Chapter 7

A REVIEW OF PERTURBATIVE APPROACHES FOR ROBUST OPTIMAL PORTFOLIO PROBLEMS

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Abstract Only a few intertemporal optimal consumption and portfolio problems in partial and general equilibrium can be solved explicitly. It is illustrated in the paper that perturbation theory is a powerful tool for deriving approximate analytical solutions for the desired optimal policies in problems where general state dynamics are admitted and a preference for robustness is present. Starting from the perturbative approach proposed recently by Kogan and Uppal it is demonstrated how robust equilibria for some formulations of a preference for robustness in the literature can be solved. A crucial requirement for this approach is the existence of a known functional form for the candidate model solutions, a condition which is not satisfied by some models of a preference for robustness. For these cases,

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recent results by Trojani and Vanini can be used to obtain a perturbative solution to the Bellman equation of the relevant benchmark model and to give some formal conditions under which the perturbative solution converges to the correct one.

Keywords: Financial Equilibrium, Merton's Model, Model Misspecification, Perturbation Theory, Robust Decision Making.

1. Introduction

It is a well-known feature of financial models that only a few intertemporal optimal portfolio problems on the single agent (partial equilibrium) level can be solved explicitly. On the general equilibrium level even greater difficulties arise, for instance when heterogeneous agents economies are considered.

The class of models that provide analytical solutions is characterized by assumptions on agent's utility functions (like for instant power utility), on the dynamics for asset prices and state variables (as for example a geometric Brownian motion price process), on the existence of intermediate consumption, and on further aspects, like for example the presence of a preference for robustness (cf. Anderson et al. (2000), AHS in the sequel). Typically, it is sufficient to weaken one of these assumptions to loose closed form solutions. This is due to the generic non-linear structure of the optimality conditions implied by the given Hamilton-Jacobi-Bellman (HJB) equation. As a consequence, exact optimal solutions can be rarely obtained. However, both from a theoretical and an applied point of view it is an important issue to characterize optimal decision rules that arise when general dynamic laws for asset prices and state variables are considered, and - for example - when some form of aversion to model misspecification is taken into account by the agent's optimal decisions.

The best that can be done when exact analytical solutions cannot be obtained is to *rely on approximation methods by which approximate analytical expressions can be achieved*. As for the natural sciences, it has been shown recently in Kogan and Uppal (2000) within the setting of standard Merton's (1969, 1971) - type models, that perturbation theory is a powerful approximation method for financial optimal decision making also. Clearly, perturbation theory is not the only approximation technique that can be used in dynamic portfolio optimization. A further one is the approach developed in Campbell (1993), which is based on a log-linearization of the HJB equation[1]. A crucial difference between perturbation theory and the log-linearization approach is that the first yields analytical solutions. Further, perturbative approaches allow for a higher generality of the analysis, permitting rich investment set specifications and admitting a quite large spectrum of portfolio constraints. Finally, a last advantage

[1] See also Campbell and Viceira (1998, 1999) and Chako and Viceira (1999).

is that they can be also used to solve general equilibria in heterogenous agents economies where the opportunity set is endogenous rather than exogenously given. On the other hand, for problems where the consumption to wealth ratio is approximately constant (the assumption on which the log-linearizazion approach is based) a log linearization of the HJB equation can produce more precise approximations, as one would (at least) partly expect[2].

The goal of this paper is to provide an introductory self-contained review on perturbative approaches for solving continuous-time optimal portfolio problems and to illustrate their usefulness with a particular focus on robustness. Control problems where the impact of an aversion to model misspecification is described by a preference for robustness are a natural application field for perturbation theory because there the implied value functions are characterized by the solution of an HJB equation that is parameterized by a single parameter. Indeed, formally many of these models are observationally equivalent to stochastic differential utility (Duffie and Epstein (1992a, 1992b)). An open question is how far perturbation theory can be applied to related approaches like multiple priors recursive utility (Chen and Epstein (2000)) in cases where value functions are characterized by the solution of some backward stochastic differential equation.

Several formulations of a preference for robustness have been proposed recently in the literature. In the sequel we will use the terminology "Minimum Entropy Robustness" (MER, AHS (2000)), "Constrained Robustness" (CR, AHS (1998) , Hansen et al. (2001)) and "Homothetic Robustness" (HR, Maenhout (1999)) to distinguish the different definitions. All these approaches to robustness are based on the idea that economic agents have an approximate benchmark model in mind by which they try to describe the probabilistic features of some underlying state variables processes, like for instance some set of security price processes. At the same time, agents consider in their decisions the possibility that the benchmark model could be bad specified. However, not all possible misspecifications are treated as being equally relevant. On the contrary, model deviations that are viewed as particularly different from the given reference model (typically measured using relative entropy as a measure of discrepancy) are penalized in their impact on the final decision. In all these models, the magnitude of this penalization is parameterized by a parameter that is interpreted as the strength of a preference for robustness[3].

On a formal level, differences between the three above formulations of robustness arise essentially through the way by which model deviations are penalized

[2]Cf. Kogan and Uppal (2000) for a numerical comparison of the accuracy of approximations based on perturbation theory and the log-linearization technique within the model of Chacko and Viceira (1999).

[3]A related approach adopted by Uppal and Wang (2001) and allowing for differences in the degree of robustness/model ambiguity of economic agents is not discussed further in this review.

in optimal decision making. MER penalizes deviations proportionally to their relative entropy with respect to the reference model, while CR puts a maximal bound on the relative entropy of a relevant candidate misspecification. However, as enlightened by Hansen et. al. (2001) MER and CR are closely related by the Lagrange Multiplier Theorem, even if they induce different preference orders[4]. An important economic difference between the two formulations is that CR depicts a form of first order risk aversion while MER (as HR) mimics second order risk aversion. Further, CR is recursive in the sense of Epstein and Chen (2000). Via statistical detection error probabilites AHS (2000) have shown how to determine empirically plausible amounts of the robustness parameter for MER and CR, an important issue for applications. Finally, a weakness of MER and CR appears to be their low analytical tractability. Indeed, so far exact partial and general equilibria have been computed in closed form only for the simplest constant opportunity set Merton (1969, 1971) model using a CR formulation (see Trojani and Vanini (2001)). Numerical partial equilibrium solutions for a stochastic opportunity set CR portfolio problem with predictability are analyzed in Lei (2001). While perturbation theory can be applied to handle also CR-based models (cf. for instance Trojani and Vanini (2001b)), we focus for brevity in our exposition on MER where no exact analytical solutions exist already for the simplest constant opportunity set model. Similarly to MER, the HR formulation penalizes relative entropy of a model deviation. However, in a way that is scaled by the current level of indirect utility and that makes HR observationally equivalent to a well-known form of stochastic differential utility[5]. A nice feature of the scaling factor defining HR is that it yields a higher analytical tractability, because it imposes homogeneity of the implied HJB equation. For instance, robust versions of models in Kim and Omberg (1996) and Chacko Viceira (1999) can be solved explicitly, and the impact of a robust motive for intertemporal hedging can be analyzed in detail.

In this review we demonstrate the usefulness of perturbation theory in deriving approximate analytical expressions for the optimal policies of intertemporal consumption/portfolio problems where general state dynamics are admitted and a preference for robustness is present. Starting from the perturbative approach of Kogan and Uppal (2000), we explain how first order approximations for the relevant optimal policies are obtained. That for we perturb the partial equilibrium solution of a log utility investor who completely believes in the given model dynamics (that is, an investor having no preference for robustness). In general equilibrium, these approximations are derived by perturbing a bench-

[4]In fact, under regularity conditions the optimal controls implied by MER can be expressed in terms of the solution to a corresponding CR problem, and vice versa.

[5]Maenhout (1999) shows that HR can be interpreted as increasing risk aversion, without changing preferences for intertemporal substitution.

mark economy where homogeneous log utility agents have no preference for robustness. Hence, in our review we focus on applications and extensions of Kogan and Uppal's (2000) approach to situations where preferences for robustness and the joint impact of aversion to risk and to model misspecification are considered.

