

# Risk, Robustness and Knightian Uncertainty in Continuous-Time, Heterogenous Agents, Financial Equilibria

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## Abstract

We analyze and compare analytically continuous-time financial equilibria where heterogeneous risk averse investors care about model misspecification through some preference for robustness and in the presence of a stochastic opportunity set. This incorporates a concern for model misspecification into equilibrium asset prices. Since no exact equilibrium computations are possible in this model setting, perturbation theory is used to provide first order asymptotics for the implied equilibria. We find that to first order robustness enhances effective risk aversion while keeping constant the preference for intertemporal substitution. Therefore, equilibrium consumption, equilibrium capital stock dynamics (in production economies) and equilibrium stock price processes (in exchange economies) are not directly modified by a preference for robustness. By contrast, robustness affects directly optimal portfolios, causing lower equilibrium interest rates - and thereby enhanced risk premia - when the speculative investment motive dominates the intertemporal hedging demand. Finally, at variance with other robustness specifications, definitions of robustness that mimic Knightian uncertainty produce state dependent effective risk aversions that generate first order risk aversion effects on optimal portfolios, equilibrium interest rates and equity premia. This yields functional forms for some key equilibrium variables like equity premia which are structurally different from those implied by standard risk aversion or other robustness definitions, which reflect all second order risk aversion. Moreover, under Knightian uncertainty the structure of an equilibrium depends strongly on the completeness of the underlying economy. For instance, within complete production economies we find that Knightian uncertainty can generate an endogenous stock market participation, a feature that cannot be obtained by the other robustness definitions. The richness of the equilibrium effects generated in our heterogenous economies suggests that definitions of robustness which mimic Knightian uncertainty can generate the largest variety of robust economic behaviours in the presence of model uncertainty.

**Keywords:** Financial Equilibria, Knightian Uncertainty, Model Misspecification, Perturbation Theory, Robust Decision Making.

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# 1 Introduction

A few approaches have been recently proposed in the literature to model continuous-time economies where agents perceive the probability law underlying asset prices as "ambiguous" or "uncertain" (cf. Anderson et al. (AHS, 1998, 2000), Chen and Epstein (2000), Epstein and Schneider (2001), Hansen, Sargent and Tallarini (1999)). In these models ambiguity/uncertainty arises because the agent's relevant probabilistic beliefs are given by a set of probability measures. A version of an intertemporal max-min expected utility approach is then used to characterize the implied optimal rules. These policies are designed to protect economic agents against unfavourable probabilistic structures of the financial environment, and introduce a form of pessimism with respect to the given set of priors beliefs. All these approaches are partly motivated by Ellsberg (1961) paradox, which illustrates that aversion to model ambiguity/uncertainty, as distinct from risk, is economically meaningful.

Ambiguity aversion has been recently motivated behaviourally in Marinacci (1999) and Ghirardato and Marinacci (1999), and has been axiomatized within multiple priors recursive utility by Epstein and Schneider (2001), and in its continuous-time version by Chen and Epstein (2000), extending the atemporal preferences in Gilboa and Schmeidler (1989). In these papers, ambiguity is modelled explicitly through the definition of an appropriate (rectangular) set of density generators. The implied preferences are dynamically consistent in the multiple priors setting, a property that is essentially equivalent to updating beliefs using a Bayes rule applied prior by prior.

A related approach pursued first by Hansen, Sargent and Tallarini (1999) and AHS (1998, 2000) to modelling aversion to model misspecification has its roots in robust control theory, and characterizes a preference for robustness through the solution of a dynamic stochastic game between nature - selecting a worst case model from the set of relevant model misspecifications - and economic agents, trying to develop optimal consumption and portfolio rules that take the malevolent actions of the first player into account. Several formulations of a preference for ro-

bustness have been proposed recently in the literature. We will use the terminologies "Minimum Entropy Robustness" (MER, AHS (2000)), "Constrained Robustness" (CR, AHS (1998) , Hansen et al. (2001)) and "Homothetic Robustness" (HR, Maenhout (1999)) to distinguish the different definitions.

This paper studies the general equilibrium effects that a preference for robustness in the form of HR, MER and CR, respectively, generates. For some examples of a continuous time, heterogenous agents economy (cf. Dumas (1989), Wang (1996) and Kogan and Uppal (2000)), we compute the corresponding general equilibria using first order perturbative approximations that characterize analytically the effects of robustness on the relevant equilibrium quantities. Based on these results, we analyze and compare the equilibrium effects implied by each single robustness specification within heterogenous agents economies with state dependent opportunity sets.

Several findings arise from our perturbative analysis. First, we observe that to first order all robustness definitions influence directly only optimal investment in risky assets via an higher effective risk aversion that produces conservative portfolio choices. As a consequence, equilibrium optimum consumption, capital stock dynamics and stock price processes are not directly affected by a preference for robustness. By contrast, equilibrium risk premia are directly affected and tend to be higher because of lower equilibrium interest rates. This last statement holds in our examples of exchange economies when the intertemporal hedging motive induced by a preference for robustness does not dominate the corresponding speculative demand.

Second, robustness behaves differently from risk aversion also in its indirect effect on the implied equilibrium variables. Moreover, its equilibrium effect depends significantly on the robustness definition used, and specifically on if the implied effective risk aversion corrections reflect either standard second order risk aversion (this is the case for HR and MER) or a more non standard first order risk aversion with respect to some specific form of Knightian (1921) uncertainty (as in the CR specification). Indeed, the equilibria implied by CR yield functional forms for some key equilibrium variables like equity premia which are structurally different from those implied either by standard

risk aversion or by HR and MER. Moreover, in the CR case the basic structure of an equilibrium depends strongly on the completeness of the underlying economy. For instance, within complete production economies we observe that CR can generate an endogenous stock market participation which cannot be obtained neither by HR or MR generally, nor by CR in situations where the underlying economy is incomplete. Thus, the richness of the equilibrium effects generated in our heterogenous economies suggests that CR is the robustness definition which induces the largest variety of robust economic behaviours in the presence of model uncertainty.

All approaches to robustness in this paper are based on the idea that economic agents have an approximate reference model in mind by which they try to describe the probabilistic features of some underlying state variables processes. At the same time, agents are concerned with the possibility that the benchmark model could be misspecified. However, not all possible misspecifications are treated as being equally relevant. Indeed, model deviations that are viewed as particularly different from the given reference model are penalized in their impact on the final decision. This happens basically by constraining nature in its choice of the corresponding worst case model by which economic agents can be damaged. The magnitude of penalization is parameterized by a parameter that is interpreted as the strength of a preference for robustness (in MER and HR models), respectively as the intensity of an aversion to a particular form of Knightian uncertainty (in CR models).

On a formal level, differences between the three above formulations of a preference for robustness arise through the specific way by which they penalize model deviations in optimal decision making. MER and HR penalize deviations proportionally to their relative entropy with respect to the reference model, where HR does it in a way that is scaled by the current level of indirect utility and that imposes homogeneity of the implied Hamilton-Jacobi-Bellman (HJB) equations<sup>1</sup>.

On the other hand, CR constrains the set of relevant model deviations by a maximal bound on the

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<sup>1</sup> Both these definitions are observationally equivalent to stochastic differential utility and are not recursive in the sense of Chen and Epstein (2000). As discussed by Hansen et al. (2001), MER is recursive in a non-standard sense if a Bellman-Isaacs condition is satisfied. Cf. also Fleming and Souganidis (1989).

relative entropy of a relevant candidate misspecification<sup>2</sup>. At variance with MER and HR, CR is not observationally equivalent to stochastic differential utility. It can be interpreted as ambiguity aversion in a multiple priors model à la Chen and Epstein (2000); see also Lei (2001a).

Economically, the differences between the preferences implied by MER, HR and CR are important because - as mentioned - the first two definitions generate effects that mimic second order risk aversion, while the latter induces first order risk aversion. Hence, CR models a specific type of aversion to Knightian uncertainty (cf. Knight (1921) and Epstein and Wang (1994)) that induces a different portfolio behaviour than MER or HR, especially at small risks<sup>3</sup>. These basic features are deduced directly from a pure partial equilibrium analysis of the optimal policies implied by MER, HR and CR, respectively. However, the economic implications of these models of a preference for robustness are understood more deeply looking also at the general equilibrium implications of the different robustness specifications. This consideration defines a major goal of this paper.

From the perspective of analytical tractability also, important differences between the different definitions arise. Indeed, for MER no closed form solution for the implied optimal rules has been derived yet, already for the simplest partial equilibrium, constant opportunity set, Merton's (1969, 1971) model. Further, CR is similarly analytically untractable. Indeed, so far only the constant opportunity set model has been solved in closed form (see Trojani and Vanini (2001a)). Finally, HR is the most tractable one, a feature that is not surprising because of the homogeneity imposed on the implied HJB equation. For instance, some robust portfolio partial equilibrium problems with stochastic opportunity set and utility over terminal wealth have been solved explicitly in Maenhout (1999). For these models, the impact of a robust motive for intertemporal hedging has been analyzed. However, also for HR general equilibrium characterizations in economies with state dependent opportunity sets are failing.

Summarizing, dynamic models of a preference for robustness are thus even less analytically

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<sup>2</sup> As enlightened by Hansen et. al. (2001), MER and CR are closely related by the Lagrange Multiplier Theorem, even if they induce different preference orders.

<sup>3</sup> Cf. also Dow and Werlang (1992) for a general discussion on first and second order risk aversion.

tractable than their standard expected utility-based counterparts. Exact optimal policies are typically not computable in closed form and exact analytical characterizations of the implied equilibria are generally impossible. However, both from a theoretical and an applied point of view characterizing the structure of equilibria where preferences for robustness are present is an important issue in financial economics.

The best that can be done when exact analytical solutions cannot be obtained is to rely on approximation methods by which approximate analytical expressions can be achieved. As for the natural sciences, it has been shown recently in Kogan and Uppal (2000) within the setting of standard (non robust) Merton's (1969, 1971)-type models, that perturbation theory is a powerful approximation method for financial optimal decision making also. Indeed, it turns out that perturbative approaches allow for a high generality of the analysis, permitting rich investment set specifications and admitting a quite large spectrum of portfolio constraints. Moreover, they can be used to solve general equilibria in heterogeneous agents economies where the opportunity set is endogenous rather than exogenously given. This last feature is clearly crucial for the main objective addressed in this work.

Focusing on economies with heterogeneities in risk aversions and preferences for robustness, we therefore adopt a perturbative approach that permits us to analyze and to compare analytically the general equilibria implied by MER, HR and CR, respectively. To achieve this goal, we perturb the general equilibria of a benchmark economy where homogeneous log utility agents have no preference for robustness. The perturbation parameters are the risk aversion and the robustness parameters in our economies.

Some authors have dealt with the existence and the characterization of heterogeneous agents equilibria in continuous-time standard (non robust) economies (see for instance Duffie and Huang (1985), Duffie and Zame (1989), Duffie et al. (1994), Dumas (1989), Karatzas et al. (1990) and Wang (1996)). However, only a few of them have derived quantitative or qualitative predictions for the relevant equilibrium entities in a continuous time setting. Moreover, when quantitative

predictions have been derived either they were computed using numerical methods (Dumas (1989)) or they were obtained in closed form only for particular values of the model parameters (Wang (1996)). Using perturbation theory, Kogan and Uppal (2000) have been able to approximate analytically the general equilibria of some extended non robust production and exchange economies of the type in Dumas (1989) and Wang (1996), where in addition incomplete markets and borrowing constraints are allowed for.

Previous work on continuous-time robust general equilibria was limited to representative agents economies with constant opportunity set motivating higher risk premia because of a concern for model misspecification (cf. again AHS (2000), Maenhout (1999), Trojani and Vanini (2001a)). However, while reinterpreting the robust representative agents results to a multiple agents setting can be straightforward in some circumstances (cf. Tallarini (1998)), in general multi agents versions of this environment are fundamentally different from their single agent counterpart (cf. Anderson (1998) and Liu (1998)). As a consequence, analyzing robust equilibria in heterogeneous economies with stochastic opportunity sets is an important task in order to understand the intrinsic role of robustness in determining the price of risky assets within dynamic economies.

In a paper related to our one, Epstein and Miao (2001) describe in closed form using a martingale approach the general equilibria of a complete heterogeneous agents economy with constant opportunity set and based on multiple priors recursive utility. The focus of that paper is on heterogeneities in aversions to (model) ambiguity and their relation to the "home bias" puzzle. Heterogeneities in risk aversions are neglected by assuming homogeneous agents with a logarithmic utility function of current consumption. On the other hand, in this paper we focus from a general perspective on heterogeneities in risk aversions and preferences for robustness, with a special emphasis on (i) their interdependence in equilibrium and (ii) the comparison of the effects implied by the different robustness specifications. Further, using perturbation theory we are able to analyze analytically the robust equilibria of complete and incomplete economies based on quite general state dynamics. Finally, at variance with Epstein and Miao (2001) in this paper only first



order approximations for the relevant equilibrium quantities are obtained

The remainder of the paper is organized as follows. Section 2 defines the basic structure of the model analyzed and introduces the different definitions of a preference for robustness addressed in this paper. In Section 3, perturbative solutions to the implied partial equilibrium optimal policies are derived. These expansions are the necessary building blocks for the following general equilibrium analysis. Section 4 develops the desired asymptotics for the relevant variables in general equilibrium while Section 5 concludes the paper with some summarizing remarks and hints for further research.

## 2 Preferences for Robustness

This section introduces the basic structure of the economies under investigation. It formalizes the set of model misspecifications relevant to a robust decision maker and defines the optimization problems behind a preference for MER, HR and CR, respectively.

### 2.1 Opportunity Set and Risk Aversions

There are two assets, a risk free asset with price  $B_t$  at time  $t$  and a risky asset with price  $P_t$  at time  $t$  whose dynamics are given by

$$\begin{aligned} dB_t &= r_t B_t dt \quad , \\ dP_t &= \alpha_t P_t dt + \sigma_t P_t dZ_t^P \quad , \end{aligned} \tag{1}$$

for given initial prices  $B_0, P_0$ . The drift and volatility  $\alpha_t = \alpha(X_t)$  and  $\sigma_t = \sigma(X_t)$  as well as the short rate  $r_t = r(X_t)$  define a stochastic opportunity set adapted to a state variable  $X_t$  with dynamics

$$dX_t = \zeta(X_t) dt + \xi(X_t) dZ_t^X \quad . \tag{2}$$

$(Z_t^P)$  and  $(Z_t^X)$  are standard one dimensional Brownian motions with joint covariation  $\rho dt := E(dZ_t^X dZ_t^P)$ . We also write  $\zeta_t = \zeta(X_t)$ ,  $\xi_t = \xi(X_t)$ .

We consider agents with time preference rate  $\delta$  and power utility  $u(\cdot)$  of current consumption  $C_t$  given by

$$u(C) = \frac{C^\gamma - 1}{\gamma} \quad , \quad \gamma < 1 \quad .$$

For  $\gamma \rightarrow 0$  the log utility case follows. Agents allocate at each date  $t$  fractions  $w_t$  ( $1 - w_t$ ) of current individual wealth  $W_t$  to risky assets (riskless assets), yielding the current wealth dynamics

$$dW_t = [w_t W_t (\alpha_t - r_t) + (r_t W_t - C_t)] dt + w_t W_t \sigma_t dZ_t^P \quad . \quad (3)$$

Defining the covariation matrix  $\Sigma_t$  of  $(dX_t, dW_t)'$  by

$$\Sigma_t = \begin{bmatrix} \xi_t^2 & \rho \xi_t w_t W_t \sigma_t \\ \rho \xi_t w_t W_t \sigma_t & w_t^2 W_t^2 \sigma_t^2 \end{bmatrix} \quad ,$$

we have  $\Sigma_t = \Lambda_t \Lambda_t'$ , where

$$\Lambda_t = \begin{bmatrix} \xi_t & 0 \\ \rho w_t W_t \sigma_t & \sqrt{1 - \rho^2} w_t W_t \sigma_t \end{bmatrix} \quad ,$$

and  $\Lambda_t'$  is the transpose matrix of  $\Lambda_t$ . Therefore, for some suitable standard Brownian motion  $(Z_t^{X^\perp})$  orthogonal to  $(Z_t^X)$  it follows

$$Z_t^P = \rho Z_t^X + \sqrt{1 - \rho^2} Z_t^{X^\perp} \quad ,$$

and

$$dY_t = \mu_t dt + \Lambda_t dZ_t \quad , \quad (4)$$

where  $Y_t = (X_t, W_t)'$ ,  $\mu_t = (\zeta_t, w_t W_t (\alpha_t - r_t) + (r_t W_t - C_t))'$  and  $Z_t = (Z_t^X, Z_t^{X^\perp})'$  is a standard two-dimensional Brownian motion.

## 2.2 Preferences for Robustness and Uncertainty Aversion

We start analyzing partial equilibria in economies populated by robust heterogeneous agents. In the sequel we will refer to model (1), (2) (or equivalently model (4)) as the "reference model" of our robust agents.

