dt = \int_{t_i}^{t^*} e^{-rt}\frac{(At)^2}{4}dt + \int_{t^*}^{t_{i+1}} e^{-rt}X_{t_i}(At - X_{t_i})dt$$

Proposition 5 *If x_i is large enough so that $t_i^* \geq t_i$, then the optimal investment time policy is unique and stationary.*

Proof. We derive the first order condition for t_is. Suppose that firm invests a fixed amount x_i at time $t_i + \varepsilon$ instead of time t_i. The only change in the revenue function due to postponing investment is between t_i and $t_i + \varepsilon$. So if x_i is large enough so that $t_i^* \geq t_i$, then the difference in the revenue function G is between a parabola whose equation is $\frac{A^2t^2}{4}$ and a straight line whose equation is $\frac{A^2}{4}[2t_{i-1}^*t - (t_{i-1}^*)^2]$ (figure 9).

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[12] Details on this are provided in Appendix B.

[13] The case when price policy involves only monopoly price regime is excluded by *Proposition 2*.

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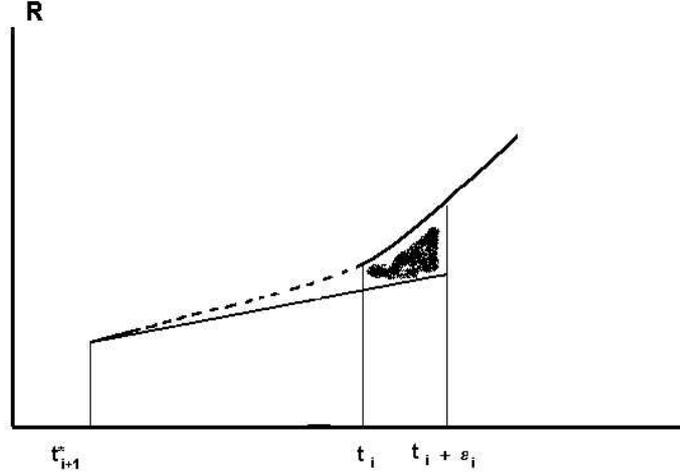
Figure 9: Difference in revenue function when investment for adding capacity is large enough.

The difference in the cost function C is

$$(K + ax_i)(e^{-rt_i} - e^{-r(t_i+\varepsilon)}) \sim \varepsilon(K + ax_i)e^{-rt_i}$$

up to terms quadratic in ε.

Then the change in the profit function is

$$\Delta V = e^{-rt_i}\varepsilon[(K + ax_i)r - \frac{A^2}{4}(t_i - t_{i-1}^*)^2]$$

From the figure 10 it is easy to see that given x_i, there is a unique t_i such that the two curves C and G intersect over x_i.**Q.E.D**.

Above the threshold value $\bar{x}$, the investment size is large enough to meet the increasing demand, the price policy involves alternating monopoly regime and constrained regime so the switching point t_i^* is greater than the decision point t_i.

Proposition 6 *If x_i is so small that $t_i^* < t_i$, then the optimal investment time policy is unique and stationary.*

Proof. As before, we derive the first order condition for t_is. Suppose that firm invests a fixed amount x_i at time $t_i + \varepsilon$ instead of time t_i. From the figure 9 the only change in the revenue function due to postponing investment is between t_i and $t_i + \varepsilon$.

If x_i is small so that $t_i^* < t_i$, then the difference in the revenue function is between two straight lines whose equations are respectively $\frac{A^2}{4}[2t_{i-1}^*t - (t_{i-1}^*)^2]$ and $\frac{A^2}{4}[2t_i^*t - (t_i^*)^2]$.

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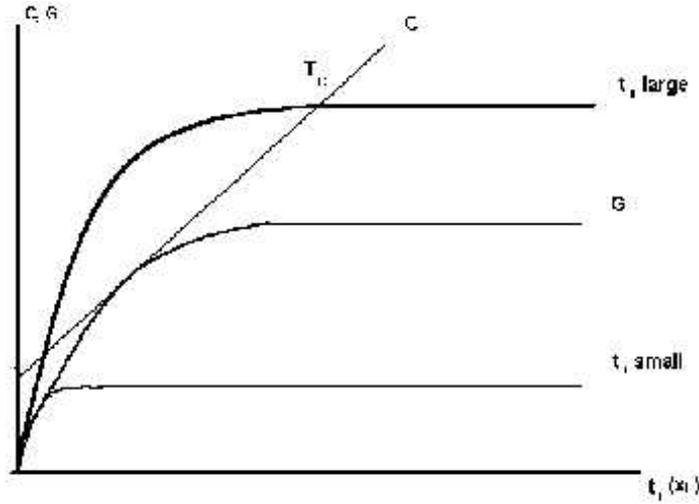
Figure 10: Stationarity of the optimal policy still holds.

Again, the difference in the cost function C is

$$(K + ax_i)(e^{-rt_i} - e^{-r(t_i+\varepsilon)}) \sim \varepsilon(K + ax_i)e^{-rt_i}$$

up to terms quadratic in ε.

Then the change in the profit function is

$$\Delta V = e^{-rt_i}\varepsilon[(K + ax_i)r - \frac{A^2}{4}(t_i^* - t_{i-1}^*)(2t_i - (t_i + t_{i-1}^*)]$$

So, given x_i, there is a unique t_i such that the two curves C and G intersect over x_i (figure 10).**Q.E.D**.

Below the threshold value $\bar{x}$, the investment size is not large enough to meet a demand increasing over time, the price policy has to dampen the demand in order to satisfy the RP property so the switching point t_i^* is strictly smaller than the decision point t_i.