As mentioned, HR imposes homotheticity of the arising value function in the corresponding robust control problem. Therefore, it allows for a direct perturbative solution where Kogan and Uppal's (2000) approach is applied simply by expanding the optimal policies with respect to the risk aversion and the robustness parameter (rather than only with respect to the first one). On the other hand, a more indirect approach is needed for the minimum entropy situation, where no exact explicit expressions for the optimal consumption and portfolio policies are known, even for the simplest constant opportunity set case. For this case a perturbative approximation to the HJB equation for the relevant benchmark model has to be supplied first. This first step is achieved by results in Trojani and Vanini (2001a). From an economic point of view this methodological exercise yields analytical expressions for the impact of a preference for robustness on partial and heterogeneous-agents general equilibria of models based on general state dynamics and including intermediate consumption. Indeed, the analytical approximations obtained apply to a broader class of robust dynamic models than those analyzed previously in the literature. For HR they permit the analysis of more general partial equilibria than those explicitly solved in Maenhout (1999) and they are useful for the analysis of heterogeneous agents continuous-time robust general equilibria, a topic that has not been largely investigated so far. For MER perturbation theory provides some first analytical partial and general equilibrium descriptions of the fundamental properties of AHS's model.

A successful rigorous perturbation theory for general financial problems has to fulfill (at least) the following requirements:

- The errors implied by a given approximation method must be (formally) quantifiable,
- Convergence of perturbation theory up to all orders has to be (formally) investigated in order to prove existence of a candidate solution.

This review focuses on an informal presentation of the basic ideas behind perturbative approaches when applied to a few models of financial robust decision making. Therefore, a formal complete analysis of these issues is behind the goal of this paper[6]. However, based on results in Trojani and Vanini (2001a) we discuss briefly some of these aspects in Section 3.5 (Theorem 3 and 4), within

[6]General results on the convergence of the approximate solution to the true solution are not currently available; for a discussion of issues related to convergence we refer to Judd (1996, 1998).

the simplest (constant-opportunity-set) AHS model. For formal proofs of all theorems and propositions in this review and for some more detailed economic interpretation of robustness we refer to the original papers in the references.

The paper is organized as follows. Section 2 synthesizes the elegant and ingenious approach of Kogan and Uppal (2000), while Section 3 introduces robustness into the analysis by describing first the direct extension of this methodology to HR. In a second step, we present the perturbative approach in Trojani and Vanini (2001a) which provides a solution way also for the model by AHS (2000). Section 4 presents perturbative solutions for a general equilibrium heterogeneous agents economy where risk aversion and preferences for robustness interplay in determining assets prices; more details on this topic can be found in Trojani and Vanini (2001b). Section 5 concludes and summarizes.

2. Standard Partial Equilibrium Problems

We start by presenting the basic ideas in Kogan and Uppal's (2000) paper that will be extended to take robustness into account.

2.1 Preferences and Objective Functions

Consider economic agents with constant relative risk aversion utility functions u of current consumption C_t

$$u(C_t) = \frac{C_t^{\gamma} - 1}{\gamma} \quad , \quad \gamma < 1 \quad . \tag{7.1}$$

For $\gamma \to 0$ the log utility case is obtained

$$\log(C_t) = \lim_{\gamma \to 0} \frac{C_t^{\gamma} - 1}{\gamma} \quad . \tag{7.2}$$

Our goal is to analyze the optimal intertemporal portfolio/consumption behaviour of investors with utilities of the form (7.1). For a given time preference rate $0 < \rho \leq 1$ (measuring depreciation of utility of consumption over time) we consider infinite horizon problems with objective functions of the form

$$V_{\gamma}(W,X) = E\left[\int_0^{\infty} e^{-\rho t} \frac{C_t^{\gamma} - 1}{\gamma} dt\right] \quad . \tag{7.3}$$

Expectations in (7.3) are taken with respect to the joint law of two state variables processes $(W_t, X_t)'$ that are defined precisely below. Again, the corresponding problem of a log-utility investor is obtained by letting $\gamma \to 0$.

2.2 Opportunity Set

There are two assets, a risky and a riskless asset with price P_t and B_t at time t, respectively, defined by the dynamics

$$dB_t = r_t B_t dt \quad , \tag{7.4}$$
$$dP_t = \alpha_t P_t dt + \sigma_t P_t dZ_t \quad . \tag{7.5}$$

The drift and volatility $\alpha_t = \alpha(X_t)$ and $\sigma_t = \sigma(X_t)$ as well as the short rate $r_t = r(X_t)$ define the (stochastic) *opportunity set* of our agents, which is assumed to be generated by a state variable process[7] (X_t) with dynamics

$$dX_t = \zeta(X_t)\, dt + \xi(X_t)\, dZ_t^X \quad , \tag{7.6}$$

where (Z_t) and $\left(Z_t^X\right)$ are both standard Brownian motions in $\mathbb{R}$, having joint covariation $\sigma_{Z^X Z} dt = E\left(dZ_t^X dZ_t\right)$, and $\zeta(X_t), \xi(X_t) \in \mathbb{R}$. We further denote by $\sigma_{XP} = \sigma\xi\sigma_{Z^X Z}$ the covariation of risky assets returns and state variables. Each agent in the model allocates at each date t a fraction w_t of current individual wealth W_t to risky assets, yielding the individual current wealth dynamics

$$dW_t = \left[w_t W_t (\alpha_t - r_t) + (r_t W_t - C_t)\right] dt + w_t W_t \sigma_t dZ_t \quad . \tag{7.7}$$

The next section introduces the standard consumption/portfolio optimization problem of an investor in Merton's (1969, 1971) model.

2.3 Single-Agent Optimization Problems

Let u be a utility function of the form (7.1). Each agent in the model solves the intertemporal optimization problem

$$(P) \quad : \quad \begin{cases} J(W,X) = \sup_{C,w} E\left[\int_0^\infty e^{-\rho t} u(C_t) dt\right] \\ \text{s.t.} \quad (7.6) \quad \text{and} \quad (7.7) \end{cases} \quad . \tag{7.8}$$

Hence, preferences and price processes on risky assets are exogenously given for the investor, who acts as a price taker optimizing lifetime expected utility of consumption. Defining by $c = \frac{C}{W}$ the consumption to wealth ratio, the HJB equation for the value function J in (P) is

$$0 = \sup_{c,w} \left\{ \frac{(Wc)^\gamma - 1}{\gamma} - \rho J + A_W J + A_X J + wW\sigma_{XP} \frac{\partial^2 J}{\partial W \partial X} \right\} , \tag{7.9}$$

[7] Prominent examples of such state variables in the context of optimal portfolio choice are economic variables describing the evolution of some potential risk factors.

where A_W, A_X, are the generators of the wealth and state dynamics (7.7), (7.6), respectively,

$$A_W = (r + w(\alpha - r) - c)\, W \frac{\partial}{\partial W} + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2}{\partial W^2} \quad , \tag{7.10}$$

$$A_X = \zeta \frac{\partial}{\partial X} + \frac{1}{2} \xi^2 \frac{\partial^2}{\partial X^2} \quad . \tag{7.11}$$

Clearly, appropriate boundary conditions on the value function have to be imposed in order to obtain well-defined solutions to (7.9). However, (P) can in principle be solved by the following procedure:

- First, by formally differentiating (7.9), the optimal policy candidates c, w are derived. At this stage, they are both functions of the unknown solution J.

- Second, insert the optimal policy candidates into the HJB equation (7.9). This leads to a non-linear partial differential equation for J (excluding trivial cases).

- Third, the partial differential equation thereby obtained has to be solved in order to obtain the value function solution J and the implied optimal rules from the first step.

As a matter of fact, the last step in this procedure can be carried-out explicitly only for a very limited class of problems[8].