For a positive random variable  $\nu$  with  $E(\nu) = 1$ , we denote by  $(\nu_t)_{t \geq 0}$  the martingale process defined by  $\nu_t := E(\nu | \mathcal{F}_t)$ , where the conditioning information set  $\mathcal{F}_t$  is generated by the current wealth and state variables dynamics up to time  $t$ . Since  $\nu_t$  is Markovian, we write  $\nu_t = \nu_t(W_t, X_t)$ . Any process  $(\nu_t)_{t \geq 0}$  represents a change of measure density from the initial reference probability to a contaminated one. The probability measure thereby induced will represent a potential model misspecifications relevant to our agents. When varying  $\nu$ , a whole spectrum of absolutely continuous model misspecifications is obtained.

Using the Cameron-Martin-Girsanov formula,  $\nu_t$  can be written as

$$\nu_t = \exp \left( - \int_0^t h_s \cdot dZ_s - \frac{1}{2} \int_0^t |h_s|^2 ds \right) ,$$

for a suitable process  $(h_s) = (h_s^X, h_s^P)'$ . For a model contamination induced by a change of density process  $(\nu_t)_{t \geq 0}$ , it then follows under the given reference model probability

$$dX_t = [\zeta_t + \xi_t h_t^X] dt + \xi_t dZ_t^X , \quad (5)$$

$$\begin{aligned} dW_t = & \left[ w_t W_t \left( \alpha_t - r_t + \sigma_t \left( \rho h_t^X + \sqrt{1 - \rho^2} h_t^P \right) \right) + (r_t W_t - C_t) \right] dt \\ & + w_t W_t \sigma_t \left( \rho dZ_t^X + \sqrt{1 - \rho^2} dZ_t^{X^\perp} \right) . \end{aligned} \quad (6)$$

Hence, the candidate dynamic misspecifications in the present setting are misspecifications of the drift terms in (1), (2). We next introduce the two-agents max-min optimization problems that define preferences for MER, HR, and CR, respectively.

- Minimum Entropy Robustness

The value function  $J^{MER}$  of the two-agents MER control problem is given by

$$J^{MER}(W, X) = \begin{cases} \sup_{C, w} \inf_h E^h \left[ \int_0^\infty e^{-\delta t} \left( \frac{C_t^\gamma - 1}{\gamma} + \frac{1}{2\vartheta} |h_t|^2 \right) dt \right] \\ \text{s.t. (4)} \end{cases} . \quad (7)$$

- Homothetic Robustness

The value function  $J^{HR}$  of the two-agents HR control problem is given by

$$J^{HR}(W, X) = \begin{cases} \sup_{C, w} \inf_h E^h \left[ \int_0^\infty e^{-\delta t} \left( \frac{C_t^\gamma - 1}{\gamma} + \frac{\gamma J^{HR}(W_t, X_t) + \frac{1}{\delta}}{2\vartheta} \cdot |h_t|^2 \right) dt \right] \\ \text{s.t. (4)} \end{cases} . \quad (8)$$

- Constrained Robustness

The value function  $J^{CR}$  of the two-agents CR control problem is given by

$$J^{CR}(W, X) = \begin{cases} \sup_{C, w} \inf_h E^h \left[ \int_0^\infty e^{-\delta t} \frac{C_t^\gamma - 1}{\gamma} dt \right] \\ \text{s.t. (4) and } \frac{1}{2} h' h \leq \eta \end{cases} . \quad (9)$$

Notice that

$$I(\nu) = E^h \left[ \int_0^\infty e^{-\delta t} \frac{|h_t|^2}{2} dt \right]$$

is a measure of relative entropy of the contaminated probability induced by  $\nu$ , relatively to the one induced by the reference model dynamics (cf. for instance Hansen et al. (2001)). Hence, in the max-min optimizations (7) and (8) model deviations are penalized in a way that is conditionally proportional to the entropy of the model misspecification under scrutiny. For MER this penalty is scaled by a constant parameter  $\vartheta$ . The higher  $\vartheta$ , the stronger the preference for robustness, respectively the higher the aversion to model misspecification. Indeed, for  $\vartheta \rightarrow \infty$  the solution of the infimization in (7) is a "worst case model" ( $\nu^{wc}$  say) yielding the lowest conditional expectation on future indirect utility  $J$  over all possible absolutely continuous contaminations of the given reference model. On the other hand,  $\vartheta \rightarrow 0$  yields a worst case model with lowest possible relative entropy, that is a model with transition densities that are equal to that of the given reference model. Hence, (7) covers the expected utility case which arises in the limit case  $\vartheta \rightarrow 0$ . On the other hand, HR uses a state dependent scaling factor

$$\frac{1}{\vartheta} \left( \gamma J^{HR}(W_t, X_t) + \frac{1}{\delta} \right) , \quad \vartheta > 0 . \quad (10)$$

This formulation of a preference for robustness imposes homotheticity on the value function arising in the corresponding optimal consumption-investment problem. Here, a preference for robustness

is state-dependent since it is inversely related to the current level of lifetime indirect utility in the given state of the world. Conditionally on the realized state, the interpretation of the parameter  $\vartheta$  is, on a pure formal level, the same as for the non-homothetic case above. Finally, CR introduces a preference for robustness through a maximal bound  $\eta$  on the rate at which relative entropy of a contaminated model is allowed to increase over time<sup>4</sup>.

### 3 Perturbed Robust Consumption/Portfolio Rules

In this section the HJB equations implied by the robust optimal control problems of the last section are introduced and their perturbative (optimal consumption-investment) solutions are presented. The results in this section are the starting point for our general equilibrium analysis in Section 4.

#### 3.1 HJB Equations and Perturbed Robust Controls

We state the HJB equations for each problem in Section 2.2 and provide the perturbative solutions for the desired optimal policies. Since in our general equilibrium analysis below we will have to focus on perturbations of an homogeneous agents economy with log utility investors, in this section perturbations are around the non-robust log-utility investor's problem. Therefore, to derive approximate solutions for a risk averse investor having a preference for robustness, we do perturbations with respect to two parameters: the risk aversion and the robustness parameter. We start with the HR case, since it is the simplest one to handle.

##### 3.1.1 HR: Perturbative Partial Equilibrium Optimal Policies.

The minimization with respect to  $h$  in (8) gives (cf. Maenhout (1999))

$$h^{HR} = -\frac{\vartheta}{\gamma J + \frac{1}{\delta}} \Lambda' \left( \frac{\partial J}{\partial Y} \right) = -\frac{\vartheta}{\gamma J + \frac{1}{\delta}} \left( \begin{array}{c} \xi J_X + \rho w W \sigma J_W \\ \sqrt{1 - \rho^2} w W \sigma J_W \end{array} \right) . \quad (11)$$

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<sup>4</sup> Indeed:

$$E^h \left[ \int_{t+\Delta t}^{\infty} \frac{|h_s|^2}{2} ds \right] - E^h \left[ \int_t^{\infty} \frac{|h_s|^2}{2} ds \right] = E^h \left[ \int_{t+\Delta t}^{\infty} \ln(\nu_s) ds \right] - E^h \left[ \int_t^{\infty} \ln(\nu_s) ds \right] .$$

Since the RHS of this equation is the variation in relative entropy over the period  $t, t + \Delta t$ , the claim follows.

The (single agent) HJB equation for the HR formulation then reads<sup>5</sup>,

$$\begin{aligned}
0 = & \sup_{c,w} \left\{ \frac{(cW)^\gamma - 1}{\gamma} - \delta J + (wW(\alpha - r) + (rW - cW)) J_W \right. \\
& + \frac{1}{2} w^2 W^2 \sigma^2 \left( J_{WW} - \frac{\vartheta}{\gamma J + \frac{1}{\delta}} \cdot J_W^2 \right) \\
& \left. + \zeta J_X + \frac{1}{2} \xi^2 \left( J_{XX} - \frac{\vartheta}{\gamma J + \frac{1}{\delta}} J_X^2 \right) + wW \sigma_{XP} \left( J_{XW} - \frac{\vartheta}{\gamma J + \frac{1}{\delta}} \cdot J_W J_X \right) \right\} \quad . \quad (12)
\end{aligned}$$

Homogeneity of  $J$  implies that the value function is of the functional form

$$J(W, X) = \frac{1}{\delta} \cdot \frac{(e^{g(X)} W)^\gamma - 1}{\gamma} \quad , \quad (13)$$

for some unknown function  $g(X)$ . Setting  $\sigma_{XP} = \rho \xi \sigma$ , for the optimal policies it follows

$$c = \frac{(J_W)^{\frac{1}{\gamma-1}}}{W} = \left( \frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{\gamma-1}} \quad (14)$$

$$\begin{aligned}
w = & - \frac{1}{1 - \frac{\vartheta}{\gamma J + \frac{1}{\delta}} \cdot \frac{J_W^2}{J_{WW}}} \cdot \frac{J_W}{W J_{WW}} \cdot \left( \frac{\alpha - r}{\sigma^2} + \frac{J_{WX}}{J_W} \frac{\sigma_{XP}}{\sigma^2} - \vartheta \frac{J_X}{\gamma J + \frac{1}{\delta}} \frac{\sigma_{XP}}{\sigma^2} \right) \\
= & \frac{1}{1 - (\gamma - \vartheta)} \cdot \left( \frac{\alpha - r}{\sigma^2} + (\gamma - \vartheta) \frac{\partial g}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2} \right) \quad . \quad (15)
\end{aligned}$$

Since the function  $g$  cannot in general be given in closed form, we approximate  $g$ . To achieve this goal, we expand  $g$  in  $(\gamma, \vartheta)$  to first order<sup>6</sup>

$$g = g_0 + \gamma g_1 + \vartheta g_2 + O^2(\gamma, \vartheta) \quad .$$

By construction  $g_0$  is the solution of (12) for  $\gamma, \vartheta \rightarrow 0$ , i.e. the solution of a standard log-utility problem. Writing  $J_{\log}(X, W)$  for this solution it follows from (13)

$$J_{\log}(X, W) = \frac{1}{\delta} (\ln(W) + g_0(X)) \quad . \quad (16)$$

Hence, for problems where  $J_{\log}$  can be computed explicitly,  $g_0$  is explicitly given. It turns out that  $g_0$  is all what we need to approximate the optimal rules in (14), (15), up to first order in  $(\gamma, \vartheta)$ .

To see this, we expand (14), (15), up to first order in  $(\gamma, \vartheta)$  yielding the next Proposition.

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<sup>5</sup> We use subscripts to denote partial derivatives with respect to the relevant argument.

<sup>6</sup> Hereafter  $O^2(\gamma, \vartheta)$  is a symbol that we use to denote terms of second order in  $(\gamma, \vartheta)$ .

**Proposition 1** The first order asymptotics for the optimal policies (14), (15) are

$$c(X) = \delta(1 - \gamma(g_0(X) - \ln(\delta)))W + O^2(\gamma, \vartheta) \quad , \quad (17)$$

$$w(X) = \frac{1}{1 - (\gamma - \vartheta)} \left( \frac{\alpha - r}{\sigma^2} + (\gamma - \vartheta) \frac{\partial g_0(X)}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2} \right) + O^2(\gamma, \vartheta) \quad . \quad (18)$$

A few remarks on Proposition 1 are appropriate. First, optimum consumptions is not altered by a preference for robustness, while optimal investment corresponds (to first order) to the policy of a standard expected utility investor with an enhanced risk aversion  $1 - (\gamma - \vartheta)$ . This is consistent with the finding that introducing HR is observationally equivalent to modifying a stochastic differential utility model by increasing "effective" risk aversion to  $1 - (\gamma - \vartheta)$  and leaving preferences for intertemporal substitution unchanged to  $\frac{1}{1-\gamma}$  (Maenhout (1999)). Robustness affects both the myopic and the hedging investment demand in the same way, by enhancing effective risk aversion

$$w = \underbrace{\frac{1}{1 - (\gamma - \vartheta)} \cdot \frac{\alpha - r}{\sigma^2}}_{\text{Myopic demand}} + \underbrace{\frac{\gamma - \vartheta}{1 - (\gamma - \vartheta)} \frac{\partial g_0}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2}}_{\text{Hedging demand}} + O^2(\gamma, \vartheta) \quad . \quad (19)$$

The hedging term in (19) can be positive or negative, depending on the signs of  $\gamma - \vartheta$  (a pure investor's characteristic), of  $\sigma_{XP}$  (a pure model characteristic) and the one of  $\frac{\partial g_0}{\partial X}$  (which is determined both by (logarithmic) investors' preferences and model structure). Finally, we remark that for  $\gamma = 0$  (a log-utility investor) robustness induces a non-standard demand for intertemporal hedging.

The impact of HR on myopic risky portfolios and on intertemporal hedging depends on the two "risk factors"  $\phi = \frac{\alpha - r}{\sigma}$  and  $\psi = \xi \frac{\partial g_0}{\partial X}$ . Indeed, with this reparameterization the myopic robust demand  $w^M(\vartheta)$  for risky assets is

$$w^M(\vartheta) = \frac{\phi}{\sigma(1 - \gamma)} \cdot \frac{1}{1 + \frac{\vartheta}{1 - \gamma}} = w^M(0) \frac{1}{1 + \frac{\vartheta}{1 - \gamma}} \quad , \quad (20)$$

while the robust demand for intertemporal hedging  $w^H(\vartheta)$  is

$$w^H(\vartheta) = \frac{\rho\psi\gamma}{\sigma(1 - \gamma)} \cdot \frac{1 - \frac{\vartheta}{\gamma}}{1 + \frac{\vartheta}{1 - \gamma}} = w^H(0) \frac{1 - \frac{\vartheta}{\gamma}}{1 + \frac{\vartheta}{1 - \gamma}} \quad . \quad (21)$$

Therefore, HR always reduces myopic risky allocations. The impact on intertemporal hedging depends on the value and the sign of the risk aversion term  $\gamma$ . For instance, for  $\gamma > 0$  robustness reduces the demand for intertemporal hedging. Moreover, the relative correction for robustness is state independent and the impact on the optimal investment policy is linear in the "risk factors"  $(\phi, \psi)$ . This linearity implies that the largest numerical corrections for HR arise when the standard optimal portfolios are highly exposed to  $\phi$  (in their myopic part) and to  $\psi$  (in their intertemporal hedging part). On the other hand, for vanishing  $\phi$  and  $\psi$ , the portfolio correction for HR disappears. Thus, HR induces effective risk aversion corrections that generate second order risk aversion effects.

Using perturbation theory, analytical expressions for the drift perturbation (11) easily follow. Indeed, expanding (11) up to first order in  $\gamma, \vartheta$ , it turns out that again only  $g_0$  is needed to describe the worst case model misspecification implied by a robustness parameter  $\vartheta$  and a risk aversion term  $\gamma$ .

**Proposition 2** The worst case drift (11) implied by a preference for HR is given by

$$h^{HR} = -\vartheta \begin{pmatrix} 1 & \frac{\rho\sigma}{\sqrt{1-\rho^2}} \\ 0 & \frac{\alpha-r}{\sigma} \end{pmatrix} \begin{pmatrix} \xi \frac{\partial g_0}{\partial X} \\ \frac{\alpha-r}{\sigma} \end{pmatrix} + O^2(\gamma, \vartheta) \quad . \quad (22)$$

Therefore,  $h^{HR}$  is a linear function in  $(\phi, \psi)$ . This result is natural, when considering that robust optimal portfolios are linearly exposed to the synthetic state variables  $\phi, \psi$ . Drift perturbations of the wealth dynamics are proportional to  $\frac{\alpha-r}{\sigma}$ , with a proportionality factor given by  $-\vartheta\sqrt{1-\rho^2}$ . Perturbations of the state variable dynamics show a dependence on  $\frac{\alpha-r}{\sigma}$  that is proportional to correlations between state variables and risky assets dynamics and to risky asset's volatility. Further, the extra term  $\xi \frac{\partial g_0}{\partial X}$  represents the contribution of the Brownian motion  $(Z_t^X)$  to the worst case drift in the state variables dynamics. As expected, in the complete markets case ( $\rho = \pm 1$ ), perturbations are completely described by a modification of the state variables drift only. For  $\rho = 0$  (orthogonal price and state variables dynamics) perturbations of the state variable dynamics modify the initial drift only by a multiple of  $\xi \frac{\partial g_0}{\partial X}$ .