4   Conclusions

In this paper we considered a plant size problem in which the demand can be manipulated by price.

We gave separate treatment to the case when the time points are exogenously given and the amount of investment is fixed

As for both strategies, we proved that prices are charged is such a way to dampen the demand and make it satisfied by the existing capacity. The optimal plant size resulting from this price policy is as follows: given the investment

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timing, the increment of capacity is strictly smaller than the one needed to serve a demand linearly growing over time; given the increment of capacity, the decision points are postponed behind the time when the existing capacity would be completely absorbed without prices changes.

Moreover, both these optimal policies are unique and stationary.

Of course, this analysis calls for being generalized. It is possible to show that optimal stationary property exists also when firm can choose both **t** and **x**. The proof will appear in Demichelis and Tarola *(in preparation)*. Although this would make the frame more complex, yet it would not change the main properties of the model. The stationarity property of investment plan involving two price regimes should still hold. Also, technological change could be introduced. Due to technological progress, equipment capital would be subject to an obsolescence process. The analysis would still hold with the only change that expansion capacity should take into account this obsolescence phenomenon. This last approach would allow for exploring the role of GPT's in a completely new setting.

5 Appendix

Appendix A: As stated in Proposition 3 and Proposition 4, the optimal investment policy can involve or alternating monopoly price regime and constrained price regime or this latter one meant as unique regime. Namely when $0 < x < \breve{x}$, the firm charges both the price regimes, otherwise when $\breve{x} < x$ constrained regime only arises.

When the growth of demand is high in comparison with the investment cost, or

$$a < A[1 - r/(e^r - 1)]/r^2$$

then both the regimes apply.

When the growth of demand is low with respect to the investment cost or

$$a > A[1 - r/(e^r - 1)]/r^2$$

then the firm refrains from investment, $x = 0$, and constrained regime only apply.

Finally when the growth of demand and investment cost exactly countervail each other or

$$a = A[1 - r/(e^r - 1)]/r^2$$

then firm starts to undertake a non-null investment $x > 0$ and the constrained regime tends to decrease. When the investment size is large enough to meet a demand linearly growing over time, say $x > \breve{x}$, then the optimal price policy involves both the regimes.

Appendix B: It is easy to see from the figure 10 that for low t_i the curves G and C do not intersect. The t_i for which G and C intersect is computed as follows.

Let $[(\frac{A}{2}t_i - X_{i-1}), (\frac{A^2}{4}(t_i - t_{i-1}^*)^2]$ be the coordinate of T_c. Then, given the straight line $(K + ax_i)r$ it follows

$$r(K + a\frac{A}{2}t_i - aX_{i-1}) = \frac{A^2}{4}(t_i - \frac{2}{A}X_{i-1})^2$$

So

$$t_i = [\frac{A}{2} \pm [(1 + 4rK)\frac{A^2}{4}]^{1/2}]/\frac{A^2}{4}$$

the only positive root of this equation is which uniquely determines t_i and so $\bar{x}$.

References

[1] Amihud Y. and H. Mendelson (1983). "Price Smoothing and Inventory", *Review of Economic Studies*, 50, 87-98.

[2] Balvers R.J. and T.R Cosimano (1990). "Actively Learning about Demand and the Dynamics of Price Adjustment", *Economic Journal* 100, 882-898.

[3] Bean J.C., J. Higle and R.L. Smith (1992). "Capacity Expansion under Stochastic Demands", *Operations Research* 40, 210-216.

[4] Bitran G. R. and S.V. Mondschein (1995). "Periodic Pricing of Seasonal Products in Retailing", *Managment Science* 43, 64-79.

[5] Chaouch B.A. and J.A. Buzacott (1994). The effects of lead time on plant timing and size, *Production and Operations Management*, 3(1), 38-54.

[6] Chenery H. (1952). "Overcapacity and the Acceleration Principle", *Econometrica*.

[7] D'Aspremont C., Gabszewicz J.J. and J.P.Vial (1972). "Capital depreciation, growing demand and economies of scale", *CORE DP* No. 6918.

[8] Davies M.H.A., Dempter M.A.H., Sethi S.P. and D. Vermes (1987). "Optimal Capacity Expansion under Uncertainty", *Advances in Applied Probability*.

[9] Demichelis S. and O. Tarola. "The Plant Size Problem and Monopoly Pricing: A Reappraisal", *work in progress*.

[10] Federgruen A. and A. Heching (1997). "Combined Pricing and Inventory Control Under Uncertainty", *Operations Research* 47, No.3, 454-475.

[11] Freidenfelds J. (1981). Capacity Expansion: Analysis of Simple Models with Applications, North-Holland, New York.

[12] Gabszewicz J.J. and J.P.Vial (1972) "Optimal Capacity Expansion Under Growing Demand and Technological Progress", *CORE DP* No.123.

[13] Manne A. S. (1961). "Capacity Expansion and Probabilistic Growth", *Econometrica*.

[14] Manne A. S. (1967). (Ed.), Investments for Capacity Expansion, London.

[15] Nichell S. (1977) "Uncertainty and lags in the investment decision of firms", *Review of Economic Studies* 44, 249-263.

[16] Ryan S. M. (2002). "Capacity Expansion for Random Exponential Demand Growth with Lead Times", *W.P.* Dep. of Industrial Manufacturing System Engineering, Iowa State University.

[17] Srinivasan T. N (1967). "Geometric rate of growth of demand", in: Investments for capacity expansion, op.cit., chapter 9.

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