2.4 Perturbative Solutions Approach

The crucial idea behind the perturbative approach in Kogan and Uppal (2000) for computing the optimal policies implied by (7.9) is as follows. Suppose first a constant opportunity set. Homogeneity of the utility function (7.1) and of the generator for the wealth-dynamics (7.7) implies that the functional form

$$J(W) = \frac{1}{\rho} \frac{\left(e^{g(\gamma)} W\right)^{\gamma} - 1}{\gamma} \quad , \tag{7.12}$$

[8] A second remark concerns the implicit differentiability assumptions. Suppose a sufficiently smooth candidate solution was found using the formal approach described above. Then, using the verification theorem, one can prove rigorously that the candidate solution is indeed a solution. If the value function is not differentiable, the formal approach is no longer meaningful. If this happens, one has to consider the viscosity solutions approach. This case arises for example in financial applications where transaction costs are considered.

is appropriate for solving (7.9)[9]. When the opportunity set is stochastic, homogeneity suggests a ”state dependent” functional form for (7.12) given by

$$J(W,X) = \frac{1}{\rho} \frac{\left(e^{g(\gamma,X)} W\right)^{\gamma} - 1}{\gamma} \quad , \tag{7.13}$$

where g is a function thas has to be determined. If we seek for an exact solution of (7.8), this functional form has to be inserted in the HJB equation (7.9) in order to start the formal procedure outlined above. However, the implied differential equation for the unknown function g is most of the times not solvable explicitly. Since $g(\gamma,X)$ can be not computed generally in closed form, Kogan and Uppal (2000) propose to approximate the implied optimal policies by expanding g in powers of a suitable parameter. In the present setup, a natural choice for this parameter is the risk aversion index γ

$$g(\gamma,X) = g_0(X) + \gamma g_1(X) + O(\gamma^2) \quad . \tag{7.14}$$

Notice that by construction[10] g_0 is implied by the value function solution for the stochastic opportunity set problem (7.8) of a log-utility agent

$$J_{\log}(W,X) = \frac{1}{\rho}(\ln(W) + g_0(X)) \quad . \tag{7.15}$$

At this point, not much seems to be gained. Even worse: it seems that we now have to determine two functions g_0, g_1, instead of a single function g. However, the key point for the analysis to follow is that g_0 *can be often obtained analytically from the value function solution of a log utility agent*[11]*, while* g_1 *is of second order in* γ *and therefore can be neglected in first order analysis.* To see this, differentiate first the HJB equation (7.9) using the functional form

[9]Indeed, it can be easily verified (cf. also Merton (1969, 1971)) that (7.12) solves (7.9) for an appropriate contant $g(\gamma)$.

[10]This is easily implied by the limit:

$$\frac{1}{\rho} \frac{\left(e^{g(\gamma,X)} W\right)^{\gamma} - 1}{\gamma} \to \frac{1}{\rho}(\ln(W) + g_0(X)) \quad ,$$

as $\gamma \to 0$.

[11]Which is typically easier to compute than that of the original (power utility) problem.

(7.13). This gives the policy candidates[12]

$$c(X) = \left(\frac{1}{\rho}e^{\gamma g(\gamma,X)}\right)^{\frac{1}{\gamma-1}} \quad , \tag{7.16}$$

$$w(X) = \frac{1}{1-\gamma}\frac{\alpha - r}{\sigma^2} + \frac{\gamma}{1-\gamma}\frac{\partial g(\gamma,X)}{\partial X}\frac{\sigma_{XP}}{\sigma^2} \quad . \tag{7.17}$$

Further, insert (7.14) in (7.16), (7.17), and expand the implied expressions up to first order in γ. It then follows

$$c(X) = \rho\,(1-\gamma(g_0(X) - \ln(\rho))) + O(\gamma^2) \quad , \tag{7.18}$$

$$w(X) = \frac{\alpha - r}{\sigma^2}(1+\gamma) + \gamma\frac{\partial g_0(X)}{\partial X}\frac{\sigma_{PX}}{\sigma^2} + O(\gamma^2) \quad . \tag{7.19}$$

This shows that g_1 does not contribute to the optimal policies up to first order in γ.

As a consequence, it is sufficient to compute g_0 in order to determine (7.18), (7.19), completely. Since this function is fully determined by the solution of the log-utility version of (P), having determined the value function $J_{\log}(W,X)$ for this problem already gives g_0 (the last unknown in the approach) from (7.15). Formally, $J_{\log}$ is defined by

$$J_{\log}(W,X) = \left\{ \begin{array}{l} \sup_{C_t,w_t} E\,[\int_0^\infty e^{-\rho t}\log(C_t)dt] \\ \quad \text{s.t.} \quad (7.6) \quad \text{and} \quad (7.7) \end{array} \right. \quad .$$

Using the perturbed policies (7.18), (7.19), for a log utility agent ($\gamma = 0$), together with the solution to the linear wealth dynamics (7.7) it then follows

$$\begin{aligned} J_{\log}(W,X) &= E\left[\int_0^\infty e^{-\rho t}\ln(\rho W_t)dt\right] \\ &= E\left\{\int_0^\infty e^{-\rho t}\ln\left[\rho W \exp\left(\int_0^t \Psi_s ds + \int_0^t \Phi_s dZ_s\right)\right]dt\right\} \end{aligned}$$

where $\Psi_s = -\frac{1}{\rho} + r_s + \frac{1}{2}\left(\frac{\alpha_s - r_s}{\sigma_s}\right)^2$ and $\Phi_s = \frac{\alpha_s - r_s}{\sigma_s}$. A final partial integration identifies g_0 as

$$g_0(X) = \ln(\rho) - 1 + E\left\{\int_0^\infty e^{-\rho t}\left[r_t + \frac{1}{2}\left(\frac{\alpha_t - r_t}{\sigma_t}\right)^2\right]dt\right\} \quad . \tag{7.20}$$

[12]Notice that c and w are wealth independent. The term

$$\frac{1}{1-\gamma}\frac{\sigma_{XP}}{\sigma^2}\frac{\partial g(\gamma,X)}{\partial X}$$

in (7.17) is a standard intertemporal hedging demand by which agents hedge against changes in the stochastic opportunity set.

After having specified the state variables and risky asset price dynamics (7.6), (7.5), explicitly, it is then possible to evaluate (7.20) in some cases analytically.

Summarizing, the above perturbative approach for computing the desired optimal policies works as follows.

1 Identify a set of parameters that parameterize the problems under scrutiny (the parameter γ in the above discussion) and a specific parameter value for which the value function solution is known explicitly (the parameter value $\gamma = 0$ in the above discussion).

2 Determine a functional form for the solution candidate (expression (7.13) above) such that to first order only functions of the solvable benchmark model enter in the optimality conditions (the function g_0 above).

3 Compute the optimal policies from the optimality conditions for the given problem using the functional form of step 2 (policies (7.16) and (7.17)).

4 Expand the implied optimal policies to first order, determine the value function for the explicitly solvable model and compute the corresponding expectations (expressions (7.18), (7.19), (7.20)).

Hence, two requisites are necessary for the above procedure:

- Existence of an explicitly solvable model within the given model parameterization,

- Existence of a functional form for the parameterized candidate model solutions.

These restrictions can be rather severe. However, in some cases analytical solutions are obtained for problems where otherwise only numerical solutions are available.

3. Robust Partial Equilibrium Problems

We now consider situations where economic agents have some doubts on the specification (7.5), (7.6), for asset prices and state variables dynamics. They rather treat model (7.5), (7.6) as an approximate description of a reality where model deviations can always be present. In the sequel we will therefore call (7.5), (7.6) the ”reference model” of our robust agents.