### 3.1.2 MER: Perturbative Partial Equilibrium Optimal Policies

The minimization with respect to  $h$  in (8) gives (cf. also AHS (2000))

$$h^{MER} = -\vartheta \Lambda' \left( \frac{\partial J}{\partial Y} \right) = -\vartheta \left( \frac{\xi J_X + \rho w W \sigma J_W}{\sqrt{1 - \rho^2 w W \sigma J_W}} \right) . \quad (23)$$

Therefore, the HJB equation for the MER case is

$$\begin{aligned} 0 = \sup_{c,w} & \left\{ \frac{(cW)^\gamma - 1}{\gamma} - \delta J + (wW(\alpha - r) + (rW - cW))J_W + \frac{1}{2}w^2W^2\sigma^2(J_{WW} - \vartheta J_W^2) \right. \\ & \left. + \zeta J_X + \frac{1}{2}\xi^2(J_{XX} - \vartheta J_X^2) + wW\sigma_{XP}(J_{XW} - \vartheta J_W J_X) \right\} , \end{aligned} \quad (24)$$

which implies the optimal rules

$$c = \frac{(J_W)^{\frac{1}{\gamma-1}}}{W} , \quad (25)$$

$$w = -\frac{1}{\left(1 - \vartheta \frac{J_W^2}{J_{WW}}\right)} \cdot \frac{J_W}{W J_{WW}} \cdot \left( \frac{\alpha - r}{\sigma^2} + \frac{J_{WX}}{J_W} \frac{\sigma_{XP}}{\sigma^2} - \vartheta J_X \frac{\sigma_{XP}}{\sigma^2} \right) . \quad (26)$$

At variance with the HR case, the HJB equation (24) is non homogeneous in current wealth  $W$ , making the functional form (13) an inadequate guess for the implied candidate solution. In the sequel we therefore adopt a direct perturbative approach on the HJB equation (24). Using results in Trojani and Vanini (2001b), a candidate functional form for a perturbative solution to (24) is given by

$$J(X, W) = \frac{1}{\delta} \left( \frac{(e^{g_0(X)} W)^\gamma - 1}{\gamma} + \vartheta g_1(\gamma, X) W^{2\gamma} \right) + O(\vartheta^2) , \quad (27)$$

where  $g_0(X)$  is implied by the log-case solution as in (16) and for some function  $g_1(\gamma, X)$ . This functional form solves (24) up to second order in  $\vartheta$  and satisfies the boundary condition that in the limit  $\gamma \rightarrow 0$  (27) is a  $\vartheta$ -first order expansion for the solution of a robust log utility agent<sup>7</sup>.

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<sup>7</sup> Indeed

$$\frac{1}{\delta} \left( \frac{(e^{g_0(X)} W)^\gamma - 1}{\gamma} + \vartheta g_1(\gamma, X) W^{2\gamma} \right) \xrightarrow{\gamma \rightarrow 0} \frac{1}{\delta} (\ln(W) + \vartheta g_1(0, X)) ,$$

where  $g_1(0, X)$  is defined by

$$\begin{aligned} J_{\log, \vartheta} &= \frac{1}{\delta} (\ln(W) + g(\vartheta, X)) , \\ g(\vartheta, X) &= g_0(X) + \vartheta g_1(0, X) + O(\vartheta^2) , \end{aligned}$$

with  $J_{\log, \vartheta}$  the solution to the robust MER problem of an agent with logarithmic utility of consumption.

Based on (27), it turns out that  $g_0$  is again all what is needed in order to approximate the optimal rules (25), (26), up to first order in  $\gamma, \vartheta$ . Indeed, inserting (27) in (25), (26), and computing the corresponding expansion the following result is obtained.

**Proposition 3** The asymptotic expansions for the optimal policies of a minimum entropy robust agent are

$$c(X) = \delta(1 - \gamma(g_0(X) - \ln(\delta))) + O^2(\gamma, \vartheta) \quad , \quad (28)$$

$$w(X, W) = \frac{1}{1 - (\gamma - \frac{\vartheta}{\delta})} \cdot \left( \frac{\alpha - r}{\sigma^2} + \left( \gamma - \frac{\vartheta}{\delta} \right) \frac{\partial g_0(X)}{\partial X} \frac{\sigma_{XP}}{\sigma^2} \right) + O^2(\gamma, \vartheta) \quad . \quad (29)$$

When comparing the optimal rules (28), (29) with those obtained in Proposition 1 for the HR case we remark that the consumption policies are identical to first order. Moreover, also the asymptotics for the optimal investment policy under MER has the same basic structure as the one obtained for HR. Indeed, even if MER affects optimal portfolios inversely to the time preference rate  $\delta$ , we note that for given  $\delta$ , it is always possible (by simple reparameterization) to write the first-order solution of a MER model as the solution of an HR model. In a similar vain, the asymptotics for the worst case drift implied by MER are given by a simple reparameterization of the corresponding HR result implied by Proposition 1

$$h^{MER} = -\frac{\vartheta}{\delta} \begin{pmatrix} 1 & \rho\sigma \\ 0 & \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} \xi \frac{\partial g_0}{\partial X} \\ \frac{\alpha - r}{\sigma} \end{pmatrix} + O^2(\gamma, \vartheta) \quad . \quad (30)$$

The similarity between the asymptotics obtained for HR and MER can surprise at first glance, since the inhomogeneity of the HJB equation (24) suggests that the implied optimal rules should show up some wealth inhomogeneity. However, it turns out that wealth inhomogeneities are at least of order two in  $\gamma$  and  $\vartheta$ . To study in detail the impact of wealth inhomogeneities caused by MER it is necessary to determine at least also the function  $g_1(\gamma, X)$  in (27). While this is a feasible task for some specific model (see for instance Trojani and Vanini (2001b)), we continue our analysis of robust equilibria based on first order approximations in  $\gamma$  and  $\vartheta$ . The advantage of this approach is to produce easily interpretable expressions for the relevant variables, which can also be easily compared across the different definitions of robustness used in this paper. However,

precisely for the inhomogenous MER case one should be aware that these approximations cannot be expected to approximate well the effects of a preference for MER when the risk aversion term  $\gamma$  largely deviates from 0.

### 3.1.3 CR: Perturbative Partial Equilibrium Optimal Policies

The minimization with respect to  $h$  in (9) gives (cf. also AHS (1998) and Trojani and Vanini (2001a))

$$\begin{aligned} h^{CR} &= -\frac{\sqrt{2\eta}}{\left(\frac{\partial J'}{\partial Y} \Lambda \Lambda' \frac{\partial J}{\partial Y}\right)^{\frac{1}{2}}} \Lambda' \frac{\partial J}{\partial Y} \\ &= -\frac{\sqrt{2\eta}}{\left(W^2 J_W^2 \sigma^2 + \xi^2 J_X^2 + 2\sigma_{XP} W J_X J_W\right)^{\frac{1}{2}}} \begin{pmatrix} \xi J_X + \rho w W \sigma J_W \\ \sqrt{1 - \rho^2} w W \sigma J_W \end{pmatrix}. \end{aligned} \quad (31)$$

Hence, the single agent HJB equation for the CR case is

$$\begin{aligned} 0 &= \sup_{c,w} \left\{ \frac{(cW)^\gamma - 1}{\gamma} - \delta J + (wW(\alpha - r) + (rW - cW)) J_W + \frac{1}{2} w^2 W^2 \sigma^2 J_{WW} \right. \\ &\quad \left. + \zeta J_X + \frac{1}{2} \xi^2 J_{XX} + wW \sigma_{XP} J_{XW} - \sqrt{2\eta} [w^2 W^2 \sigma^2 J_W^2 + \xi^2 J_X^2 + 2wW \sigma_{XP} J_W J_X]^{\frac{1}{2}} \right\}. \end{aligned} \quad (32)$$

Homogeneity of  $J$  implies an homogeneous functional form of the type (13) also for the candidate solution under CR. Hence, the implied optimal policies are given by

$$c = \frac{(J_W)^{\frac{1}{\gamma-1}}}{W} = \left( \frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{\gamma-1}}, \quad (33)$$

$$w = \frac{1}{1 - \sqrt{\frac{2\eta}{\Gamma(w)} \frac{J_W^2}{J_{WW}}}} \cdot \frac{1}{1 - \gamma} \left( \frac{\alpha - r}{\sigma^2} + \gamma \frac{\partial g}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2} - \sqrt{\frac{2\eta}{\Gamma(w)}} J_X \frac{\sigma_{XP}}{\sigma^2} \right), \quad (34)$$

where

$$\begin{aligned} \Gamma(w) &= W^2 J_W^2 \left( \sigma^2 w^2 + \frac{\xi^2 J_X^2}{W^2 J_W^2} + 2w \sigma_{XP} \frac{J_X}{W J_W} \right) \\ &= W^2 J_W^2 \left( \sigma^2 w^2 + \xi^2 \left( \frac{\partial g}{\partial X} \right)^2 + 2w \sigma_{XP} \frac{\partial g}{\partial X} \right) \\ &= W^2 J_W^2 G(w), \end{aligned}$$

with  $G$  defined accordingly. Using the last expression for  $\Gamma$ , the optimal policy (34) reads more compactly

$$w = \frac{1}{1 - \left(\gamma - \sqrt{\frac{2\eta}{G(w)}}\right)} \cdot \left( \frac{\alpha - r}{\sigma^2} + \left( \gamma - \sqrt{\frac{2\eta}{G(w)}} \right) \frac{\partial g}{\partial X} \frac{\sigma_{XP}}{\sigma^2} \right) . \quad (35)$$

We remark, that by contrast with HR and MER, the optimal investment policy of a CR investor is given by the solution of an implicit equation in  $w$ . In order to expand (33), (35), up to first order, we write

$$w(X) = \frac{\alpha - r}{\sigma^2} + \gamma w_1(X) + \sqrt{2\eta} w_2(X) + O^2(\gamma, \sqrt{\eta}) , \quad (36)$$

$$g(X) = g_0(X) + \gamma g_1(X) + \sqrt{2\eta} g_2(X) + O^2(\gamma, \sqrt{\eta}) . \quad (37)$$

Expanding (33), (35), while making use of (36) and (37), it turns out that again  $g_0$  is all what is needed in order to approximate the optimal policies of an agent with CR preferences for robustness up to first order. The next result summarizes these findings.

**Proposition 4** The asymptotic expansions for the optimal policies of a constrained robust agent are

$$c(X) = \delta (1 - \gamma (g_0(X) - \ln(\delta))) + O^2(\gamma, \sqrt{\eta}) , \quad (38)$$

$$w(X) = \frac{1}{1 - \left(\gamma - \sqrt{\frac{2\eta}{G_0(X)}}\right)} \left( \frac{\alpha - r}{\sigma^2} + \left( \gamma - \sqrt{\frac{2\eta}{G_0(X)}} \right) \frac{\partial g_0}{\partial X} \frac{\sigma_{XP}}{\sigma^2} \right) , \quad (39)$$

where

$$G_0(X) = \left( \frac{\alpha - r}{\sigma} \right)^2 + \left( \xi \frac{\partial g_0}{\partial X} \right)^2 + 2 \left( \frac{\alpha - r}{\sigma} \right) \rho \xi \frac{\partial g_0}{\partial X} .$$

It follows that to first order optimum consumption is not affected directly by a preference for CR. Conversely, on the optimal investment side, robustness influences as usual both the myopic and the hedging demand for risky assets. However, corrections for robustness are now state dependent through the function  $G_0(X)$ . Specifically, we see that the risky allocation in (39) can be interpreted as the portfolio strategy that would be pursued by an investor with a state dependent effective risk aversion  $1 - \left(\gamma - \sqrt{\frac{2\eta}{G_0}}\right)$ , where the term  $\sqrt{\frac{2\eta}{G_0}}$  penalizes states where model uncertainty can cause particularly important damages.

The state dependent effective risk aversion correction depends on the state  $X$  only through the "risk factors"  $\phi = \frac{\alpha-r}{\sigma}$  and  $\psi = \xi \frac{\partial g_0}{\partial X}$  in  $G_0$ . The myopic robust portfolio demand  $w^M(\eta)$  implied by a CR parameter  $\eta$  reads

$$w^M(\eta) = w^M(0) \frac{1}{1 + \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0}}} \quad . \quad (40)$$

Similarly, the robust demand  $w^H(\eta)$  for intertemporal hedging is given by

$$w^H(\eta) = w^H(0) \frac{1 - \frac{1}{\gamma} \sqrt{\frac{2\eta}{G_0}}}{1 + \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0}}} \quad . \quad (41)$$

From these expressions we see that CR reduces the myopic demand for risky assets, as it was the case for HR and MER. The qualitative impact on intertemporal hedging depends on the sign of  $\gamma$  in the same way as it depends for HR and MER. However, the numerical impact on both myopic and hedging portfolios is now state dependent in a non linear way through the function  $G_0(X)$ . It follows that the relative impact of a preference for CR as a function of the underlying state vector  $(\phi, \psi)$  is nonlinear and bounded. Indeed, at variance with HR and MER we notice that by (40), (41) the largest relative portfolio corrections because of CR are realized when  $\phi, \psi \rightarrow 0$ , in a neighborhood of the origin in  $(\phi, \psi)$ -space. Hence, CR affects optimal portfolios significantly, precisely when the standard myopic and intertemporal demands for risky assets are small, that is when the standard risk exposure of a corresponding non robust agent is low. This is a first order risk aversion effect.

An interesting and easily interpretable special case of Proposition 4 arises in the complete markets case  $\rho = \pm 1$ . In this case  $G_0$  is a perfect square and  $\sqrt{G_0}$  collapses to an absolute value expression, leading to the next immediate consequence of Proposition 4.

**Corollary 5** If markets are complete ( $\rho = \pm 1$ ), the asymptotics of the optimal investment policy are:

1. If  $\frac{\alpha-r}{\sigma} \pm \frac{\partial g_0}{\partial X} \xi \geq 0$  and  $\frac{\alpha-r}{\sigma} \pm \gamma \frac{\partial g_0}{\partial X} \xi - \sqrt{2\eta} > 0$

$$w(X) = \frac{1}{\sigma^2(1-\gamma)} \left( \alpha - r \pm \gamma \frac{\partial g_0}{\partial X} \xi - \sqrt{2\eta} \sigma \right) + O^2(\gamma, \sqrt{\eta}) \quad . \quad (42)$$

2. If  $\frac{\alpha-r}{\sigma} \pm \frac{\partial g_0}{\partial X} \xi < 0$  and  $\frac{\alpha-r}{\sigma} \pm \gamma \frac{\partial g_0}{\partial X} \xi + \sqrt{2\eta} < 0$

$$w(X) = \frac{1}{\sigma^2(1-\gamma)} \left( \alpha - r \pm \gamma \frac{\partial g_0}{\partial X} \sigma \xi + \sqrt{2\eta} \sigma \right) + O^2(\gamma, \sqrt{\eta}) \quad . \quad (43)$$

3. If neither 1. nor 2. are satisfied  $w(X) = O^2(\gamma, \sqrt{\eta})$ .

In the complete financial markets case, the impact of CR on optimal portfolio choice is particularly easy to interpret. Indeed, we can think of the implied optimal investment rules as those of a standard optimal portfolio where a precautionary investor has excess returns  $\alpha - r$  modified by a factor  $\sqrt{2\eta}\sigma$  which penalizes uncertainty about the underlying reference model for asset prices. Compared to the incomplete markets case, the portfolio correction induced by CR now only concerns myopic optimal portfolio allocations. For instance the simplest "constant opportunity set" case in Trojani and Vanini (2001a) yields the optimal policy (setting  $\frac{\partial g_0}{\partial X} = 0$ )

$$w = \frac{1}{\sigma^2(1-\gamma)} \left[ \alpha - r - \text{sign}(\alpha - r) \sqrt{2\eta} \sigma \right] \quad .$$

Hence, in this case the rule is simply to invest in risky assets when a correction for model uncertainty yields a precautionary risk premium that points in the same direction as the risk premium implied by the given reference model.

If we finally compute the worst case drift (31) implied by CR, the next result follows.

**Proposition 6** The worst case drift implied by a preference for CR is given by

$$h^{CR} = - \frac{\sqrt{2\eta}}{\left[ \left( \frac{\alpha-r}{\sigma} \right)^2 + \left( \xi \frac{\partial g_0}{\partial X} \right)^2 + 2 \left( \frac{\alpha-r}{\sigma} \right) \rho \xi \frac{\partial g_0}{\partial X} \right]^{\frac{1}{2}}} \begin{pmatrix} 1 & \rho \sigma \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \xi \frac{\partial g_0}{\partial X} \\ \frac{\alpha-r}{\sigma} \end{pmatrix} + O^2(\gamma, \sqrt{\eta}) \quad (44)$$

Both drift distortions in the components of  $h^{CR}$  are given up to first order by the ratio of a linear and an hyperbolic function of  $\phi = \frac{\alpha-r}{\sigma}$  and  $\psi = \xi \frac{\partial g_0}{\partial X}$ . As a consequence, the largest drift distortions are obtained near to the origin in  $(\phi, \xi)$ -space, which corresponds to low standard (non robust) portfolio expositions. For  $\phi, \xi$  tending to infinity, the drift corrections converge to 0. This behaviour is consistent with the optimal portfolio rules in Proposition 4 and the discussion thereafter.

A more explicit discussion of the formulas derived in this section can be developed for models where  $g_0$  can be computed explicitly. An example is provided in the next section.

### 3.2 Some Explicit Partial Equilibrium Computations

We now shortly investigate a version of Kim and Omberg's (1996) model allowing for intermediate consumption. This allows us to analyze the structure of the robust optimal policies implied by HR, MER and CR, as well as the associated worst case perturbations in a fully analytical setting where the function  $g_0$  can be computed in closed form.