Given this cautious perception of the reality, the goal in robust portfolio/consumption decision making is to develop policies that perform well not only at the given reference model but also across a set of competing relevant specifications. This leads naturally to embed robust portfolio selection into some kind of max-min expected utility theory (cf. Gilboa and Schmeidler (1989)) using optimization objectives of the form

$$\inf_{Q\in\mathcal{Q}} E_Q\left[\int_0^\infty e^{-\rho t}\frac{C_t^\gamma-1}{\gamma}dt\right] \quad . \tag{7.21}$$

These criteria extend (7.3) to consider situations where model uncertainty in the form of a set $\mathcal{Q}$ of canditate relevant models (containing the reference model probability) is present. Criteria like (7.21) appear as objective function also in recursive multiple priors utility (cf. Chen and Epstein (2000)) and robust control theory. These two approaches are based on a similar motivation but differ crucially in the way by which $\mathcal{Q}$ is specified and in the corresponding behavioural implications[13]. In the first one, $\mathcal{Q}$ is constructed explicitly through the definition of an appropriate (rectangular) set of density generators. In the second approach $\mathcal{Q}$ is parameterized typically only implicitly, through some positive penalty parameter that penalizes a statistical perturbation of the reference model probability implied by (7.5), (7.6); see below. This parameter parameterizes in a parsimonious way a (one parameter) set of model misspecifications with quite rich alternative dynamics. As a consequence, robust control problems are particulartly adequate for a perturbative solution approach that perturbs the standard (non robust) model solution with respect to this parameter.

This review focuses on perturbative solutions of HJB equations implied by some robust control problems that have been recently formulated for a few models of robust intertemporal consumption/portfolio choice. Some specific objects and definitions used in robust control theory are now shortly introduced. More details are given in Hansen, Sargent and Tallarini (1999), AHS (2000) and Hansen et al. (2001) for the basic theory, and Maenhout (1999) and Trojani and Vanini (2001) for the more specific optimal consumption-portfolio perspective.

3.1 Model Misspecifications and Measures of Model Discrepancy

For a positive random variable ν such that $E(\nu)=1$ denote by $(\varsigma)_{t\geq 0}$ the martingale process obtained by setting $\varsigma_t := E(\nu|\mathcal{F}_t)$, the conditional expectation of ν conditionally on the information $\mathcal{F}_t$ generated by the current wealth and state variables dynamics up to time t. By the Markov property we write without loss of generality $\varsigma_t=\varsigma_t(W_t,X_t)$. Contaminations of the dynamics (7.5), (7.6) are described by a family $(T_t^\nu)_{t\geq 0}$ of conditional expectation operators defined

[13] While the preferences implied by multiple priors recursive utility are recursive those behind robust control theory are not in the usual sense. They justify a recursive solution by relating a solution of a date zero commitment game to a Markov perfect game in which the decision of both agents are functions of the underlying state vector. See Chen and Epstein (2000), Epstein and Schneider (2001, 2001a) and Hansen et al. (2001) for more details on this point.

by the distortion law

$$T_t^{\nu}(\phi)(W,X) = E\left[\varsigma_t \phi(W_t, X_t) \,|W_0 = W, X_0 = X\right] \quad , \tag{7.22}$$

for suitable functions $\phi : \mathbb{R}^2 \to \mathbb{R}$. Hence, a contaminating model ν consists simply of an absolutely continuous change of measure with respect to the initial reference model probability. By Girsanov Theorem the class of dynamic misspecifications induced by model contaminations of the form (7.22) are therefore mispecifications of the drift in (7.5), (7.6)[14].

Naturally, model misspecifications that are a priori less easily detectable should induce a more cautious behaviour of a robust economic agent. As mentioned in the introduction, this aspect is taken into account in robust control theory making use of relative entropy as a measure of discrepancy between the reference model and a model contamination[15]. Given a candidate contaminated model ν, relative entropy $I_t(\nu)$ at time t is defined by

$$I_t(\nu)(W,X) = E\left(\varsigma_t \cdot \log(\varsigma_t) | W_0 = W, X_0 = X\right) \quad . \tag{7.23}$$

I_t is not a metric. However, it measures the discrepancy of the finite dimensional densities under scrutiny by the information inequality (cf. for instance White (1996))[16].

3.2 Preferences for Robustness and Objective Functions

Preferences for robustness are modelled by introducing a pessimistic view in the computation of the current certainty equivalent of future indirect utility of consumption of a robust agent. Pessimism is embodied by a max-min expected utility approach where a malevolent player (nature, say) selects a worst case model ν^{wc} from the set of model misspecifications (7.22) that a robust decision maker considers as relevant. The set of misspecifications relevant to a robust decision maker is constrained by a (possibly state dependent) penalization parameter that penalizes misspecifications ν with ”particularly high” relative entropy (7.23).

Specifically, let $\psi : \mathbb{R} \to \mathbb{R}^+$ be a positively valued function and define $I_t'(\nu) = \frac{\partial}{\partial t} I_t(\nu)$. We consider worst case (robust) objective functions (com-

[14] Under regularity conditions, explicit expressions for the drift under the misspecified model are obtained by representing the process $(\varsigma)_{t\geq 0}$ as an exponential martingale.

[15] This incorporates an asymmetric treatment which embodyies the idea that a robust decision maker tendentially believes to the given reference model.

[16] Further $I_t(\nu)$ can be interpreted as the mean surprise experienced (over the time period $[0,t]$) when believing that (7.5), (7.6), describe the model dynamics and being informed that in fact these are described by a contaminated model ν; cf. Renyi (1961, 1971) for a deeper discussion of this point. Finally note that $I_t(\nu) = 0$ for all $t \geq 0$ if and only if ν is equal to 1 everywhere.

pare also with (7.3)) of the form

$$V_{\gamma,\vartheta}(W,X) = \inf_{\nu}\left[E^{\nu}\left(\int_0^{\infty}\exp(-\rho t)\left(u(C_t)+\frac{1}{\psi_t}I_t'(\nu)\right)dt\right)\right], \tag{7.24}$$

where

$$\psi_t = \psi\left(V_{\gamma,\vartheta}(W_t,X_t)\right) ,$$

and expectations $E^{\nu}(\cdot)$ are with respect to the joint law for wealth and state variables induced by a model contamination ν through (7.5), (7.6) and (7.22). The infimum with respect to ν in (7.24) determines a worst case model ν^{wc} and a corresponding worst case (pessimistic) expected utility of lifetime consumption.

In this formulation ψ_t depicts a (possibly state dependent) preference for robustness. Choosing

$$\psi_t = \vartheta \quad , \quad \vartheta > 0 \quad , \tag{7.25}$$

yields the minimum-entropy objective function

$$V_{\gamma,\vartheta}(W,X) = \inf_{\nu}\left[E^{\nu}\left(\int_0^{\infty}e^{-\rho t}\left(u(C_t)+\frac{1}{\vartheta}I_t'(\nu)\right)dt\right)\right] \quad , \tag{7.26}$$

in AHS (2000). In this case preferences for robustness are state independent. The higher ϑ the stronger the preference for robustness, respectively the higher the aversion to model misspecification. Indeed, when $\vartheta \to \infty$ the solution of the infimization in (7.26) is a "worst case model" (ν^{wc} say) yielding the lowest conditional expectation on future indirect utility over all possible absolutely continuous contaminations (7.22) of the given reference model. On the other hand, when $\vartheta \to 0$ this yields a worst case model with lowest possible relative entropy. That is, a model with transition densities that are equal to that of the given reference model. Hence, (7.26) covers the objective function (7.3) as a particular case which arises as the limit case $\vartheta \to 0$. As discussed by Hansen et. al (2001) the preferences represented by criteria of the type (7.26) are recursive in a non-standard sense if a Bellman-Isaacs condition is satisfied[17]. This condition defines a Bellman equation for a two-player zero-sum game (between a robust agent and nature, say) in which both players decide at time 0 and recursively and it is needed to relate solutions of a date zero commitment two-agents game to a Markov perfect game where the decision rules of both agents are functions of the underlying state vector. For criteria of the type

[17]Cf. also Fleming and Souganidis (1989).

(7.26) the Bellman-Isaacs condition is equivalent to finding a solution for the HJB equation (7.38) below in this review[18].