To achieve this goal, we specify price and state dynamics (1), (2) to

$$dB_t = rB_t dt, \quad (45)$$

$$dP_t = \alpha_t P_t dt + \sigma P_t dZ_t^P, \quad (46)$$

$$dX_t = \lambda (\bar{X} - X_t) dt + \xi dZ_t^X, \quad (47)$$

where  $r, \sigma, \xi, \lambda, \bar{X} > 0$ , and  $\alpha_t = r + \sigma X_t$ . The function  $g_0$  is then given by<sup>8</sup>

$$g_0 = a_0 + a_1 X + \frac{1}{2} a_2 X^2, \quad (48)$$

with

$$\begin{aligned} a_0 &= \ln(\delta) - 1 + \frac{r}{\delta} + \frac{\xi^2}{2\delta(\delta + 2\lambda)} + \frac{(\lambda\bar{X})^2}{\delta(\delta + \lambda)(\delta + 2\lambda)}, \\ a_1 &= \frac{\lambda\bar{X}}{(\delta + \lambda)(\delta + 2\lambda)} > 0, \\ a_2 &= \frac{1}{\delta + 2\lambda} > 0. \end{aligned}$$

We remark that (48) implies that the two relevant risk factors  $\phi = \frac{\alpha - r}{\sigma}$ ,  $\psi = \xi \frac{\partial g_0}{\partial X}$  are perfectly correlated. Using (48) we can compute the relevant quantities implied by preferences for robustness in the present setting. The next section presents the analytical expressions obtained for HR<sup>9</sup> and CR.

#### 3.2.1 Homothetic Robustness

The relevant analytical expressions implied by a preference for HR in model (45)-(47) are presented in the next proposition.

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<sup>8</sup> A proof is given in the Appendix for completeness.

<sup>9</sup> By reparameterization this gives a set of first order asymptotics for the MER case also.

**Proposition 7** In the model (45)-(47) for a robust agent with preferences for HR,

1. The optimal consumption and investment asymptotics are

$$c(X) = \delta \left( 1 - \gamma \left( a_0 + a_1 X + \frac{1}{2} a_2 X^2 - \ln(\delta) \right) \right) + O^2(\gamma, \vartheta) \quad (49)$$

$$w(X) = \frac{1}{1 - (\gamma - \vartheta)} \left( \frac{X}{\sigma} + (\gamma - \vartheta) \frac{\sigma_{XP}(a_1 + a_2 X)}{\sigma^2} \right) + O^2(\gamma, \vartheta) \quad (50)$$

2. The worst case drift perturbation is

$$h^{MER} = -\vartheta \left( \frac{\xi a_1 + (\xi a_2 + \rho \sigma) X}{\sqrt{1 - \rho^2} X} \right) + O^2(\gamma, \vartheta) \quad (51)$$

Hence, the optimal consumption/wealth ratio is a quadratic function of the underlying state variable. On the other hand, the optimal risky asset proportion is a linear function of  $X$ , a feature that is caused by the linearity of the drift (51) as a function of the current state level. Explicitly (20), (21) read

$$w^M(\vartheta) = \frac{X}{\sigma(1 - \gamma)} \cdot \frac{1}{1 + \frac{\vartheta}{1 - \gamma}} \quad , \quad w^H(\vartheta) = \frac{\rho \xi (a_1 + a_2 X) \gamma}{\sigma(1 - \gamma)} \cdot \frac{1 - \frac{\vartheta}{\gamma}}{1 + \frac{\vartheta}{1 - \gamma}} \quad (52)$$

As a consequence, we notice that HR modifies globally the profile of both  $w^M(\vartheta)$  and  $w^H(\vartheta)$  as linear functions of  $X$ , through the same multiplicative factor for all  $X$ .

### 3.2.2 Constrained Robustness

The relevant analytical expressions implied by a preference for CR in model (45)-(47) are given in the next result.

**Proposition 8** In the model (45)-(47) for a robust agent with preferences for CR,

1. The optimal consumption and investment asymptotics are

$$c(X) = \delta \left( 1 - \gamma \left( a_0 + a_1 X + \frac{1}{2} a_2 X^2 - \ln(\delta) \right) \right) + O^2(\gamma, \sqrt{\eta}) \quad (53)$$

$$w(X) = \frac{1}{1 - \left( \gamma - \sqrt{\frac{2\eta}{G_0}} \right)} \left( \frac{X}{\sigma} + \left( \gamma - \sqrt{\frac{2\eta}{G_0}} \right) \frac{\sigma_{XP}(a_1 + a_2 X)}{\sigma^2} \right) + O^2(\gamma, \sqrt{\eta}) \quad (54)$$

where

$$G_0 = (\xi^2 a_2^2 + 2\rho \xi a_2 + 1) X^2 + 2a_1 \xi (\rho + \xi a_2) X + \xi^2 a_1^2 \quad (55)$$

2. The worst case drift perturbation is

$$h^{CR} = -\sqrt{\frac{2\eta}{G_0}} \left( \frac{\xi a_1 + (\xi a_2 + \rho \sigma) X}{\sqrt{1 - \rho^2} X} \right) + O^2(\gamma, \sqrt{\eta}) \quad (56)$$



As expected, optimal consumption is identical to the one obtained for the HR case. On the other hand, optimal risky portfolios and worst case distortions in Proposition 8 differ from those under HR. Indeed, expressing (40) and (41), explicitly, it follows

$$w^M(\eta) = \frac{X}{\sigma(1-\gamma)} \cdot \frac{1}{1 + \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0}}} , \quad (57)$$

$$w^H(\eta) = \frac{\rho\xi(a_1 + a_2X)\gamma}{\sigma(1-\gamma)} \cdot \frac{1 - \frac{1}{\gamma} \sqrt{\frac{2\eta}{G_0}}}{1 + \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0}}} , \quad (58)$$

with  $G_0$  given in (55). Hence, the corrections in myopic optimal portfolio positions implied by CR are largest for  $G_0 \rightarrow 0$ , while for  $G_0 \rightarrow \infty$  they become negligible: For extreme  $X$  values, the effect of CR vanishes, while at moderate risk exposures - for  $X$  in the "middle" of its state space - the highest portfolio corrections are obtained. Similar arguments apply for the intertemporal hedging portfolio of a CR agent. In the same vain, the largest drift corrections in (56) arise "locally", for moderate risk exposures, rather than as for HR where large corrections arise in the tails of the stationary distribution of the underlying state variable.

Figure 1 compares the optimal policies and drift distortions implied by HR and CR for some particular choices of the model parameters.

Insert Figure 1 about here

## 4 Robust General Equilibria

We consider general equilibria in economies populated by agents with heterogenous preferences for HR and CR, who have the same reference model for asset prices. Even if in principle one could use perturbation theory to investigate differences in the given reference model too, we fix it in order to simplify the analysis to follow. Notice that because of the heterogeneity in the preferences for robustness, each agent in the economy will select a different worst case model when determining optimal consumption and portfolio rules, despite having the same reference model.

An important remark on the equilibrium setting used below is related to the question of why

heterogenous robust agents do have different model beliefs (that is different perceptions of the relevant model misspecifications around the reference model) despite observing a common price process. Formal continuous time models that describe learning and that allow for the presence of model ambiguity have not been largely developed yet<sup>10</sup>. However, the Law of Large Numbers result in Marinacci (1999) for beliefs represented by a set of priors, suggests that model ambiguity will not disappear even asymptotically when agents learn about the underlying data generating process. Indeed, in this setting the connection between empirical frequencies and asymptotic beliefs turns out to be weakened to a degree that depends on the extent of diversity in prior beliefs. Therefore, a model where agents have different beliefs and where they observe the same price process is compatible with general equilibrium<sup>11</sup>.

## 4.1 Homothetically Robust General Equilibria

In this section we discuss heterogeneous economies where there are two agents with utilities of current consumption given by

$$u(C) = \frac{C^\gamma - 1}{\gamma} \quad , \quad u\left(C^{(1)}\right) = \log\left(C^{(1)}\right) \quad , \quad \gamma < 1 \quad , \quad (59)$$

preferences for HR

$$\vartheta > 0 \quad , \quad \vartheta^{(1)} = 0 \quad , \quad (60)$$

and the same time preference rate  $\delta$ . In the sequel we analyze first an example of a production economy and in a second step an example of an exchange economy.

### 4.1.1 Production Economies

We consider a robust version of the complete production economy in Dumas (1989) and assume existence of a single constant returns to scale technology yielding the dynamics

$$dS_t = \left( \alpha S_t - C_t - C_t^{(1)} \right) dt + \sigma S_t dZ_t \quad ,$$

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<sup>10</sup> A few references on this topic are Cagetti et al. (2000), Epstein and Schneider (2001b) and Lei (2001b).

<sup>11</sup> Cf. also the discussion in Chen and Epstein (2000), Section 1.3.

for the aggregate capital stock, with  $\alpha, \sigma > 0$ . The risky asset is a stock on the production technology with cumulative return process

$$dP_t = \alpha P_t dt + \sigma P_t dZ_t \quad .$$

The number of shares in this economy is equal to the aggregate capital stock. The riskless asset is a short term bond with price dynamics

$$dB_t = r_t B_t dt \quad ,$$

where  $r_t$  is an interest rate that has to be determined in equilibrium.

**Definition 9** We call a process  $(S_t, r_t, w_t, w_t^{(1)}, c_t, c_t^{(1)})$  an homothetically robust equilibrium if:

- The individual portfolio and consumption rules  $w_t, w_t^{(1)}, c_t, c_t^{(1)}$  are optimal up to first order, i.e they satisfy (17) and (18),
- The financial markets clear

$$\frac{w_t W_t + w_t^{(1)} W_t^{(1)}}{W_t + W_t^{(1)}} = 1 \quad .$$

In the present general equilibrium setting the relevant state variable  $X$  is the cross-sectional wealth distribution  $\omega_t$  in the given economy

$$X_t = \omega_t := \frac{W_t}{W_t + W_t^{(1)}} \quad .$$

Furthermore, in general equilibrium the function  $g_0$  in (17) and (18) is now endogenous to the economy, i.e. it depends on  $\gamma$  and  $\vartheta$ . However, it can be further expanded in a neighbourhood of the representative agent value function solution of an homogeneous economy with log utility non-robust investors ( $\gamma = \gamma^{(1)} = \vartheta = \vartheta^{(1)} = 0$ ). Denoting this value function by

$$J_{\log,0}(X, W) = \frac{1}{\delta} (\log(W) + g_{00}(X)) \quad ,$$

it turns out that in order to characterize the desired rules completely we only need to determine  $r$  and  $g_{00}$ . Moreover, since  $g_{00}$  is completely determined by the value function of a representative log utility agent in the given production economy, it is a constant. For this benchmark economy

$\alpha - \sigma^2$  is the implied constant equilibrium interest rate. By formula (41) in Kogan and Uppal it follows<sup>12</sup>

$$g_0(X) = g_{0,0} + O(\gamma, \vartheta) = \log(\delta) - 1 + \frac{\alpha - \sigma^2/2}{\delta} + O(\gamma, \vartheta) \quad . \quad (61)$$

The implied robust equilibrium asymptotics are given in the next proposition.

**Proposition 10** In the given production economy we have:

1. Equilibrium interest rate asymptotics

$$r_t = \alpha - \sigma^2 + \sigma^2(\gamma - \vartheta)\omega_t + O^2(\gamma, \vartheta) \quad .$$

2. Optimum consumption and portfolio asymptotics

$$C_t^{(1)} = \delta W_t^{(1)} \quad , \quad C_t = (\delta - \gamma(\alpha - \sigma^2/2 - \delta))W_t + O^2(\gamma, \vartheta) \quad ,$$

$$w_t^{(1)} = 1 - (\gamma - \vartheta)\omega_t + O^2(\gamma, \vartheta) \quad , \quad w_t = 1 + (\gamma - \vartheta)(1 - \omega_t) + O^2(\gamma, \vartheta) \quad .$$

3. Asymptotics for cross-sectional wealth dynamics

$$d\omega_t = \gamma\omega_t(1 - \omega_t)(\alpha - \sigma^2/2 - \delta)dt + (\gamma - \vartheta)\omega_t(1 - \omega_t)\sigma dZ_t + O^2(\gamma, \vartheta) \quad .$$

4. Asymptotics for capital stock dynamics

$$dS_t = [\alpha - \delta + \gamma(\alpha - \sigma^2/2 - \delta)\omega_t]S_tdt + \sigma S_t dZ_t + O^2(\gamma, \vartheta) \quad .$$

The asymptotics for the optimal policies show a basic difference between the pure risk averse ( $\vartheta = 0$ ) and the robust ( $\vartheta > 0$ ) solutions. First of all, we see that while risk aversion affects directly all decision variables of the investor, the robustness parameter influences directly only optimal investment to risky assets. Hence, in equilibrium robustness affects optimum consumption only indirectly, through an altered conditional variance of the equilibrium process for cross-sectional wealth.

The marginal qualitative impact of HR on equilibrium portfolio holdings depends on whether  $\gamma > 0$  or  $\gamma < 0$ <sup>13</sup>. In the first case, HR reduces the incentive of the investor to borrow in order to invest in leveraged portfolios. This induces lower equilibrium interest rates through a lower equilibrium demand for leveraged positions in the the risky asset. In the second case,

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<sup>12</sup>  $O(\gamma, \vartheta)$  is a symbol that we use to denote terms of order  $O(\|(\gamma, \vartheta)\|)$ .

<sup>13</sup> Note that (as in Kogan and Uppal (2000)) no equilibrium intertemporal hedging position arises because the variance of the only relevant state variables to the investors in this economy (namely  $\omega_t$ ) is of order  $O^2(\gamma, \vartheta)$ .

HR enhances the incentive of a robust investor to lend in the riskless asset. This induces lower equilibrium interest rates through a higher equilibrium excess supply for the riskless asset. The arising equilibrium interest rate is between that of an heterogeneous standard economy where no preference for robustness arises and an heterogeneous robust economy with homogeneous log utility investors, i.e.

$$\alpha - \sigma^2 (1 + \vartheta \omega_t) \leq r_t \leq \alpha - \sigma^2 (1 - \gamma \omega_t) \quad .$$

Furthermore, the lower interest rate is supported by an equilibrium open interest

$$OI_t = \frac{1}{2} \left( |1 - w_t| \omega_t + |1 - w_t^{(1)}| (1 - \omega_t) \right) = |\gamma - \vartheta| \omega_t (1 - \omega_t) \quad ,$$

which is higher (lower) than in the absence of robustness when  $\gamma < 0$  ( $\gamma > 0$ ).

We notice that the drift

$$\gamma \omega_t (1 - \omega_t) \left( \alpha - \frac{\sigma^2}{2} - \delta \right) \quad ,$$

in the cross-sectional wealth dynamics is not affected by the preference for robustness<sup>14</sup> . This implies that for  $\gamma < 0$  ( $\gamma > 0$ ) the long run distribution of  $\omega_t$  is degenerated at  $\omega_t = 0$  or  $\omega_t = 1$ , so that either only the robust risk averse or the log utility agent is going to invest in the risky asset in that case. Hence, in this model HR does not avoid to first order a non stationary distribution of cross sectional wealth, as for the standard pure risk aversion case in Kogan and Uppal (2000). The conditional variance of  $\omega_t$

$$(\gamma - \vartheta)^2 (\omega_t (1 - \omega_t))^2 \sigma^2 \quad ,$$

is lower (higher) in the presence of robustness if  $\gamma > 0$  ( $\gamma < 0$ ). In particular, lower (higher) volatilities

$$\sigma^3 (\gamma - \vartheta)^2 (\omega_t (1 - \omega_t))$$

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<sup>14</sup> This is because the impact of the optimal portfolio policies on the drift of  $\omega_t$  is of order no less than two (see the proof in the Appendix).

of equilibrium interest rates then arise<sup>15</sup>. Therefore, the lower equilibrium interest rate can be supported both by higher and lower conditional volatilities in the presence of robustness, which however converge to zero under the corresponding long run distribution.

Finally, an important economic difference between the impact of HR and MER is that in the latter case equilibrium interest rates, optimal portfolios and the volatility of cross-sectional wealth depend on time preferences<sup>16</sup>.

#### 4.1.2 Exchange Economies

We consider a robust version of an incomplete heterogeneous exchange economy of the type discussed in Kogan and Uppal (2000) where the risky asset is a claim on the aggregate endowment process  $(e_t)$  with dynamics given by:

$$\begin{aligned} de_t &= Y_t e_t dt + \sigma_e e_t dZ_t^e, \\ dY_t &= \lambda (\bar{Y} - Y_t) dt + \sigma_Y dZ_t^Y, \end{aligned}$$

where  $\sigma_e, \lambda, \bar{Y}, \sigma_Y > 0$ , and  $(Z_t^e), (Z_t^Y)$  are both standard Brownian motions in  $\mathbb{R}$  with covariation  $E(dZ_t^e dZ_t^Y) = \delta_{eY} dt$ . Hence,  $(Y_t)$  is a mean reverting Ornstein Uhlenbeck process for expected endowment returns. The cumulative return process of the risky asset is given by

$$dP_t = \alpha_t P_t dt + \sigma_t P_t dZ_t, \quad ,$$

where  $\alpha_t, \sigma_t$ , are drift and diffusion parameters that have to be determined in equilibrium. Further, the riskless asset is a short term bond with price dynamics

$$dB_t = r_t B_t dt, \quad ,$$

where  $r_t$  is an interest rate that has to be determined in equilibrium.