A scaled version of (7.26) is obtained by setting:

$$\psi_t = \frac{\vartheta}{\gamma V_{\gamma,\vartheta}(W_t, X_t) + \frac{1}{\rho}} \quad , \quad \vartheta > 0 \qquad . \tag{7.27}$$

This formulation of a preference for robustness arises naturally if homogeneity of the HJB equation for optimum consumption-investment problems with power utility functions is imposed. At variance with the minimum entropy case above, the homogeneity of the implied HJB equation permits to determine analytical solutions for the value function of virtually all optimum portfolio problems where explicit solutions for the standard (expected utility) formulation exist. Therefore, HR is by construction analytically more tractable than MER. The implied objective function is given by

$$V_{\gamma,\vartheta}(W,X) \quad = \quad \inf_{\nu} \left[E^{\nu} \left(\int_0^{\infty} e^{-\rho t} \left(u(C_t) + \frac{\gamma V_{\gamma,\vartheta} + \frac{1}{\rho}}{\vartheta} I_t'(\nu) \right) dt \right) \right] \quad . \tag{7.28}$$

Here, a preference for robustness is state-dependent since it is inversely related to the current level of lifetime indirect utility in the given state of the world. Conditionally on the realized state, the interpretation of the parameter ϑ is, on a pure formal level, the same as for the non homothetic case above. Finally, notice that while the interpretation of the parameter ϑ in (7.28) is natural based on the given functional form and compared to (7.26), a more detailed analysis of the preference structure implied by objectives of the type (7.28) has not been pursued yet in the literature (to our knowledge). Specifically, a discussion of the sense by which these preferences can be recursive - in a similar vain as for instance in Hansen et al. (2001) - is absent[19].

---

[18]Notice that in general the value function implied by these stochastic control problems will satisfy the given HJB equation in a weak viscosity sense (cf. Fleming and Souganidis (1989)). Existence of a classical solution will typically require specific arguments and model assumptions, as for instance for the MER solutions obtained below in Theorem 3 and 4.

[19]A starting point to this problem could be to formulate a type of Bellman-Isaacs condition for the HR formulation and to analyze if it is sufficient to relate solutions of a date zero commitment two-agents game to a Markov perfect game in which the decision rules of both agents are functions of the underlying state vector. However, a detailed discussion of these aspects is clearly behind the goal of this review.

3.3 Robust Single-Agent Optimization Problems

Each agent solves a worst case (robust) optimization problem of the form (cf. also problem (P))

$$\begin{cases} J(W,X) = \sup_{C,w} \inf_{\nu} \left[E^{\nu} \left(\int_0^{\infty} e^{-\rho t} \left(u(C_t) + \frac{1}{\psi(J)} I_t'(\nu) \right) dt \right) \right] \\ \text{s.t.} \quad (7.6) \quad \text{and} \quad (7.7) \end{cases} . \tag{7.29}$$

Hence, the HJB characterization of the value function J in continuous time robust models reads

$$0 = \sup_{c,w} \inf_{\nu>0} \left\{ \frac{(Wc)^{\gamma}-1}{\gamma} - \rho J + A_W^{\nu} J + A_X^{\nu} J + wW\sigma_{XP} \frac{\partial^2 J}{\partial W \partial X} + \frac{1}{\psi(J)} \cdot I_t'(\nu) \right\} , \tag{7.30}$$

where A_W^{ν}, A_X^{ν} are the generators for the asset prices and state variables dynamics under the law induced by (7.22)[20]. This equation represents the zero-sum game between a malevolent player (selecting the worst case model ν^{wc}) and a robust agent (choosing optimal consumption and investment rules C, w) who is rationally taking into account the possibility that the first agent will indeed hurt her by selecting a least favourable model from the set of relevant model misspecifications.

Calculating A_W^{ν}, A_X^{ν}, (cf. also AHS (2000)) and solving for the implied worst case model ν^{wc}, the HJB equation (7.30) is equivalent to the *single agent* HJB (cf. also (7.9)):

$$0 = \sup_{c,w} \left\{ \frac{(cW)^{\gamma}-1}{\gamma} - \rho J + A_W J + A_X J + wW\sigma_{XP} \frac{\partial^2 J}{\partial W \partial X} - \frac{\psi(J)}{2} \left(\left(wW\sigma \frac{\partial J}{\partial W} \right)^2 + \left(\xi \frac{\partial J}{\partial X} \right)^2 + wW\sigma_{XP} \frac{\partial J}{\partial W} \frac{\partial J}{\partial X} \right) \right\} . \tag{7.31}$$

Given a set of appropriate boundary conditions, this equation can be in principle solved using the procedure outlined at the end of Section 2.3. However, this is again explicitly possible only for a very limited number of special cases. We therefore rely on perturbation theory to derive in the next sections approximate solutions for the implied optimal rules.

[20]Remember that by Girsanov Theorem the given model contaminations simply modify the drift of the joint process for asset prices and state variables.

3.4 Perturbative Solutions Approach: Homothetic Preferences for Robustness

Due to the homogeneity of the implied value function Kogan and Uppal's (2000) approach can be applied to solving the HR case with only a few slight modifications. Indeed, homogeneity suggests that a functional form of the type (7.13) should be appropriate to develop a perturbative approach similar to that used for the standard (non robust) case. However, in the present situation the class of problems under scrutiny is parameterized by two parameters (γ and ϑ) rather than by only one as in non robust problems. Therefore, the function g in (7.13) has to depend in the robust setting on both γ and ϑ.

The HJB equation (7.31) for the HR formulation reads (using subscripts to denote partial derivatives with respect to the relevant argument)

$$\begin{aligned} 0 &= \sup_{c,w}\left\{ \frac{(cW)^{\gamma}-1}{\gamma} - \rho J + (wW(\alpha-r)+(rW-cW))J_W \right. \\ &\quad + \frac{1}{2}w^2W^2\sigma^2\left(J_{WW} - \frac{\vartheta J_W^2}{\gamma J+\frac{1}{\rho}}\right) + \zeta J_X \\ &\quad \left. + \frac{1}{2}\xi^2\left(J_{XX} - \frac{\vartheta J_X^2}{\gamma J+\frac{1}{\rho}}\right) + wW\sigma_{XP}\left(J_{XW} - \frac{\vartheta J_W J_X}{\gamma J+\frac{1}{\rho}}\right)\right\}. \end{aligned}$$

To start the perturbative approach, the homogeneity of J implies the following functional form for a candidate solution

$$J(W,X) = \frac{1}{\rho}\cdot\frac{\left(e^{g(\gamma,\vartheta,X)}W\right)^{\gamma}-1}{\gamma} \quad . \tag{7.32}$$

As usual the exact optimal policies are obtained by the first order conditions implied by the corresponding HJB equation, using (7.32)

$$c(W,X) = \left(\frac{e^{\gamma g(\gamma,\vartheta,X)}}{\rho}\right)^{\frac{1}{\gamma-1}}, \tag{7.33}$$

$$w(W,X) = \frac{1}{1-\frac{\vartheta}{\gamma-1}}\cdot\left(\frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2} + \frac{\gamma-\vartheta}{1-\gamma}\frac{\partial g(\gamma,\vartheta,X)}{\partial X}\cdot\frac{\sigma_{XP}}{\sigma^2}\right). \tag{7.34}$$

In the second step, g is expanded up to first order in the risk aversion parameter γ *and* the robustness parameter ϑ

$$g(X) = g_0(X) + \gamma g_1(X) + \vartheta g_2(X) + O\left(\|(\gamma,\vartheta)\|^2\right) \quad . \tag{7.35}$$

$g_0(X)$ is again obtained from the value function of an agent with logarithmic utility ($\gamma \to 0$) and no preference for robustness ($\vartheta \to 0$)[21]. Hence, the model that is being perturbed is the same as in the standard (non robust) case discussed above. The next statement summarizes the result relevant for our exposition.

Theorem 2. *The asymptotic expansions for the optimal policies of an homothetically robust agent are:*

$$c(W,X) = \rho(1-\gamma(g_0(X)-\ln(\rho))) + O\left(\|(\gamma,\vartheta)\|^2\right), \quad (7.36)$$

$$w(W,X) = \frac{\alpha-r}{\sigma^2}(1+\gamma-\vartheta) + (\gamma-\vartheta)\frac{\partial g_0(X)}{\partial X}\cdot\frac{\sigma_{XP}}{\sigma^2} + O\left(\|(\gamma,\vartheta)\|^2\right). \quad (7.37)$$

Notice that again only g_0 appears in the optimal rules. Hence, the choice of the functional form (7.32) to develop a perturbative approach turns out to be successful.