**Definition 11** We call a process  $(P_t, r_t, w_t, w_t^{(1)}, c_t, c_t^{(1)})$  an homothetically robust equilibrium if

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<sup>15</sup> As in standard economies, the highest interest rate volatilities are observed when aggregate wealth is evenly distributed across agents.

<sup>16</sup> This follows by replacing the HR parameter  $\vartheta$  with the ratio  $\frac{\vartheta}{\delta}$  in the MER case.

- The individual portfolio and consumption rules  $w_t, w_t^{(1)}, c_t, c_t^{(1)}$  are optimal to first order, i.e they satisfy (17) and (18),
- The financial markets clear:

$$\frac{w_t W_t + w_t^{(1)} W_t^{(1)}}{W_t + W_t^{(1)}} = 1 \quad ,$$

- The good markets clear:

$$e_t = C_t + C_t^{(1)} \quad .$$

In the present general equilibrium setting the relevant state variables for investors' decisions are both expected endowment returns  $Y_t$  and the cross-sectional wealth distribution  $\omega_t$ . Therefore the function  $g(Y, \omega)$  implied by the corresponding (exact) value function solution now depends on two arguments. Nevertheless, the same approximation methodology as in the above production economy example applies and  $g$  can be expanded in a neighborhood of the solution implied by the value function of a logarithmic investor in an homogeneous agents, non-robust exchange economy. By construction this solution does not depend on  $\omega$ . Therefore, in the sequel we can still work with a function  $g_{00}$  that depends only on the state variable  $X_t := Y_t$ . For the present specific model, Proposition 5.1 in Kogan and Uppal yields

$$g_0(Y, \omega) = g_{0,0}(X) + O(\gamma, \vartheta) = a + bY + \log \delta + O(\gamma, \vartheta) \quad , \quad (62)$$

with

$$a = -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\delta + \lambda)} \quad , \quad b = \frac{1}{\delta + \lambda} \quad . \quad (63)$$

Using this result we can derive the desired equilibrium asymptotics for our robust exchange economy.

**Proposition 12** In the given exchange economy we have:

1. Equilibrium stock prices process asymptotics

$$\begin{aligned} P_t &= p(Y_t, \omega_t) e_t \quad , \\ p(Y, \omega) &= \frac{1}{\delta} (1 + \gamma \omega (a + bY)) + O^2(\gamma, \vartheta) \quad , \\ \frac{dP_t + e_t dt}{P_t} &= \left[ Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta e_Y \sigma_e \sigma_Y}{\delta + \lambda} \right) \right] dt \\ &\quad + \sigma_e dZ_t^e + \gamma \omega_t \frac{\sigma_Y}{\delta + \lambda} dZ_t^Y + O^2(\gamma, \vartheta) \quad . \end{aligned}$$

## 2. Equilibrium interest rate asymptotics

$$\begin{aligned} r_t = & \left( Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\lambda + \delta} \right) \right) - \left( \sigma_e^2 + \frac{2\sigma_e \gamma \omega_t \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \\ & + (\gamma - \vartheta) \left( \sigma_e^2 + \frac{\sigma_e \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \omega_t + O^2(\gamma, \vartheta) \quad . \end{aligned}$$

## 3. Optimum consumption and portfolio asymptotics

$$\begin{aligned} C_t^{(1)} &= \delta W_t^{(1)} \quad , \\ C_t &= \delta \left( 1 - \gamma \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) \right) W_t + O^2(\gamma, \vartheta) \quad , \\ w_t^{(1)} &= 1 - (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \omega_t + O^2(\gamma, \vartheta) \quad , \\ w_t &= 1 + (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) (1 - \omega_t) + O^2(\gamma, \vartheta) \quad . \end{aligned}$$

## 4. Asymptotics for cross-sectional wealth dynamics

$$\begin{aligned} d\omega_t = & \gamma \omega_t (1 - \omega_t) \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) dt \\ & + (\gamma - \vartheta) \omega_t (1 - \omega_t) \sigma_e \left( 1 + \frac{\delta_{eY} \sigma_Y}{\sigma_e(\lambda + \delta)} \right) dZ_t + O^2(\gamma, \vartheta) \quad . \end{aligned}$$

In the same general vein as for the results in the production economy of Proposition 10, equilibrium optimal consumption and stock price dynamics are affected only indirectly by the presence of HR, through the altered conditional variance in the cross-sectional wealth dynamics. By contrast, HR influences equilibrium portfolio holdings, interest rates and cross sectional wealth dynamics directly.

Similarly to the production economy case, the impact on equilibrium portfolio holdings depends on whether in an equivalent non robust economy the robust investor is a borrower or a lender. This in turn happens if and only if the conditions

$$\gamma \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) > 0 \quad , \quad \gamma \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) < 0 \quad , \quad (64)$$

respectively, are satisfied. In the first case, robustness can reduce equilibrium leverage positions in the risky asset and lower thereby equilibrium interest rates, but only if

$$\left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) > 0 \quad . \quad (65)$$



That is, if standard intertemporal hedging does not overcompensate myopic portfolio allocations.

Finally, as in the above production economy HR leaves the drift of cross-sectional wealth unchanged to first order, while it can reduce or enhance its conditional variance, depending on which combination of the conditions in (64) is verified. By the linearity of equilibrium interest rates in  $\omega_t$ , analogous implications for the impact of HR on interest rates variances follow.

## 4.2 Constrained Robust Equilibria

In this section we perform the above general equilibrium analysis for a situation where aversion to model uncertainty is modelled by a preference for CR. We consider again the robust production economy and the robust exchange economy of the last section. An important feature of these two model settings from the perspective of analyzing the impact of CR is that the first one is an example of a complete economy while the second one is not.

For this section, utilities of current consumption are given by (59), while preferences for CR are parameterized by the two parameters:

$$\eta > 0 \quad , \quad \eta^{(1)} = 0 \quad . \quad (66)$$

Homogeneous time preferences are again parameterized by the parameter  $\delta$ .

### 4.2.1 Production Economies

The same production technology and price processes of Section 4.1.2 are used, with an equilibrium Definition 11 modified by requiring optimality to first order to be satisfied with respect to the consumption rule (38) and to the investment rules in Corollary 5.

Similarly to the HR computations, we get

$$g_0(X) = g_{00}(X) + O(\gamma, \sqrt{\eta}) = \log(\delta) - 1 + \frac{\alpha - \sigma^2/2}{\delta} + O(\gamma, \sqrt{\eta}) \quad , \quad (67)$$

because the homogenous agents standard economy that is being perturbed is the same of Section 4.1.1.

To characterize the CR equilibrium in the given economy, we need to decompose the state space of  $(\omega_t)$  using the set  $A$  below

$$A = \left\{ \omega \mid \omega > 1 - \frac{\sigma}{\sqrt{2\eta}} \right\} . \quad (68)$$

$A$  identifies the states of  $\omega_t$  for which in equilibrium the robust agent is holding a long position in the risky asset. On the other hand,  $A^c$  defines the set of states of  $\omega_t$  for which in equilibrium the robust agent invests her entire wealth only in the riskless investment. Intuitively, set  $A$  is larger either when risk (measured by  $\sigma$ ) is high, or when model uncertainty (measured by  $\eta$ ) is low. Indeed, while risk and standard risk aversion incentivate investors to trade in order to diversify risk, model uncertainty reduces the incentive to trade and creates in equilibrium a no risky investment zone for the robust agent defined by the set  $A^c$ .

The characterization of the general equilibrium under CR in the given production economy is provided by the next proposition.

**Proposition 13** In the given production economy we have:

1. Equilibrium interest rate asymptotics. On the set  $A$

$$r_t = \alpha - \sigma^2 \left( 1 - \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) \omega_t \right) + O^2(\gamma, \sqrt{\eta}) \quad (69)$$

and on the set  $A^c$

$$r_t = \alpha - \frac{\sigma^2}{1 - \omega_t} + O^2(\gamma, \sqrt{\eta}) . \quad (70)$$

2. Optimum consumption asymptotics.

$$C_t^{(1)} = \delta W_t^{(1)} , \quad C_t = (\delta - \gamma(\alpha - \sigma^2/2 - \delta)) W_t + O^2(\gamma, \sqrt{\eta}) .$$

3. Optimum portfolio asymptotics. On the set  $A$

$$\begin{aligned} w_t^{(1)} &= 1 - \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) \omega_t + O^2(\gamma, \sqrt{\eta}) , \\ w_t &= 1 + \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) (1 - \omega_t) + O^2(\gamma, \sqrt{\eta}) , \end{aligned} \quad (71)$$

and on the set  $A^c$

$$w_t^{(1)} = \frac{1}{1 - \omega_t} + O^2(\gamma, \sqrt{\eta}) , \quad w_t = O^2(\gamma, \sqrt{\eta}) . \quad (72)$$

4. Asymptotics for cross-sectional wealth dynamics. On the set  $A$

$$d\omega_t = \gamma\omega_t(1-\omega_t)(\alpha - \sigma^2/2 - \delta)dt + \left(\gamma - \frac{\sqrt{2\eta}}{\sigma}\right)\omega_t(1-\omega_t)\sigma dZ_t + O^2(\gamma, \sqrt{\eta}) \quad , \quad (73)$$

and on the set  $A^c$

$$d\omega_t = \gamma\omega_t(1-\omega_t)\left[\alpha - \sigma^2/2 - \delta - \frac{1}{\gamma}\frac{\sigma^2\omega_t}{(1-\omega_t)^2}\right]dt - \omega_t\sigma^2 dZ_t + O^2(\gamma, \sqrt{\eta}) \quad . \quad (74)$$

5. Asymptotics for capital stock dynamics.

$$dS_t = [\alpha - \delta + \gamma(\alpha - \sigma^2/2 - \delta)\omega_t]S_t dt + \sigma S_t dZ_t + O^2(\gamma, \sqrt{\eta}) \quad .$$

To investigate the impact of CR on the above equilibrium in more detail, we notice that similarly to the HR case the effect on equilibrium consumption and capital stock dynamics is of order no less than two, leaving these entities unchanged when compared with the non robust analogous economy. On the other hand, equilibrium interest rates, optimal portfolios and cross-sectional wealth dynamics are directly influenced by the presence of CR. The specific way by which they are affected depends in equilibrium on (i) whether the robust agent is long in the risky asset (on the set  $A$ ) or (ii) if she invests her whole wealth only in the riskless investment (on the set  $A^c$ ).

Compared to an economy without robustness ( $\eta \rightarrow 0$  and  $A^c \rightarrow \emptyset$ ), we see from (69) that lower equilibrium interest rates are obtained for states  $\omega_t \in A$ . As mentioned, on this set the robust agent is holding a long position in the risky asset, exactly as would do a comparable non robust investor. However, CR reinforces the incentive to invest in the riskless asset (reduces the incentive to invest in the risky portfolios) and causes a higher equilibrium net supply of bonds if  $\gamma < 0$  (a lower net leveraged position in risky assets if  $\gamma > 0$ ), which in turn induces the lower interest rate. On the set  $A^c$  equilibrium interest rates are lower than those in an economy populated by homogeneous, log-utility agents. They do not depend on the parameters  $\gamma, \sqrt{\eta}$ , because market clearing is achieved in this case by the non robust log-utility agent alone. The behaviour of the equilibrium interest rate  $r_t$  as a function of  $\omega_t$  is illustrated in Figure 2 for a specific parameter choice.

Insert Figure 2 about here

The equilibrium interest rate in the presence of robustness is supported by an open interest

$$OI_t = \frac{1}{2} \left( |1 - w_t| \omega_t + \left| 1 - w_t^{(1)} \right| (1 - \omega_t) \right) = \left| \gamma - \frac{\sqrt{2\eta}}{\sigma} \right| \omega_t (1 - \omega_t) \quad ,$$

on set  $A$ , and

$$OI_t = \frac{1}{2} \left( |1 - w_t| \omega_t + \left| 1 - w_t^{(1)} \right| (1 - \omega_t) \right) = \omega_t \quad ,$$

on the set  $A^c$ . Therefore, on the set  $A$  the open interest decreases linearly with  $\omega_t$ .

On the set  $A$ , the direction of the impact of CR on the optimal holding of risky assets is similar to the one obtained under HR. At variance with HR, the impact of CR depends on the risky asset volatility  $\sigma$  and is higher (lower) for lower (higher) volatility parameters, showing again a first order risk aversion effect. This induces a contribution of CR to equilibrium interest rates that is proportional to volatilities, and hence equity premia that incorporate a premium for first order risk aversion.

On the set  $A^c$ , no equilibrium robust risky allocation arises and the standard log utility investor borrows from the robust agent her entire wealth, which becomes a leveraged position of the standard investor in the risky asset. Remark, that this happens independently of the coefficient of risk aversion  $1 - \gamma$ . Therefore, in this case the log utility investor is going to be leveraged in the risky asset independently of the degree of risk aversion of the robust agent. This produces in equilibrium a no-risky assets investment zone for the robust agent, even in the absence of transaction costs or portfolio constraints. This result agrees with Dow and Werlang (1992) finding that equilibrium stock allocations in the presence of uncertainty aversion can be zero in equilibrium even when expected returns are positive. The definition of the set  $A$  gives precise conditions under which such a situation is compatible with general equilibrium in the model: model uncertainty parameterized by  $\eta$  must be sufficiently large with respect to first order risk, as measured by  $\sigma$ .

The direction of the impact of CR on the dynamics of  $(\omega_t)$  is similar to the one obtained under HR, when  $\omega_t \in A$ . At variance with HR, the impact depends again on risky assets volatilities. Since on the set  $A$  equilibrium interest rates are linear in  $\omega_t$ , similar effects arise for the implied

interest rates volatilities in the presence of CR when  $\omega_t \in A$ . On the set  $A^c$ , both the drift and the volatility of cross-sectional wealth are altered in equilibrium. If  $\gamma < 0$ , the drift of  $\omega_t$  is negative and the long run cross sectional wealth distribution will degenerate to 0. However, if  $\gamma > 0$  - that is, the robust investor is less risk averse than the non robust agent - the sign of the drift in  $\omega_t$ 's dynamics can change as  $\omega_t$  approaches the boundary of set  $A^c$ . In this case the long run distribution of  $\omega_t$  could be stationary for parameter choices where the support of  $\omega_t$  is the set  $A^c$  (for instance for values of  $\gamma$  near to 0). A more precise discussion of these issues would however require an higher order analysis of the above equilibria.

#### 4.2.2 Exchange Economies

The same endowment and price processes of Section 4.1.2 are assumed, while the equilibrium Definition 11 is modified by requiring optimality to first order to be satisfied with respect to the consumption and investment rules (38), (39).

Similarly to the previous models we have

$$g_0(Y, \omega) = g_{0,0}(X) + O(\gamma, \sqrt{\eta}) := a + bY + \log \delta + O(\gamma, \sqrt{\eta}) \quad , \quad (75)$$

where  $a, b$ , are given by (62). Moreover, the function  $G_0$  in (39) depends on  $\gamma$  and  $\eta$ , through  $\frac{\partial g_0}{\partial X}$  and a function  $\phi_0$  implied by the equilibrium market price of risk in the model

$$\phi_0(Y_t, \omega_t) = \frac{\alpha_t + e_t - r_t}{\sigma_t} \quad .$$

Therefore, we expand also  $\phi_0$  in the neighborhood of the equilibrium market price of risk in an homogeneous economy with log utility non-robust investors

$$\phi_0(Y, \omega) = \phi_{00}(Y, \omega) + \gamma \phi_{01}(Y, \omega) + \sqrt{2\eta} \phi_{02}(Y, \omega) + O^2(\gamma, \sqrt{\eta}) \quad , \quad (76)$$

where  $\phi_{00}(Y, \omega) = \sigma_e$ , by Proposition 12. This yields immediately the next proposition.