3.5 Perturbative Solutions Approach: Minimum Entropy Preferences for Robustness

The HJB equation (7.31) for the minimum entropy formulation reads:

$$0 = \sup_{c,w}\left\{\frac{(cW)^\gamma-1}{\gamma} - \rho J + (wW(\alpha-r)+(rW-cW))J_W + \frac{1}{2}w^2W^2\sigma^2(J_{WW}-\vartheta\cdot J_W^2) + \zeta J_X + \frac{1}{2}\xi^2(J_{XX}-\vartheta J_X^2) + wW\sigma_{XP}(J_{XW}-\vartheta J_W J_X)\right\}, \quad (7.38)$$

with optimal robust policies

$$c = \frac{(J_W)^{\frac{1}{\gamma-1}}}{W}, \quad (7.39)$$

$$w = -\frac{J_W}{\left(1-\vartheta\cdot\frac{J_W^2}{J_{WW}}\right)WJ_{WW}}\left(\frac{\alpha-r}{\sigma^2}+\frac{J_{WX}}{J_W}\frac{\sigma_{XP}}{\sigma^2}-\vartheta J_X\frac{\sigma_{XP}}{\sigma^2}\right), \quad (7.40)$$

[21] This is again easily implied by the limit:

$$\frac{1}{\rho}\frac{\left(e^{g(\gamma,\vartheta,X)}W\right)^\gamma-1}{\gamma} \to \frac{1}{\rho}(\ln(W)+g_0(X)),$$

as $\gamma,\vartheta \to 0$.

that follow as usual.

At variance with the HR case where explicit solutions are known for a few models, for this nonlinear partial differential equation no explicit solutions are currently known when $\gamma \neq 0$, already in the simplest constant opportunity set situation[22]. Specifically, the difficulty in finding a solution to this problem derives from the fact that a candidate solution has to be nonhomogeneous in current wealth W_t, implying that the functional form (7.32) is not adequate for the setting of this section. To obtain a functional form on which Kogan and Uppal's (2000) approach can be applied in the minimum entropy situation we start from results in Trojani and Vanini (2001a) which characterize the perturbative solution of the HJB equation (7.38) for the simplest constant-opportunity-set AHS (2000) model. Specifically, let

$$\alpha_t = \alpha \quad , \quad r_t = r \quad , \quad \sigma_t = \sigma \quad , \tag{7.41}$$

be constant. This gives a constant-opportunity set version of the HJB equation (7.38) with candidate solution J, say. The perturbative solution approach to this problem is based on a power series of the form:

$$J(W) = \sum_{i=0}^{\infty} \frac{\vartheta^i}{i!} J^{(i)}(W) \quad , \tag{7.42}$$

with a hierarchy $\left(J^{(i)}\right)_{i\in\mathbb{N}}$ of functions that are determined recursively starting from the zero order term $J^{(0)}$, the well-known solution in a standard (non robust) Merton's-type model. Theorem 1 in Trojani and Vanini (2001a) shows that each function in the hierarchy (7.42) has to be a solution to a second order Euler equation with an homogenous part that is invariant with respect to the stage i of the hierarchy. Moreover, computing the first order term $J^{(1)}$ explicitly the following result is obtained[23].

Theorem 3. *If a classical solution for the robust HJB equation (7.38) under assumption (7.41) and under appropriate boundary conditions exists, it is given by*

$$J = J^{(0)} + \vartheta J^{(1)} + O\left(\vartheta^2\right) \quad , \tag{7.43}$$

with

$$J^{(1)}(W) = \frac{\left(e^{g_{\gamma,0}} W\right)^{2\gamma}}{C} \quad ,$$

[22] For the log utility case the solution is a logarithmic function of wealth (as in the non robust case); cf. Schroder and Skiadas (1999).

[23] Clearly, in order to obtain the result in Theorem 3 appropriate boundary conditions have to be imposed. As a non-standard boundary condition we require that for any order i the candidate solution converges as $\gamma \to 0$ to the known explicit solution of the robust problem (7.38), (7.41), when agents are of the logarithmic utility type.

where

$$
\begin{aligned}
C &= \rho^2 \left(2\gamma(2\gamma+\delta_1-1)+\delta_2\right) \quad , \quad \delta_1 = -\frac{a_1}{a_2} \quad , \quad \delta_2 = -\frac{\rho}{a_2} \quad , \\
a_1 &= \left(\frac{e^{\gamma g_{\gamma,0}}}{\rho}\right)^{\frac{1}{\gamma-1}} - r + \frac{(\alpha-r)^2}{(\gamma-1)\,\sigma^2} \quad , \\
g_{\gamma,0} &= \ln(\rho) + \frac{\gamma-1}{\gamma}\ln\left(\frac{\gamma}{1-\gamma}\left(\frac{1}{\gamma}+\frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2(\gamma-1)\,\rho}-\frac{r}{\rho}\right)\right) \quad , \\
a_2 &= \frac{(\alpha-r)^2}{2\,(\gamma-1)^2\,\sigma^2} \quad .
\end{aligned}
$$

A main open question is wether the series (7.42) converges. Sufficient conditions for that are implied by Theorem 4 in Trojani and Vanini (2001a); these conditions are summarized by the next theorem.

Theorem 4. *If*

$$
\textit{(i)}\ a_1 > 0 \quad , \quad \textit{(ii)}\ \gamma < \frac{2}{3} \quad , \quad \textit{(iii)}\ \frac{1}{2}\left(3+\frac{a_1}{a_2}\right) > \frac{\gamma}{1-\gamma} \quad ,
$$

the power series (7.42) converges on compact subsets of $(0,\infty)$.

Notice, that while condition (ii) is a pure technical one, condition (i) implies that for $\gamma \to 0$ one should have $\rho \geq r$. This further constraint is implicit in the conditions of Theorem 4.

Given the above functional form, the wealth-inhomogeneity of the value function for the MER case is up to ϑ-first order proportional to $J^{(1)}$, a function behaving like $W^{2\gamma}$. Therefore, in a state dependent situation a natural functional form for the solution to (7.38) is

$$
J(W,X) = \frac{1}{\rho}\left(\frac{\left(e^{g(\gamma,X)}W\right)^{\gamma}-1}{\gamma} + \vartheta g_2(\gamma,X)\,W^{2\gamma}\right) + O\left(\vartheta^2\right), \qquad (7.44)
$$

where

$$
\begin{aligned}
g(\gamma,X) &= g_0(X) + \gamma g_1(X) + O\left(\gamma^2\right) \quad , & (7.45)\\
g_2(\gamma,X) &= g_2(X) + \gamma g_{21}(X) + O\left(\gamma^2\right) \quad , & (7.46)
\end{aligned}
$$

for some $g_{21}(X)$. Expanding the optimal rules (7.39), (7.40) and using the functional form (7.44) together with (7.35) finally gives the desired approximate policies.

Theorem 5. *The asymptotic expansions for the optimal policies of a minimum entropy robust agent are:*

$$\begin{aligned} c(W,X) &= \rho\left(1-\gamma(g_0(X)-\ln(\rho))\right)+O\left(\|(\gamma,\vartheta)\|^2\right), & (7.47)\\ w(W,X) &= \frac{\alpha-r}{\sigma^2}\left(1+\gamma-\frac{\vartheta}{\rho}\right)+\left(\gamma-\frac{\vartheta}{\rho}\right)\frac{\partial g_0(X)}{\partial X}\cdot\frac{\sigma_{XP}}{\sigma^2}\\ &\quad +O\left(\|(\gamma,\vartheta)\|^2\right). & (7.48)\end{aligned}$$

Again, only g_0 appears in these optimal rules. Hence, the choice of the functional form (7.44) to develop a perturbative approach turns out to successful.