**Proposition 14** Let

$$G_{00} = \sigma_e^2 + \frac{2\delta_{eY}\sigma_Y\sigma_e}{\lambda + \delta} + \left( \frac{\sigma_Y}{\lambda + \delta} \right)^2 \quad .$$

In the given exchange economy we have:

### 1. Equilibrium stock prices process asymptotics

$$\begin{aligned}
P_t &= p(Y_t, \omega_t) e_t \quad , \\
p(Y, \omega) &= \frac{1}{\delta} (1 + \gamma \omega (a + bY)) + O^2(\gamma, \vartheta) \quad , \\
\frac{dP_t + e_t dt}{P_t} &= \left[ Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\delta + \lambda} \right) \right] dt \\
&\quad + \sigma_e dZ_t^e + \gamma \omega_t \frac{\sigma_Y}{\delta + \lambda} dZ_t^Y + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 2. Equilibrium interest rate asymptotics

$$\begin{aligned}
r_t &= \left( Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\lambda + \delta} \right) \right) - \left( \sigma_e^2 + \frac{2\sigma_e \gamma \omega_t \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \\
&\quad + \left( \gamma - \sqrt{\frac{2\eta}{G_{00}}} \right) \left( \sigma_e^2 + \frac{\sigma_e \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \omega_t + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 3. Optimum consumption and portfolio asymptotics

$$\begin{aligned}
C_t^{(1)} &= \delta W_t^{(1)} \quad , \\
C_t &= \delta \left( 1 - \gamma \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) \right) W_t + O^2(\gamma, \sqrt{\eta}) \quad , \\
w_t^{(1)} &= 1 - \left( \gamma - \sqrt{\frac{2\eta}{G_{00}}} \right) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \omega_t + O^2(\gamma, \sqrt{\eta}) \quad , \\
w_t &= 1 + \left( \gamma - \sqrt{\frac{2\eta}{G_{00}}} \right) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) (1 - \omega_t) + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 4. Asymptotics for cross-sectional wealth dynamics

$$\begin{aligned}
d\omega_t &= \gamma \omega_t (1 - \omega_t) \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) dt \\
&\quad + \left( \gamma - \sqrt{\frac{2\eta}{G_{00}}} \right) \omega_t (1 - \omega_t) \sigma_e \left( 1 + \frac{\delta_{eY} \sigma_Y}{\sigma_e (\lambda + \delta)} \right) dZ_t + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

From a pure formal perspective the impact of CR on the relevant equilibrium quantities in the given exchange economy is similar to that obtained in the presence of HR. Indeed, by the simple reparameterization  $\vartheta = \sqrt{\frac{2\eta}{G_{00}}}$  the equilibria obtained under CR and HR are in a one-to-one relationship. Specifically, to first order CR implies in equilibrium state independent effective risk aversions

$$1 + \left( \gamma - \sqrt{\frac{2\eta}{\sigma_e^2 + \frac{2\delta_{eY} \sigma_Y \sigma_e}{\lambda + \delta} + \left( \frac{\sigma_Y}{\lambda + \delta} \right)^2}} \right)$$

in the given economy. This in turn, derives from the fact that the state dependent part of the equilibrium market price of risk (76) and the equilibrium function  $\frac{\partial g_{00}}{\partial X}$  have an impact of an higher

order on general equilibrium in this model. The interpretation of the correction factor  $\sqrt{\frac{2\eta}{G_{00}}}$  is however economically different from the one of the parameter  $\vartheta$  in the HR case. Indeed, under CR the impact of robustness is inversely related to a first-order measure of risk

$$\sqrt{\sigma_e^2 + \frac{2\delta_{eY}\sigma_Y\sigma_e}{\lambda + \delta} + \left(\frac{\sigma_Y}{\lambda + \delta}\right)^2}$$

in the economy, showing again a first order risk aversion effect and robust equity premia components that are related to equity volatilities rather than to variances. This result is consistent with the one obtained above for production economies. However, to first order one of the partial equilibrium distinguishing features of CR in incomplete markets - that is a state dependent correction for risk aversion - is lost in general equilibrium. It is easily seen (for instance from the proof of Proposition 14) that in neighborhoods of an homogeneous log-utility agent economy, a state dependent effective risk aversion because of CR can arise to first order in the class of models considered only in the presence of a stochastic volatility of endowment returns. The last section of the paper addresses precisely such a model setting.

#### 4.2.3 Exchange Economies with Stochastic Volatility

We consider a robust exchange economy where the risky asset is a claim on the aggregate endowment process  $(e_t)$ , with dynamics given by

$$\begin{aligned} de_t &= Y_t e_t dt + \sigma_e \sqrt{Y_t} e_t dZ_t^e, \\ dY_t &= \lambda (\bar{Y} - Y_t) dt + \sqrt{Y_t} \sigma_Y dZ_t^Y, \end{aligned}$$

where  $\sigma_e, \lambda, \bar{Y}, \sigma_Y > 0$ , and  $(Z_t^e), (Z_t^Y)$  are both standard Brownian motions in  $\mathbb{R}$  with covariation  $E(dZ_t^e dZ_t^Y) = \delta_{eY} dt$ .  $(Y_t)$  is a mean reverting Bessel process which affects both expected endowment returns and endowment returns volatilities.

The cumulative return process of the risky asset is given by

$$dP_t = \alpha_t P_t dt + \sigma_t P_t dZ_t,$$

and the riskless asset is a short term bond with price dynamics

$$dB_t = r_t B_t dt \quad .$$

The equilibrium concept used in this section is the same of the one used in the last section. Again, we approximate  $g_0$  as

$$g_0(Y, \omega) = g_{0,0}(Y, \omega) + O(\gamma, \sqrt{\eta}) \quad , \quad (77)$$

with a function  $g_{0,0}$  that is determined by the value function solution of a representative log-utility investor in a standard economy. The form of  $g_{0,0}$  in the present model setting is given by the next proposition.

**Proposition 15** In the given economy it follows

$$g_{0,0}(Y, \omega) = a + bY + \ln(\delta) \quad ,$$

where

$$a = \frac{\lambda(1 - \frac{1}{2}\sigma_e^2)\bar{Y}}{\delta(\delta + \lambda)} \quad , \quad b = \frac{1 - \frac{1}{2}\sigma_e^2}{\delta + \lambda} \quad .$$

Furthermore, when expanding the market price of risk  $\phi$  in the given economy it follows

$$\phi(Y, \omega) = \phi_{00}(Y, \omega) + \gamma\phi_{01}(Y, \omega) + \sqrt{2\eta}\phi_{02}(Y, \omega) + O^2(\gamma, \sqrt{\eta}) \quad , \quad (78)$$

where

$$\phi_{00}(Y, \omega) = \sigma_e \sqrt{Y} \quad ,$$

is the market price of risk in a standard comparable homogeneous economy populated by log-utility agents. The asymptotics for the general equilibrium in the above exchange economy are presented in the next last proposition of the paper.

**Proposition 16** Let

$$G = \sigma_e^2 + 2\delta_{eY}\sigma_e\sigma_Y \frac{1 - \frac{1}{2}\sigma_e^2}{\delta + \lambda} + \sigma_Y^2 \left( \frac{1 - \frac{1}{2}\sigma_e^2}{\delta + \lambda} \right)^2 \quad .$$

In the given exchange economy we have:



### 1. Equilibrium stock prices process asymptotics

$$\begin{aligned}
P_t &= p(Y_t, \omega_t) e_t \quad , \\
p(Y, \omega) &= \frac{1}{\delta} \left( 1 + \gamma \omega \left( a + \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) Y \right) \right) + O^2(\gamma, \sqrt{\eta}) \quad , \\
\frac{dP_t + e_t dt}{P_t} &= \left[ \delta + Y_t \left( 1 - \gamma \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \omega_t (\lambda + \delta - \delta_{eY} \sigma_e \sigma_Y) \right) \right] dt \\
&\quad + \sigma_e \sqrt{Y_t} dZ_t^e + \gamma \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \omega_t \sigma_Y \sqrt{Y_t} dZ_t^Y + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 2. Equilibrium interest rate asymptotics

$$\begin{aligned}
r_t &= \left( \delta + \left( 1 - \gamma \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \omega_t (\lambda + \delta - \delta_{eY} \sigma_e \sigma_Y) \right) Y_t \right) \\
&\quad - \left( \sigma_e^2 + 2\sigma_e \gamma \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \omega_t \sigma_Y \delta_{eY} \right) Y_t \\
&\quad + \left( \gamma - \sqrt{\frac{\eta}{GY_t}} \right) \left( \sigma_e^2 + \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \sigma_e \sigma_Y \delta_{eY} \right) Y_t \omega_t + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 3. Optimum consumption and portfolio asymptotics

$$\begin{aligned}
C_t^{(1)} &= \delta W_t^{(1)} \quad , \\
C_t &= \delta \left( 1 - \gamma \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \left( \frac{\lambda \bar{Y}}{\delta} + Y_t \right) \right) W_t + O^2(\gamma, \sqrt{\eta}) \quad , \\
w_t^{(1)} &= 1 - \left( \gamma - \sqrt{\frac{\eta}{GY_t}} \right) \left( 1 + \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \omega_t + O^2(\gamma, \sqrt{\eta}) \quad , \\
w_t &= 1 + \left( \gamma - \sqrt{\frac{\eta}{GY_t}} \right) \left( 1 + \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) (1 - \omega_t) + O^2(\gamma, \sqrt{\eta}) \quad .
\end{aligned}$$

### 4. Asymptotics for cross-sectional wealth dynamics.

$$\begin{aligned}
d\omega_t &= \gamma \omega_t (1 - \omega_t) \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \left( \frac{\lambda \bar{Y}}{\delta} + Y_t \right) dt \\
&\quad + \left( \gamma - \sqrt{\frac{\eta}{GY_t}} \right) (1 - \omega_t) \sigma_e \sqrt{Y_t} \left( 1 + \left( \frac{1 - \frac{1}{2} \sigma_e^2}{\delta + \lambda} \right) \frac{\delta_{eY} \sigma_Y}{\sigma_e} \right) dZ_t \quad .
\end{aligned}$$

As expected, the state dependent effective risk aversion corrections implied by CR now affect explicitly the arising general equilibrium in the expressions for the equilibrium holdings of risky assets, the equilibrium interest rate and the equilibrium cross sectional wealth dynamics. Specifically, we notice that while risk aversion implies a linear dependence of interest rates on expected endowment returns variances  $Y_t$ , CR affects equilibrium interest rates in a way that is proportional to the volatility  $\sqrt{Y_t}$  of expected endowment returns. This is a last first order risk

aversion effect yielding equity premia for CR that are proportional to the volatilities of expected endowment returns, with a functional form different from the one that arises in the presence of other robustness definitions.

## 5 Conclusions

We explored the relation between risk aversion, robustness and Knightian uncertainty in heterogeneous agents economies, by means of an extended Kogan and Uppal (2000) perturbative approach, which characterizes to first order the implied equilibria for several definitions of robustness recently proposed in the literature. To first order, we observed that definitions of robustness that mimic Knightian uncertainty induce the largest variety of robust economic behaviours in the presence of model uncertainty, including equity premia that reflect first order risk aversion and an endogenous stock market participation in one of our explicit examples.

Strictly speaking, our asymptotics are well suited to investigate situations where  $\gamma, \vartheta$  and  $\sqrt{\eta}$  are close to 0. For the robustness parameter this is a quite natural assumption since too strong preferences for robustness can induce unrealistically pessimistic worst case models. With respect to the risk aversion term  $\gamma$  this however implies that our analysis is limited basically to values of  $\gamma$  (or equivalently differences in relative risk aversions) in a neighborhood of 0 and to an analysis of the direct interplay between risk and model uncertainty aversion.

To analyze the higher order properties of the equilibria induced by models of a preference for robustness one would have to develop a more direct approach that solves perturbatively the implied robust HJB equations, similarly to the approach followed in Trojani and Vanini (2001b). To an higher order these perturbative approaches will differ more crucially than the ones used in the present paper, making a direct comparison between the different definition of robustness more complex and involved. Therefore, we leave this analysis as a topic for future research.

## 6 Appendix

Proof of Proposition 3. Using the functional form (27) it follows

$$\begin{aligned}
-\vartheta \frac{J_W^2}{J_{WW}} &= \frac{\vartheta (e^{g_0(X)} W)^\gamma}{\delta (1-\gamma)} + O^2(\gamma, \vartheta) = \frac{\vartheta}{\delta} \frac{1}{1-\gamma} + O^2(\gamma, \vartheta) \quad , \\
-\frac{J_W}{W J_{WW}} &= \frac{1}{1-\gamma} + O^2(\gamma, \vartheta) \quad , \\
\frac{J_{WX}}{J_W} &= \gamma \frac{\partial g_0(X)}{\partial X} + O^2(\gamma, \vartheta) \quad , \\
-\vartheta J_X &= -\frac{\vartheta}{\delta} (e^{g_0(X)} W)^\gamma \frac{\partial g(X)}{\partial X} + O^2(\gamma, \vartheta) = -\frac{\vartheta}{\delta} + O^2(\gamma, \vartheta) \quad .
\end{aligned}$$

This proves the claim. ■

Proof of Proposition 4 . Setting

$$\begin{aligned}
w(X) &= \frac{\alpha - r}{\sigma^2} + \gamma w_1(X) + \sqrt{2\eta} w_2(X) + O^2(\gamma, \sqrt{2\eta}) \quad , \\
g(X) &= g_0(X) + \gamma g_1(X) + \sqrt{2\eta} g_2(X) + O^2(\gamma, \sqrt{2\eta}) \quad ,
\end{aligned}$$

it follows

$$\begin{aligned}
w &= \frac{1}{\left(1 + \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G(w)}}\right)} \frac{1}{1-\gamma} \left( \frac{\alpha - r}{\sigma^2} + \gamma \frac{\partial g}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2} - \sqrt{\frac{2\eta}{G(w)}} \frac{\partial g}{\partial X} \frac{\sigma_{XP}}{\sigma^2} \right) + O^2(\gamma, \sqrt{2\eta}) \\
&= \frac{1}{1 - \left(\gamma - \sqrt{\frac{2\eta}{G_0}}\right)} \left( \frac{\alpha - r}{\sigma^2} + \left(\gamma - \sqrt{\frac{2\eta}{G_0}}\right) \frac{\partial g_0}{\partial X} \frac{\sigma_{XP}}{\sigma^2} \right) + O^2(\gamma, \sqrt{2\eta}) \quad ,
\end{aligned}$$

with

$$G_0 = \left(\frac{\alpha - r}{\sigma}\right)^2 + \left(\xi \frac{\partial g_0}{\partial X}\right)^2 + 2\rho \left(\frac{\alpha - r}{\sigma}\right) \xi \frac{\partial g_0}{\partial X} \quad .$$

This proves the proposition. ■

Proof of (48). Using formula (19) in Kogan and Uppal (2000) it follows

$$\begin{aligned}
g_0(X) &= \ln(\delta) - 1 + E \left[ \int_0^\infty e^{-\delta t} \left( r + \frac{1}{2} \left( \frac{\alpha_t - r}{\sigma} \right)^2 \right) dt \middle| X_0 = X \right] \\
&= \ln(\delta) - 1 + \left[ \int_0^\infty e^{-\delta t} \left( r + \frac{1}{2} E(X_t^2 | X_0 = X) \right) dt \right] \quad .
\end{aligned}$$

Since

$$\begin{aligned} E(X_t^2 | X_0 = X) &= \text{Var}(X_t | X_0 = X) + (E(X_t | X_0 = X))^2 \\ &= \xi^2 \cdot \frac{1 - e^{-2\lambda t}}{2\lambda} + [e^{-\lambda t} (X + \bar{X} (e^{\lambda t} - 1))]^2, \end{aligned}$$

a final integration gives

$$\begin{aligned} g_0(X) &= \ln(\delta) - 1 + \int_0^\infty e^{-\delta t} \left( r + \frac{\xi^2}{2} \cdot \frac{1 - e^{-2\lambda t}}{2\lambda} + \frac{1}{2} [e^{-\lambda t} (X + \bar{X} (e^{\lambda t} - 1))]^2 \right) dt \\ &= \ln(\delta) - 1 + \int_0^\infty e^{-\delta t} \left( r + \frac{\xi^2}{2} \cdot \frac{1 - e^{-2\lambda t}}{2\lambda} + \frac{(e^{-\lambda t} \bar{X} (e^{\lambda t} - 1))^2}{2} \right) dt \\ &\quad + X \cdot \bar{X} \int_0^\infty (e^{-\delta t} (e^{\lambda t} - 1)) dt + \frac{X^2}{2} \cdot \int_0^\infty e^{-\delta t} e^{-2\lambda t} dt. \end{aligned}$$

By computing these expectations explicitly, the claim of the proposition is obtained. ■

Notation: In the sequel we adopt the symbol  $\doteq$  to denote equality up to terms of order  $O^2(\gamma, \vartheta)$  and  $O^2(\gamma, \sqrt{\eta})$ , respectively.

**Proof of Proposition 10.** Aggregate market clearing, together with the portfolio asymptotics (18) and the functional form (61), implies

$$1 \doteq w_t \frac{W_t}{W_t + W_t^{(1)}} + w_t^{(1)} \frac{W_t^{(1)}}{W_t + W_t^{(1)}} \doteq w_t^{(1)} + (w_t - w_t^{(1)}) \omega_t \doteq \frac{(\alpha - r_t)}{\sigma^2} [1 + (\gamma - \vartheta) \omega_t].$$

Solving for  $r_t$  and performing a final expansion proves 1, i.e.

$$r_t \doteq \alpha - \frac{\sigma^2}{1 + (\gamma - \vartheta) \omega_t} \doteq \alpha - \sigma^2 + \sigma^2 (\gamma - \vartheta) \omega_t. \quad (79)$$

The consumption asymptotics (17) implies

$$c_t^{(1)} = \delta, \quad c_t \doteq \delta - \gamma \delta (g_0(X) - \ln(\delta)) \doteq \delta - \gamma (\alpha - \sigma^2/2 - \delta). \quad (80)$$

Moreover, using (79) we get:

$$\begin{aligned} w_t^{(1)} &= \frac{\alpha - r_t}{\sigma^2} \doteq 1 - (\gamma - \vartheta) \omega_t \\ w_t &\doteq \frac{\alpha - r_t}{\sigma^2} (1 + (\gamma - \vartheta)) \doteq (1 - (\gamma - \vartheta) \omega_t) (1 + (\gamma - \vartheta)) \doteq 1 + (\gamma - \vartheta) (1 - \omega_t), \end{aligned}$$

which proves 2.

To prove 3, we apply first Itô's Lemma to derive the cross sectional wealth dynamics

$$\begin{aligned} d\omega_t &\doteq \omega_t (1 - \omega_t) \left[ \left( w_t - w_t^{(1)} \right) \left[ (\alpha - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma_t^2 \right] \right. \\ &\quad \left. - \left( c_t - c_t^{(1)} \right) \right] dt + \omega_t (1 - \omega_t) \left( w_t - w_t^{(1)} \right) \sigma dZ_t \quad . \end{aligned}$$

From (80)

$$- \left( c_t - c_t^{(1)} \right) \doteq \gamma \left( \alpha - \sigma^2/2 - \delta \right) \quad ,$$

and

$$\left( w_t - w_t^{(1)} \right) \sigma \doteq (\gamma - \vartheta) \sigma \quad . \quad (81)$$

Finally, using the financial markets clearing condition and (79),

$$(\alpha - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma^2 \doteq (\alpha - r_t) - \sigma^2 \doteq -\sigma^2 (\gamma - \vartheta) \omega_t \quad .$$

Using (81) this implies:

$$\left( w_t - w_t^{(1)} \right) \left[ (\alpha - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma^2 \right] \doteq 0 \quad .$$

Putting the results together, we proved 3. We remark that this last equation also implies an asymptotic contribution of order no less than 2 for the optimal portfolio policies to the drift of the cross sectional wealth dynamics.

To prove 4, we get from (80)

$$\begin{aligned} dS_t &\doteq \left( \alpha S_t - \left( C_t + C_t^{(1)} \right) \right) dt + \sigma S_t dZ_t \\ &\doteq \left[ \alpha S_t - \left( (\delta - \gamma (\alpha - \sigma^2/2 - \delta)) W_t + \delta W_t^{(1)} \right) \right] dt + \sigma S_t dZ_t \\ &\doteq \left[ \alpha - \delta + \gamma (\alpha - \sigma^2/2 - \delta) \omega_t \right] S_t dt + \sigma S_t dZ_t \quad . \end{aligned}$$

This proves 4 and the whole proposition. ■

**Proof of Proposition 12.** To prove 1, we note that in our economy  $W_t + W_t^{(1)} = P_t$  and

$$W_t = \omega_t P_t \quad , \quad W_t^{(1)} = (1 - \omega_t) P_t \quad .$$

Using the consumption asymptotics (17) with  $g_0$  replaced by  $g_{0,0}$ , aggregate good markets clearing implies

$$\begin{aligned}\frac{e_t}{\delta} &\doteq \frac{1}{\delta} \left( c_t + c_t^{(1)} \right) \doteq \frac{1}{\delta} \left( \delta (1 - \omega_t) + \delta (1 - \gamma (g_{0,0} (X_t) - \log (\delta))) \omega_t \right) P_t \\ &\doteq (1 - \gamma (g_{0,0} (X_t) - \log (\delta)) \omega_t) P_t \quad .\end{aligned}$$

From (62) we have:

$$\begin{aligned}P_t &\doteq \frac{e_t}{\delta} \frac{1}{1 - \gamma (g_{0,0} (X_t) - \log (\delta)) \omega_t} \doteq \frac{e_t}{\delta} (1 + \gamma (g_{0,0} (X_t) - \log (\delta)) \omega_t) \\ &\doteq \frac{e_t}{\delta} (1 + \gamma (a + bY_t) \omega_t) \quad .\end{aligned}\tag{82}$$

Defining  $p_t = p(Y_t, \omega_t) = \frac{1}{\delta} (1 + \gamma (a + bY_t) \omega_t)$ , Itô's Lemma yields

$$\begin{aligned}\frac{dP_t}{P_t} &\doteq \frac{1}{P_t} p_t de_t + \frac{1}{P_t} e_t dp_t + \left[ \frac{dp_t}{p_t}, \frac{de_t}{e_t} \right] \doteq \frac{de_t}{e_t} + \frac{dp_t}{p_t} + \left[ \frac{dp_t}{p_t}, \frac{de_t}{e_t} \right] \\ &\doteq Y_t dt + \sigma_e dZ_t^e + \frac{p_\omega(Y_t, \omega_t)}{p_t} d\omega_t + \frac{p_Y(Y_t, \omega_t)}{p_t} dY_t + \frac{1}{2} \frac{p_{\omega\omega}(Y_t, \omega_t)}{p_t} [d\omega_t, d\omega_t] \\ &\quad + \frac{1}{2} \frac{p_{YY}(Y_t, \omega_t)}{p_t} [dY_t, dY_t] + \frac{p_{\omega Y}(Y_t, \omega_t)}{p_t} [d\omega_t, dY_t] + \frac{p_\omega(Y_t, \omega_t)}{p_t e_t} [d\omega_t, de_t] \\ &\quad + \frac{p_Y(Y_t, \omega_t)}{p_t e_t} [dY_t, de_t] \quad .\end{aligned}$$

Moreover, we note

$$\begin{aligned}p_\omega(Y_t, \omega_t) &= \frac{\gamma}{\delta} (a + bY_t) = O(\gamma, \vartheta) \quad , \\ p_{\omega\omega}(Y_t, \omega_t) &= p_{YY}(Y_t, \omega_t) = 0 \quad , \\ p_{\omega Y}(Y_t, \omega_t) &= \frac{\gamma b}{\delta} = O(\gamma, \vartheta) \quad .\end{aligned}$$

Expanding  $d\omega_t$  in a neighborhood of the (constant) cross sectional wealth dynamics implied by an homogeneous non robust ( $\vartheta = 0$ ) economy with log utility investors ( $\gamma = 0$ ) gives

$$d\omega_t = O(\gamma, \vartheta) \quad , \quad [d\omega_t, dY_t] = O(\gamma, \vartheta) \quad , \quad [d\omega_t, de_t] = O(\gamma, \vartheta) \quad .$$

Inserting these results in the above expression for  $\frac{dP_t}{P_t}$  gives

$$\begin{aligned}\frac{dP_t}{P_t} &\doteq Y_t dt + \sigma_e dZ_t^e + \frac{p_Y(Y_t, \omega_t)}{p_t} dY_t + \frac{p_Y(Y_t, \omega_t)}{p_t e_t} [dY_t, de_t] \\ &\doteq (Y_t + \gamma b \omega_t (\lambda (\bar{Y} - Y_t) + \delta_{eY} \sigma_e \sigma_Y)) dt + \sigma_e dZ_t^e + \gamma b \omega_t \sigma_Y dZ_t^Y \quad .\end{aligned}\tag{83}$$

With (82) and (62), the cumulative return is

$$\begin{aligned}
\frac{dP_t + e_t dt}{P_t} &\doteq (Y_t - \gamma b \omega_t (\lambda (Y_t - \bar{Y}) - \delta_{eY} \sigma_e \sigma_Y)) dt + \delta (1 - \gamma \omega_t (a + b Y_t)) dt \\
&\quad + \sigma_e dZ_t^e + \gamma b \omega_t \sigma_Y dZ_t^Y \\
&\doteq \left( Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\lambda + \delta} \right) \right) dt + \sigma_e dZ_t^e + \gamma b \omega_t \sigma_Y dZ_t^Y \quad .
\end{aligned}$$

This proves 1.

To prove 2 we use the financial market clearing condition and the asymptotics (18) for the optimal portfolio positions:

$$\begin{aligned}
1 &\doteq w_t \frac{W_t}{W_t + W_t^{(1)}} + w_t^{(1)} \frac{W_t^{(1)}}{W_t + W_t^{(1)}} \doteq w_t \omega_t + w_t^{(1)} (1 - \omega_t) \doteq w_t^{(1)} + (w_t - w_t^{(1)}) \omega_t \\
&\doteq \frac{\alpha_t + e_t - r_t}{\sigma_t^2} [1 + (\gamma - \vartheta) \omega_t] + \frac{\partial g_{0,0}(X_t)}{\partial X} \frac{\sigma_{XP}}{\sigma_t^2} (\gamma - \vartheta) \omega_t \quad .
\end{aligned}$$

Hence, it follows

$$\begin{aligned}
r_t &\doteq \alpha_t + e_t - \frac{1}{[1 + (\gamma - \vartheta) \omega_t]} \left( \sigma_t^2 - \frac{\partial g_{0,0}(X_t)}{\partial X} \sigma_{XP} (\gamma - \vartheta) \omega_t \right) \\
&\doteq \alpha_t + e_t - \sigma_t^2 + (\gamma - \vartheta) \left( \sigma_t^2 + \frac{\partial g_{0,0}(X_t)}{\partial X} \sigma_{XP} \right) \omega_t \quad .
\end{aligned}$$

Using the above cumulative return dynamics and (62) this finally leads to

$$\begin{aligned}
r_t &\doteq \left( Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\lambda + \delta} \right) \right) - (\sigma_e^2 + 2\sigma_e \gamma b \omega_t \sigma_Y \delta_{eY}) \\
&\quad + (\gamma - \vartheta) \left( \sigma_e^2 + \frac{1}{\lambda + \delta} (\sigma_e \sigma_Y \delta_{eY} + \gamma b \omega_t \sigma_Y^2) \right) \omega_t \\
&\doteq \left( Y_t + \delta - \gamma \omega_t \left( Y_t - \frac{\sigma_e^2}{2} - \frac{\delta_{eY} \sigma_e \sigma_Y}{\lambda + \delta} \right) \right) - \left( \sigma_e^2 + \frac{2\sigma_e \gamma \omega_t \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \\
&\quad + (\gamma - \vartheta) \left( \sigma_e^2 + \frac{\sigma_e \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \omega_t \quad ,
\end{aligned}$$

which proves 2.

Claim 3 is based on the optimum consumption asymptotics (17) together with (62):

$$\begin{aligned}
c^{(1)} &= \delta \quad , \\
c &\doteq \delta (1 - \gamma (g_{0,0}(X_t) - \log(\delta))) \doteq \delta \left( 1 - \gamma \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) \right) \quad .
\end{aligned}$$

Moreover

$$w^{(1)} \doteq \frac{\alpha_t + e_t - r_t}{\sigma_t^2} \doteq 1 - (\gamma - \vartheta) \left( 1 + \frac{\partial g_{0,0}(X_t)}{\partial X} \frac{\sigma_{XP}}{\sigma_t^2} \right) \omega_t \doteq 1 - (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \omega_t ,$$

and

$$\begin{aligned} w &\doteq \frac{\alpha_t + e_t - r_t}{\sigma_t^2} (\gamma - \vartheta) + (\gamma - \vartheta) \frac{\partial g_{0,0}(X_t)}{\partial X} \frac{\sigma_{XP}}{\sigma_t^2} \\ &\doteq 1 + (\gamma - \vartheta) - (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \omega_t + (\gamma - \vartheta) \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \\ &\doteq 1 + (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) (1 - \omega_t) \quad . \end{aligned}$$

This proves 3.

We next consider 4. The equilibrium cross-sectional wealth dynamics implied by Itô's Lemma for the given exchange economy are of the form

$$\begin{aligned} d\omega_t &= \omega_t (1 - \omega_t) \left[ \left( w_t - w_t^{(1)} \right) \left[ (\alpha_t + e_t - r_t) - \sigma_t^2 \right] - \left( c_t - c_t^{(1)} \right) \right] dt \\ &\quad + \omega_t (1 - \omega_t) \left( w_t - w_t^{(1)} \right) \sigma_t dZ_t \quad . \end{aligned}$$

Moreover

$$\left( w - w_t^{(1)} \right) \sigma_t \doteq (\gamma - \vartheta) \left( 1 + \frac{1}{\lambda + \delta} \frac{\sigma_Y \delta_{eY}}{\sigma_e} \right) \sigma_e \quad ,$$

and

$$- \left( c_t - c_t^{(1)} \right) \doteq \gamma \left( -\frac{\sigma_e^2}{2\delta} + \frac{\lambda \bar{Y}}{\delta(\lambda + \delta)} + \frac{Y_t}{\lambda + \delta} \right) \quad .$$

Finally, using the results in the proof of 2 we have:

$$\begin{aligned} (\alpha_t + e_t - r_t) - \sigma_t^2 &\doteq -(\gamma - \vartheta) \left( \sigma_t^2 + \frac{\partial g_{0,0}(X_t)}{\partial X} \sigma_{XP} \right) \omega_t \\ &\doteq -(\gamma - \vartheta) \left( \sigma_e^2 + \frac{\sigma_e \sigma_Y \delta_{eY}}{\lambda + \delta} \right) \omega_t \quad , \end{aligned}$$

yielding

$$\left( w_t - w_t^{(1)} \right) \left[ (\alpha_t + e_t - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma_t^2 \right] = O^2(\gamma, \vartheta) \quad .$$

This proves 4 in Proposition 12. ■



Proof of Proposition 13. Aggregate market clearing implies

$$1 \doteq w_t^{(1)} + (w_t - w_t^{(1)}) \omega_t \quad .$$

Since in the present setting  $\frac{\partial g_{00}(X)}{\partial X} = 0$  (cf. again (61)), the complete markets asymptotics for optimal investment in Corollary 5 implies

$$w_t \doteq \begin{cases} \frac{1}{\sigma^2(1-\gamma)} (\alpha - r_t - \sqrt{2\eta}\sigma) & , \text{ if } \alpha - r_t - \sqrt{2\eta}\sigma \geq 0 \\ \frac{1}{\sigma^2(1-\gamma)} (\alpha - r_t + \sqrt{2\eta}\sigma) & , \text{ if } \alpha - r_t + \sqrt{2\eta}\sigma \leq 0 \\ 0 & \text{ otherwise} \end{cases} \quad .$$

In general equilibrium, the relevant cases are:

$$\text{Case 1: } \frac{\alpha - r_t}{\sigma} > \sqrt{2\eta} \quad , \quad \text{Case 2: } 0 < \frac{\alpha - r_t}{\sigma} \leq \sqrt{2\eta} \quad .$$

For Case 1 follows

$$1 \doteq w_t^{(1)} + (w_t - w_t^{(1)}) \omega_t \doteq \frac{(\alpha - r_t)(1 - \gamma(1 - \omega_t)) - \sqrt{2\eta}\sigma\omega_t}{(1 - \gamma)\sigma^2} \quad ,$$

implying

$$\frac{\alpha - r_t}{\sigma} \doteq \frac{(1 - \gamma)\sigma + \sqrt{2\eta}\omega_t}{1 - \gamma(1 - \omega_t)} \quad .$$

Hence, for Case 1

$$\frac{\alpha - r_t}{\sigma} > \sqrt{2\eta} \iff (1 - \gamma)\sigma + \sqrt{2\eta}\omega_t > (1 - \gamma)\sqrt{2\eta} + \gamma\sqrt{2\eta}\omega_t \iff \omega_t > 1 - \frac{\sigma}{\sqrt{2\eta}} \quad .$$

Solving for  $r_t$ , we get

$$r_t \doteq \alpha - \sigma \frac{(1 - \gamma)\sigma + \sqrt{2\eta}\omega_t}{1 - \gamma(1 - \omega_t)} \doteq \alpha - \sigma^2 \left( 1 - \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) \omega_t \right) \quad . \quad (84)$$

Similarly, for Case 2

$$1 \doteq w_t^{(1)}(1 - \omega_t) \doteq \frac{\alpha - r_t}{\sigma^2}(1 - \omega_t) \quad ,$$

which implies

$$r_t = \alpha - \frac{\sigma^2}{1 - \omega_t} \quad . \quad (85)$$

This proves 1. of the proposition.