3.6 Qualitative Discussion of the Robust Optimal Policies

The optimal policies for the homothetic and minimum entropy formulation are of the form:

$$\begin{aligned} c(W,X) &= \rho\left(1-\gamma(g_0(X)-\ln(\rho))\right)+O\left(\|(\gamma,\vartheta^*)\|^2\right), & (7.49)\\ w(W,X) &= \frac{\alpha-r}{\sigma^2}(1+\gamma-\vartheta^*)+(\gamma-\vartheta^*)\frac{\partial g_0(X)}{\partial X}\cdot\frac{\sigma_{XP}}{\sigma^2}\\ &\quad +O\left(\|(\gamma,\vartheta^*)\|^2\right), & (7.50)\end{aligned}$$

where $\vartheta^*=\vartheta$, $\frac{\vartheta}{\rho}$, in the HR and the MER case, respectively. We remark the following distinguishing features of these first order asymptotics. First, the functional form for optimum consumption is exactly the same as in the standard non robust situation (cf. (7.18)). Hence, to (γ,ϑ)-first order a preference for robustness does not affect optimum consumption planes directly. Second, optimal allocations to risky assets are altered when a preference for robustness is present. Indeed, a higher *effective* risk aversion amounting to

$$1-(\gamma-\vartheta^*) \quad ,$$

is obtained, compared to the standard expected utility situation. This enhanced risk aversion affects both the myopic and the intertemporal hedging demands for risky assets. Notice that a non standard hedging component arises in (7.50), which is purely driven by a preference for robustness. Indeed, the term

$$-\vartheta^*\frac{\partial g_0(X)}{\partial X}\frac{\sigma_{XP}}{\sigma^2} \quad ,$$

is a first order asymptotic for a hedging demand caused by a concern of a robust agent for the quadratic variation of the underlying value function J. This term disappears only when $\vartheta^*=0$, that is in the absence of a preference for robustness. Indeed, in the case $\gamma = 0$ (that is a log utility investor) this hedging component is still non-zero. Hence, stochastic opportunity sets generate non myopic investment policies for log utility agents when robustness is present. This point is important for the way of developing our asymptotic general equilibrium analysis below. The sensitivity of the optimal risky allocations with respect to the robustness and risk aversion parameters can be positive or negative, depending on the sign of $\frac{\partial g_0(X)}{\partial X} \cdot \frac{\sigma_{XP}}{\sigma^2}$. Finally, notice that at variance with the HR case in the MER formulation effective risk aversion is related to the time preference parameter ρ.

4. Robust General Equilibrium Problems

While several authors have already dealt with the existence and the characterization of heterogeneous agents equilibria in standard (non robust) economies (see for instance Duffie and Huang (1985), Duffie and Zame (1989), Duffie et al. (1994), Dumas (1989), Karatzas et al. (1990) and Wang (1996)) only a few of them have derived quantitative or qualitative predictions for the relevant entities in a continuous time setting. Moreover, when quantitative predictions have been derived either they were computed using numerical methods (Dumas (1989)) or they where obtained in closed form only for particular values of the model parameters (Wang (1996)). Using perturbation theory, Kogan and Uppal (2000) have been able to compute analytically the general equilibria of heterogenous non robust production and exchage economies where in excess incomplete markets and borrowing constraints are allowed for.

To our knowledge only two papers have so far considered heterogeneous agents general equilibria in continuous-time economies where model misspecification is taken into account in optimal decision making. Epstein and Miao (2001) have described in closed form equilibria for a complete model based on recursive multiple priors utility using a martingale approach[24]. That model focus on heterogeneities in aversions to (model) ambiguity and leaves outside heterogenities in agent's attitutes to risk. On the other hand, Trojani and Vanini (2001b) have solved by a perturbative approach a robust version of Dumas (1989) and Wang (1996) models where heterogeneities arise both in aversions to risk and preferences for robustness[25]. In this section we illustrate this methodology by computing the relevant quantities in a robust version of the complete heterogenous-agents production economy of[26] Dumas (1989).

[24]Since the martingale approach is based on market completeness, extensions of this model to allow for incomplete markets are not immediate.

[25]As for standard non-robust economies the perturbative approach permits the analysis of incomplete markets equilibria.

[26]A further reference relevant for this section is Anderson (1998).

4.1 The General Equilibrium Economy

There are two groups of agents in the economy with utilities of current consumption given by

$$u(C_t) = \frac{C_t^\gamma - 1}{\gamma} \quad , \quad u_1\left(C_t^1\right) = \log\left(C_t^1\right) \quad , \quad \gamma < 1 \quad , \tag{7.51}$$

preferences for homothetic/minimum entropy robustness[27] $\vartheta > 0$, $\vartheta^1 = 0$, and identic time preferences $\rho = \rho^1$. There further exists a single constant returns to scale technology yielding the dynamics

$$dS_t = \left(\alpha S_t - C_t - C_t^1\right) dt + \sigma S_t dZ_t \quad ,$$

for the aggregate capital stock, where α, $\sigma > 0$. The risky asset is a stock on the production technology with cumulative return process (P_t) given by

$$dP_t = \alpha P_t dt + \sigma P_t dZ_t \quad .$$

The number of shares in this economy is equal to the aggregate capital stock. The riskless asset is a short term bond with price dynamics

$$dB_t = r_t B_t dt \quad ,$$

where r_t is a (possibly stochastic) interest rate that has to be determined in equilibrium. The equilibrium definition used is given next.

Definition We call a process $\left(S_t, r_t, w_t, w_t^1, C_t, C_t^1\right)$ such that:

- The individual portfolio and consumption rules w_t, w_t^1, C_t, C_t^1 are optimal to first order, i.e they satisfy (7.49) and (7.50),
- Financial markets clear, i.e. aggregate wealth $W_t + W_t^1$ is completely invested in the given production technology:

$$\frac{w_t W_t + w_t^1 W_t^1}{W_t + W_t^1} = 1 \quad , \tag{7.52}$$

a robust equilibrium.

Remark that in this definition of equilibrium feed-backs between the sets of model specifications relevant to each single (robust and non robust) agent in the economy are excluded. Implicitly, it is assumed that the set of relevant

[27]For simplicity of notation just set $\vartheta = \frac{\vartheta^*}{\rho}$, for some $\vartheta^* > 0$ in the MER case.

model misspecifications is determined by agent specific characteristics (the parameter ϑ depicting a preference for robustness) and hence it is not determined endogenously by the equilibrium. We have basically two reasons to make use of this assumption. First, we are not focusing on modelling game theoretical issues where each agent is strategically planning to develop an optimal strategy, given that he knows the set of model specifications relevant to her counterpart. We are rather interested in equilibria were groups of "beliefs-takers" develop optimal consumption/portfolio plans in the presence of misspecifications and where heterogenous aversions to risk and preferences for robustness interplay in determining asset prices. Second, from a more methodological point of view introducing feed-backs between agents beliefs necessitates to solve some kind of multi-agents stochastic game where each robust agent interplays with the malevolent player (the nature) and with each other agent in the economy in determining the optimal policy of a max-min optimization problem. Solving this problem is an highly nontrivial task; it is an open question how far perturbation theory can be applied to compute analytical approximate solutions for this kind of problems. A second important remark on the above equilibrium definition is related to the question of why in the above economy agents do have different model beliefs (that is different perceptions of the relevant model misspecifications) despite observing a common price process. Formal continuous time models that describe learning and that allow for the presence of model ambiguity have not been largely developed yet[28]. However, the LLN result in Marinacci (1999) for beliefs represented by a set of priors suggests that model ambiguity will not disappear even asymptotically when agents learn about the underlying data generating process. Indeed, in this setting the connection between empirical frequencies and asymptotic beliefs turns out to be weakened to a degree that depends on the extent of diversity in prior beliefs. Therefore, it is very well plausible that agents with different prior beliefs will still have posterior different beliefs, even if they observe the same price process[29].

4.2 Perturbative Solutions Approach

In general equilibrium the function g_0 in (7.49), (7.50) is now endogenous to the economy, i.e. it depends on γ and ϑ. However, it can be further expanded in a neighbourhood of the representative agent value function of an homogeneous production economy with log utility non-robust investors ($\gamma = \vartheta = \vartheta^1 = 0$):

$$g_0(X) = g_{0,0}(X) + \gamma g_{0,1}(X) + \vartheta g_{0,2}(X) + O\left(\|(\gamma,\vartheta)\|^2\right) \quad ,$$

[28]Work in progress related to this topic is Epstein and Schneider (2001a).
[29]Cf. also the discussion in Chen and Epstein (2000), Section 1.3.

implying

$$\begin{aligned}
c(W,X) &= \rho\left(1-\gamma\left(g_{00}(X)-\ln(\rho)\right)\right)+O\left(\|(\gamma,\vartheta)\|^2\right),\\
c^1(W,X) &= \rho\ ,\\
w(W,X) &= \frac{\alpha-r}{\sigma^2}(1+\gamma-\vartheta)+(\gamma-\vartheta)\frac{\partial g_{00}(X)}{\partial X}\cdot\frac{\sigma_{XP}}{\sigma^2}\\
&\quad +O\left(\|(\gamma,\vartheta)\|^2\right)\ ,\\
w^1(W,X) &= \frac{\alpha-r}{\sigma^2}\ .
\end{aligned}$$

To characterize these rules completely we need to determine r and g_{00}. Notice that since g_{00} is completely determined by the value function of a representative log utility agent in the given production economy, it is a constant. For this benchmark economy it follows in equilibrium that $\alpha-\sigma^2$ is the implied constant interest rate. As a consequence, (7.20) gives

$$g_{0,0}=\ln(\rho)-1+\frac{\alpha-\sigma^2/2}{\rho}\quad . \tag{7.53}$$

The equilibrium interest rate r in now completely determined by the market clearing condition (7.52), that can be expressed as

$$w_t\omega_t+w_t^1(1-\omega_t)=1\quad ,$$

with $\omega_t=\frac{W_t}{W_t+W_t^1}$ the cross-sectional wealth distribution in the given economy. This gives the last result of the paper.

Theorem 6. *In the above production economy equilibrium interest rates are given by*

$$r_t=\alpha-\sigma^2+\sigma^2(\gamma-\vartheta)\,\omega_t+O\left(\|(\gamma,\vartheta)\|^2\right)\quad . \tag{7.54}$$

The individual equilibrium optimal policies are

$$c_t^1 = \rho\quad , \tag{7.55}$$

$$c_t = \rho-\gamma\left(\alpha-\sigma^2/2-\rho\right)+O\left(\|(\gamma,\vartheta)\|^2\right)\quad , \tag{7.56}$$

$$w_t^1 = 1-(\gamma-\vartheta)\,\omega_t+O\left(\|(\gamma,\vartheta)\|^2\right)\quad , \tag{7.57}$$

$$w_t = 1+(\gamma-\vartheta)(1-\omega_t)+O\left(\|(\gamma,\vartheta)\|^2\right)\quad . \tag{7.58}$$

Finally, the equilibrium cross-sectional wealth and capital stock dynamics are given by

$$\begin{aligned}
d\omega_t &= \gamma\omega_t(1-\omega_t)\left[\left(\alpha-\sigma^2/2-\rho\right)\right]dt+(\gamma-\vartheta)\,\omega_t(1-\omega_t)\,\sigma dZ_t\\
&\quad +O\left(\|(\gamma,\vartheta)\|^2\right),
\end{aligned}$$

and

$$dS_t = \left[\alpha - \rho + \gamma\left(\alpha - \sigma^2/2 - \rho\right)\omega_t\right] S_t dt + \sigma S_t dZ_t + O\left(\|(\gamma,\vartheta)\|^2\right).$$

4.3 Qualitative Discussion of the Equilibrium Variables

In the sequel we illustrate for brevity the implications of Theorem 5 in the case $\gamma > \vartheta$.

The asymptotics for the optimal policies in Proposition 6 show a basic difference between the pure risk averse ($\vartheta = 0$) and the robust ($\vartheta > 0$) solutions. Indeed, we see that while risk aversion affects directly all decision variables of the investor, the robustness parameter influences directly only optimal investment to risky assets. However, robustness still affects optimum consumption indirectly, through the altered equilibrium process for cross-sectional wealth. Further, robustness tends to reduce heterogeneities in the individual portfolio positions. Note that (as in Kogan and Uppal (2000)) no equilibrium intertemporal hedging position arises because the variance of the only relevant state variables to the investors in this economy (namely ω_t) is of order $O\left(\|(\gamma,\vartheta)\|^2\right)$. Moreover, robustness lowers equilibrium interest rates (by given cross-sectional wealth distribution ω_t). The arising equilibrium interest rate is between that of an heterogeneous standard economy where no preference for robustness arises and an heterogeneous robust economy with homogeneous log utility investors.

$$\alpha - \sigma^2(1 + \vartheta\omega_t) \le r_t \le \alpha - \sigma^2(1 - \gamma\omega_t) \quad .$$

Compared to standard economies this lower interest rate reflects a lower demand for riskless assets (by given cross sectional wealth distribution ω_t) caused by the higher "effective" risk aversion $1 - \gamma + \vartheta$ in the partial equilibrium asymptotics (7.50) for the optimal portfolio strategy. In fact, the equilibrium open interest in the bond market is:

$$OI_t = \frac{1}{2}\left(|1 - w_t|\,\omega_t + \left|1 - w_t^1\right|(1 - \omega_t)\right) = (\gamma - \vartheta)\,\omega_t\,(1 - \omega_t) \quad ,$$

and is lower than in a non-robust economy.

Finally, robustness affects the cross sectional wealth distribution, through a reduction of the volatility

$$(\gamma - \vartheta)\,\omega_t\,(1 - \omega_t)\,\sigma \quad ,$$

of (ω_t), but does not alter the corresponding drift which is given by

$$\gamma\omega_t\,(1 - \omega_t)\left(\alpha - \frac{\sigma^2}{2} - \rho\right) \quad .$$

This is because the impact of the optimal portfolio policies on the drift of ω_t is of order no less than two. In particular, (since equilibrium interest rates are linearly linked to ω_t) lower volatilities of equilibrium interest rates, given by

$$\sigma^3 (\gamma - \vartheta)^2 \omega_t (1 - \omega_t) \quad ,$$

are obtained. As in standard economies, the highest interest rate volatilities are observed when aggregate wealth is evenly distributed across agents. Finally, an important difference between HR and MER is that in the latter case equilibrium interest rates, optimal portfolios and the volatily of cross-sectional wealth depend on time preferences.

5. Conclusions

We demonstrated the usefulness of perturbation theory in deriving approximate analytical expressions for the optimal policies of intertemporal consumption/portfolio problems where general state dynamics are admitted and a preference for robustness is present. We illustrated the methodology proposed in Kogan and Uppal (2000) within several economic settings, starting from partial equilibrium standard expected utility economies to general equilibrium models where general stochastic opportunity sets are allowed for and an aversion to model misspecification is present. The approach was applicable to a large class of models and the implied equilibrium characterizations were particularly simple. Moreover, an even larger class of models than those discussed here could be easily handled by the methodology. For instance, robust intertemporal consumption/portfolio problems with transaction costs can be solved in the same general vain of Kogan and Uppal (2000) or models using further formulations of a preference for robustness (specifically, a constrained formulation) can be analyzed analytically (see again Trojani and Vanini (2001b)).

Kogan and Uppal (2000) methodology is based essentially on two crucial assumptions:

- Knowledge of the explicit solution of a benchmark model within the given parameterization,

- Existence of an appropriate functional form for a candidate value function solution.

Since for the MER formulation no benchmark exact explicit solution has been derived yet, we used results on perturbative solutions of HJB equations in non-homothetic robust decision making (Trojani and Vanini (2001 a)) to guess the appropriate functional form for a candidate value function. After this preliminary step, Kogan and Uppal (2000) approach could be applied successfully also to this case without further significant difficulties.

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