As for the HR case, the consumption asymptotics for CR in Proposition 4 (which are valid also for the complete markets case) implies

$$c_t^{(1)} = \delta \quad , \quad c_t \doteq \delta - \gamma \delta (g_0(X) - \log(\delta)) \doteq \delta - \gamma (\alpha - \sigma^2/2 - \delta) \quad .$$

Moreover, from (84) and (85)

$$w_t^{(1)} = \frac{\alpha - r_t}{\sigma^2} \doteq 1 - \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) \omega_t \quad ,$$

in Case 1, and

$$w_t^{(1)} = \frac{1}{1 - \omega_t} \quad ,$$

in Case 2. In the same vain (again by (84),(85))

$$w_t \doteq \frac{1}{\sigma^2(1 - \gamma)} \left( \alpha - r_t - \sqrt{2\eta}\sigma \right) \doteq 1 + \left( \gamma - \frac{\sqrt{2\eta}}{\sigma} \right) (1 - \omega_t)$$

in Case 1, and  $w_t \doteq 0$  otherwise. This proves 2. of the proposition. To prove 3., we consider as in the HR setting, the Itô's dynamics of  $\omega_t$

$$\begin{aligned} d\omega_t &\doteq \omega_t (1 - \omega_t) \left[ \left( w_t - w_t^{(1)} \right) \left[ (\alpha - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma^2 \right] \right. \\ &\quad \left. - \left( c_t - c_t^{(1)} \right) \right] dt + \omega_t (1 - \omega_t) \left( w_t - w_t^{(1)} \right) \sigma dZ_t \quad . \end{aligned}$$

and distinguish again the two Cases considered above. We have

$$- \left( c_t - c_t^{(1)} \right) \doteq \gamma (\alpha - \sigma^2/2 - \delta) \quad ,$$

and

$$\begin{aligned} \left( w_t - w_t^{(1)} \right) \sigma &\doteq \sigma \gamma - \sqrt{2\eta} \quad , \quad \text{Case 1} \quad , \\ \left( w_t - w_t^{(1)} \right) \sigma &\doteq -\frac{\sigma}{1 - \omega_t} \quad , \quad \text{Case 2} \quad . \end{aligned}$$

Using the financial markets clearing condition we have

$$(\alpha - r_t) - \left( \omega_t w_t + (1 - \omega_t) w_t^{(1)} \right) \sigma^2 \doteq (\alpha - r_t) - \sigma^2 \quad ,$$

implying

$$\left(w_t - w_t^{(1)}\right) \left[(\alpha - r_t) - \left(\omega_t w_t + (1 - \omega_t) w_t^{(1)}\right) \sigma^2\right] \doteq 0 \quad ,$$

for Case 1. On the other hand, for Case 2 it follows

$$\left(w_t - w_t^{(1)}\right) [(\alpha - r_t) - \sigma^2] \doteq -\frac{1}{1 - \omega_t} \left(\frac{\sigma^2}{1 - \omega_t} - \sigma^2\right) = -\frac{\sigma^2 \omega_t}{(1 - \omega_t)^2} \quad .$$

Putting the results together we proved 3. We remark that this last equation also implies an asymptotic contribution of order no less than 2 for the optimal portfolio policies to the drift of the cross sectional wealth dynamics in the Cases 1. and 2., respectively.

To prove 4, it is sufficient to note that the optimal consumption policy under HR and CR are the same, thereby implying the same asymptotics for  $dS_t$  as in Proposition 10. ■

**Proof of Proposition 14.** To prove the proposition, we note that

$$\sqrt{\frac{2\eta}{G_0(X_t)}} = \sqrt{\frac{2\eta}{\sigma_e^2 + \frac{2\delta_{eY}\sigma_e\sigma_Y}{\lambda+\delta} + \left(\frac{\sigma_Y}{\lambda+\delta}\right)^2}} + O^2(\gamma, \sqrt{\eta})$$

holds from Proposition 4, (76), and Proposition 12 (with  $\gamma = \vartheta = 0$ ). Therefore the proof of Proposition 12 applies, readily by replacing  $\vartheta$  with

$$\sqrt{\frac{2\eta}{G_{00}}} := \sqrt{\frac{2\eta}{\sigma_e^2 + \frac{2\delta_{eY}\sigma_e\sigma_Y}{\lambda+\delta} + \left(\frac{\sigma_Y}{\lambda+\delta}\right)^2}} \quad .$$

■

**Proof of Proposition 15.** The value function of the representative log-utility agents is

$$\frac{1}{\delta} (\log(W_0) + g_{0,0}(Y_0)) = E_0 \left[ \int_0^\infty e^{-\delta t} \log(e_t) dt \right] \quad . \quad (86)$$

Computing the expectation on the RHS of (86) it follows

$$\begin{aligned} E_0 \left[ \int_0^\infty e^{-\delta t} \log(e_t) dt \right] &= E_0 \left[ \int_0^\infty e^{-\delta t} \left( \log(e_0) + \left(1 - \frac{1}{2}\sigma_e^2\right) \int_0^t Y_s ds \right) dt \right] \\ &= \log(e_0) \int_0^\infty e^{-\delta t} dt + \left(1 - \frac{1}{2}\sigma_e^2\right) \int_0^\infty e^{-\delta t} \left( \int_0^t E_0(Y_s) ds \right) dt \quad . \end{aligned}$$

Using the formula

$$E_0(Y_s) = \bar{Y} + (Y_0 - \bar{Y})e^{-\lambda s} \quad ,$$

we then have

$$\begin{aligned} E_0 \left[ \int_0^\infty e^{-\delta t} \log(e_t) dt \right] &= \frac{1}{\delta} \ln(e_0) + \left( 1 - \frac{1}{2} \sigma_e^2 \right) \int_0^\infty e^{-\delta t} \left( \bar{Y}t + (Y_0 - \bar{Y}) \int_0^t e^{-\lambda s} ds \right) dt \\ &= \frac{1}{\delta} \ln(e_0) + \frac{1 - \frac{1}{2} \sigma_e^2}{\delta(\delta + \lambda)} \left( Y_0 + \frac{\lambda \bar{Y}}{\delta} \right) \quad . \end{aligned}$$

At the same time,  $W_0$  is the aggregate wealth of the economy, which is equal to the price of the stock,  $\delta^{-1}e_0$ . Thus,

$$g_{0,0}(Y_0) = a + bY_0 + \ln(\delta) \quad ,$$

with

$$a = \frac{\lambda(1 - \frac{1}{2}\sigma_e^2)}{\delta(\delta + \lambda)} \bar{Y} \quad , \quad b = \frac{1 - \frac{1}{2}\sigma_e^2}{\delta + \lambda} \quad .$$

■

**Proof of Proposition 16.** Using the consumption asymptotics (17), with  $g_0$  replaced by  $g_{00}$ , aggregate good markets clearing implies

$$P_t \doteq \frac{e_t}{\delta} (1 + \gamma(a + bY_t)\omega_t) \quad . \quad (87)$$

Defining  $p_t = p(Y_t, \omega_t) = \frac{1}{\delta} (1 + \gamma(a + bY_t)\omega_t)$ , it follows

$$\begin{aligned} \frac{dP_t}{P_t} &\doteq Y_t dt + \gamma b \omega_t \left( dY_t + \left[ dY_t, \frac{de_t}{e_t} \right] \right) + \sigma_e \sqrt{Y_t} dZ_t^e \\ &= (Y_t + \gamma b \omega_t (\lambda(\bar{Y} - Y_t) + Y_t \delta_{eY} \sigma_e \sigma_Y)) dt + \sigma_e \sqrt{Y_t} dZ_t^e + \gamma b \omega_t \sigma_Y \sqrt{Y_t} dZ_t^Y \quad . \end{aligned}$$

The cumulative return then is

$$\begin{aligned} \frac{dP_t + e_t dt}{P_t} &\doteq (Y_t + \gamma b \omega_t (\lambda(\bar{Y} - Y_t) + Y_t \delta_{eY} \sigma_e \sigma_Y)) dt + \delta(1 - \gamma \omega_t(a + bY_t)) dt \\ &\quad + \sigma_e \sqrt{Y_t} dZ_t^e + \gamma b \omega_t \sigma_Y \sqrt{Y_t} dZ_t^Y \\ &\doteq (Y_t + \delta - \gamma b \omega_t Y_t (\lambda + \delta - \delta_{eY} \sigma_e \sigma_Y)) dt + \sigma_e \sqrt{Y_t} dZ_t^e + \gamma b \omega_t \sigma_Y \sqrt{Y_t} dZ_t^Y \\ &= (\delta + Y_t(1 - \gamma b \omega_t (\lambda + \delta - \delta_{eY} \sigma_e \sigma_Y))) dt + \sigma_e \sqrt{Y_t} dZ_t^e + \gamma b \omega_t \sigma_Y \sqrt{Y_t} dZ_t^Y \quad . \end{aligned}$$

This proves 1. To prove 2, remark that  $\sqrt{\frac{\eta}{G_0}} \doteq \sqrt{\frac{\eta}{G_{00}}}$ , where

$$G_{00}(Y) = \phi_{00}^2(Y) + 2\phi_{00}(Y)\delta_{eY}\sigma_Y\sqrt{Y}b + \left(\sigma_Y\sqrt{Y}b\right)^2 = Y(\sigma_e^2 + 2\delta_{eY}\sigma_e\sigma_Yb + \sigma_Y^2b^2) \quad .$$

Financial markets clearing gives

$$\begin{aligned} r_t &\doteq \alpha_t + e_t - \sigma_t^2 + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(\sigma_t^2 + \frac{\partial g_{0,0}(Y_t)}{\partial X} \sigma_{XP}\right) \omega_t \\ &\doteq \alpha_t + e_t - \sigma_t^2 + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) (\sigma_t^2 + b\sigma_{XP}) \omega_t \quad . \end{aligned}$$

Using the above cumulative return dynamics this leads to

$$\begin{aligned} r_t &\doteq (\delta + (1 - \gamma b \omega_t (\lambda + \delta - \delta_{eY} \sigma_e \sigma_Y)) Y_t) - (\sigma_e^2 + 2\sigma_e \gamma b \omega_t \sigma_Y \delta_{eY}) Y_t \\ &\quad + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) (\sigma_e^2 + b\sigma_e \sigma_Y \delta_{eY}) Y_t \omega_t \quad . \end{aligned}$$

which proves 2.

The first part of Claim 3 is based on the optimum consumption asymptotics (8) together with

(15)

$$c_t^{(1)} = \delta \quad , \quad c_t \doteq \delta (1 - \gamma (g_{00}(Y_t) - \log(\delta))) \doteq \delta (1 - \gamma (a + bY_t)) \quad .$$

Moreover,

$$w^{(1)} \doteq \frac{\alpha_t + e_t - r_t}{\sigma_t^2} \doteq 1 - \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(1 + b \frac{\sigma_Y \delta_{eY}}{\sigma_e}\right) \omega_t \quad ,$$

and

$$\begin{aligned} w &\doteq w^{(1)} \left(1 + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right)\right) + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \frac{\partial g_{0,0}(Y_t)}{\partial X} \frac{\sigma_{XP}}{\sigma_t^2} \\ &\doteq 1 + \left(1 + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right)\right) - \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(1 + b \frac{\sigma_Y \delta_{eY}}{\sigma_e}\right) \omega_t + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) b \frac{\sigma_Y \delta_{eY}}{\sigma_e} \\ &\doteq 1 + \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(1 + b \frac{\sigma_Y \delta_{eY}}{\sigma_e}\right) (1 - \omega_t) \end{aligned}$$

This proves 3.

We next consider 4. The equilibrium cross-sectional wealth dynamics implied by Itô's Lemma

for the given exchange economy are of the form

$$\begin{aligned} dw_t &= \omega_t (1 - \omega_t) \left[ (w_t - w_t^{(1)}) [(\alpha_t + e_t - r_t) - \sigma_t^2] - (c_t - c_t^{(1)}) \right] dt \\ &\quad + \omega_t (1 - \omega_t) (w_t - w_t^{(1)}) \sigma_t dZ_t \quad . \end{aligned}$$

Moreover

$$\left(w - w_t^{(1)}\right) \sigma_t \doteq \left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(1 + b \frac{\sigma_Y \delta_e Y}{\sigma_e}\right) \sigma_e \sqrt{Y_t} \quad ,$$

and

$$-\left(c_t - c_t^{(1)}\right) \doteq \delta \gamma (a + b Y_t) \quad .$$

Finally, using the results in the proof of 2 we have

$$\begin{aligned} (\alpha_t + e_t - r_t) - \sigma_t^2 &\doteq -\left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) \left(\sigma_t^2 + \frac{\partial g_{0,0}(Y_t)}{\partial X} \sigma_{XP}\right) \omega_t \\ &\doteq -\left(\gamma - \sqrt{\frac{\eta}{G_{00}}}\right) (\sigma_e^2 + b \sigma_e \sigma_Y \delta_e Y) Y_t \omega_t \quad , \end{aligned}$$

yielding

$$\left(w_t - w_t^{(1)}\right) \left[(\alpha_t + e_t - r_t) - \left(\omega_t w_t + (1 - \omega_t) w_t^{(1)}\right) \sigma_t^2\right] = O^2(\gamma, \sqrt{\eta}) \quad .$$

This proves 4 in Proposition 12. ■

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Kin and Öberg (1996) model dynamics

$$dB_t = -rB_tdt, \tag{1}$$

$$dP_t = -\alpha_tP_tdt + \sigma P_t dZ_t^P, \tag{2}$$

$$dX_t = -\lambda \overline{X} + X_t^\gamma dt + \xi dZ_t^X, \tag{3}$$

where  $r, \sigma, \xi, \lambda, \overline{X} > 0$ , and  $\alpha_t = r + \sigma X_t Y = \overline{X}$ .

Parameter Definition:

$$r = 0.05$$

$$\rho = 0.5$$

$$\delta = 0.06$$

$$\sigma = 0.15$$

$$\xi = 0.037$$

$$\lambda = 0.0423$$

$$Y = 0.0942$$

$$\gamma = 0.05$$

$$\eta = 0.05$$

$$\vartheta = 0.1$$

$$a_0 = \ln(\delta) - 1 + \frac{r}{\delta} + \frac{\xi^2}{2\delta(\delta+2\lambda)} + \frac{(\lambda Y)^2}{\delta(\delta+\lambda)(\delta+2\lambda)}$$

$$a_1 = \frac{\lambda Y}{(\delta+\lambda)(\delta+2\lambda)}$$

$$a_2 = \frac{1}{\delta+2\lambda}$$

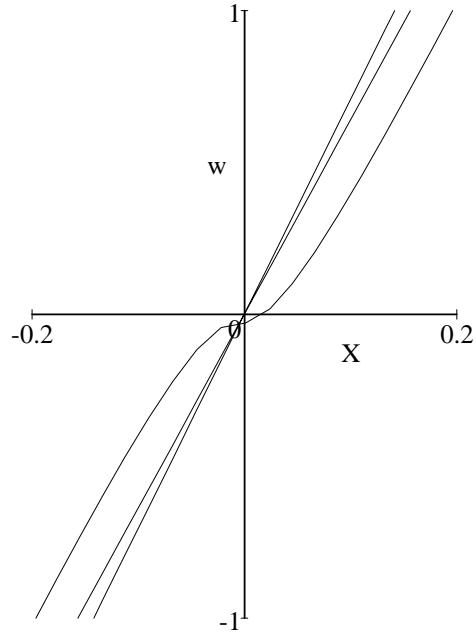


Figure 1: Optimal risky allocations  $w$  as a function of the state  $X$  for the parameter choice  $r = 0.05$ ,  $\rho = 0.5$ ,  $\delta = 0.06$ ,  $\sigma = 0.15$ ,  $\xi = 0.037$ ,  $\lambda = 0.0423$ ,  $Y = 0.0942$ ,  $\gamma = 0.05$ ,  $\vartheta = 0.1$ . The black straight curve is the optimal risky allocation under CR, the black straight line the one under HR, and the black dashed line the one in the absence of a preference for robustness.

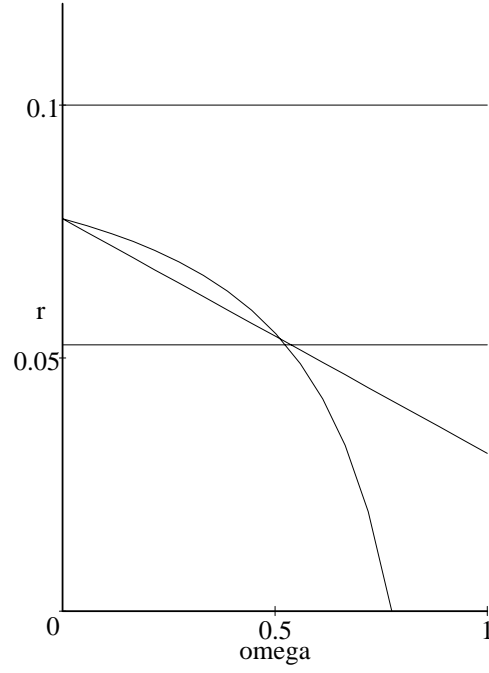


Figure 2: Equilibrium interest rate  $r$  as a function of cross sectional wealth  $\omega$ . The dotted curve represents the equilibrium interest rate function for states  $\omega \in A$ , while the black straight line represents the equilibrium interest rate function for states  $\omega \in A^c$ . The parameter choice is  $\alpha = 0.1$ ,  $\sigma = 0.15$ ,  $\gamma = 0.05$ ,  $\eta = 0.05$ . For this parameter choice the set  $A$  lies to the right of the intersection points of the two equilibrium interest rate functions.