

Applying and testing VaR estimation methods for non-linear portfolios

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Abstract

In the first part of this work, some of the most known VaR estimation methods are described. In such a description we try to focus mainly on the application problems of each approach, which are modified to take into account non-linear positions. In particular, some generalizations of the standard historical simulation and of the filtered historical simulation methods are shown. The second part is devoted to measuring the performances of the different estimation methods by testing them on the last two years data (including the recent NASDAQ crash). The observed variables are the number of times that the loss exceeds VaR and the mean of the ratio between the losses which exceed VaR and the VaR itself. For linear portfolios all the methods give quite accurate results, while for portfolios containing options the best performances are given by the non-parametric and the semi-parametric methods. The main contribution of this paper is to show how the generalized historical simulation methods perform better than the standard ones during high volatility periods.

PREFACE

Since the first half of 90's, Value at Risk has been an important tool to control market risk. Its diffusion, mainly due by the popularity of JP Morgan's RiskMetrics, doesn't fail to affect also the decisions of Basel Committee which, with the 1996 amendment of the Capital Accord of July 1988, has allowed financial institutions to use internal models to measure market risk.

In such a setting Banca del Gottardo decided to enforce its first method to systematically measure market risk. In 1996 a first instrument for this task was introduced and since that date the evolution has never interrupted till to generate a widely diffuse internal culture on market risk control.

This work represent only a further effort on this direction. A greater and greater part of the portfolio of many institutions is invested in non linear instruments such as options. Banca del Gottardo is not an exception and its interest on this matter has increased during the last years.

Moreover, recent crashes on the high technology markets drastically raised the attention on how the different VaR estimation methods work in an extreme setting. Indeed a large market movement and the presence of large non linear leveraged positions in the portfolio can be considered a dangerous mixture for a financial institution.

The will of Banca del Gottardo is to be on the safe side also in such extreme scenarios. To this aim a deep knowledge of the instruments to measure market risk represents the first necessary step.

The present work is the most recent effort to increase the knowledge on the risk held during a market activity. However, the will of Banca del Gottardo to know everyday a little bit more on market risk enables us to say that this surely will not be the last effort on this direction.

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1 Introduction

The increase in the dimension of the traded portfolio and the rise of the market volatility have made the market risk measurement a more and more important issue for a financial institution. In recent years Value at Risk (VaR) has become one of the most used instrument to measure such a risk both for regulatory purposes and internal control motivations. Defined as the maximum portfolio loss that we expect to have with a certain probability and within a time interval, VaR has the attractive feature to summarize a complex market risk exposure with a single number.

Different methods are in general used to estimate the VaR. Many of this methods are based on some assumptions on the asset returns distribution, the so called parametric methods, while some others give a sort of non-parametric VaR estimate. There is a long literature that tries to evaluate the accuracy of the different methods, especially for equity portfolios. Maybe Hendricks [14] and Pritsker [20] are the most famous paper on this subject.

Neither the first kind of models nor the second one can be considered free of drawbacks. Indeed, the parametric methods are strongly dependent on the assumptions made on the returns distribution. As formerly noted by Mandelbrot [17] and Fama [11] the normality assumption is unrealistic since the return distribution seems to be more fat-tailed than the normal. As documented by a wide literature¹ the fat tail problem produce an underestimate of VaR.

The above problems increase when the portfolio contains positions such as options. Indeed, in such a situation the relation between the derivative price and the underlying price is not linear and it is not clear, given the distribution of the underlying, what the distribution of the option returns is.

On the other hand, the non parametric approaches have the implicit assumptions that the returns are identically and independently distributed (iid). Also such an assumption is violated by the evidences that the volatility changes over time. The wrong assumption of iid returns leads to an inconsistent estimate of VaR.

Some techniques are used to reduce the problem. In the first one we relax the standard assumption that all the past returns have the same probability to occur again. By putting an higher probability to occur to the most recent observations, we should have a VaR estimate more sensitive to sudden changes on market risk.

Moreover, Barone-Adesi, Bourgoin and Giannopoulos [3] face the non iid problem by a semi-parametric method. The main idea is to standardize the returns by assuming a model for the volatility so to have an iid series of standard residuals. Inside the series a bootstrap is done to obtain a

¹See Duffie and Pan [9] for an overview on the subject.

simulated distribution for returns.

The bootstrap is usually done by assuming that all the standardized residuals have the same probability to occur. This would have been the best thing to do if the volatility model had been correctly specified. If we take into account a possible misspecification of the volatility model, it will be better to give a higher probability to occur to the most recent observations.

Another problem of the non-parametric methods is that a time series of option prices with the same characteristics is in general not available. The problem is decided by evaluating the option price by using the historical or the simulated underlying prices in the Black-Scholes formula and then by introducing the Black-Scholes assumptions which partially reduce the main benefits of the non parametric approach.

The aim of this work is to show how some of the most known VaR estimation methods can be enforced and modified to take into account non linear positions. The different methods which we are going to describe will be applied to portfolios containing equities and options. To measure the performances of the estimation methods, they will be tested on the last two years, a period that includes the recent crash of the NASDAQ.

The choice of such a period was not made by chance. Indeed, our attention is also on the VaR sensitivity to sudden changes of risk due to market crashes. It seems natural to require this kind of risk sensitivity to the used risk measure but, as we will see, not all the estimation methods supply this feature.

In section 2 there is a general definition of portfolio VaR, while sections 3 to 5 are devoted to describe the different methods that are applied. In particular, in section 3 a parametric method based on the quadratic approximation of the non-linear assets is shown.

In section 4 we describe the non-parametric method based on historical data, the so called historical simulation. To face the risk sensitivity problem, in the same section a generalization of the standard historical simulation is presented. The filtered historical simulation method described in section 5 has the aim to face the inconsistency in VaR estimate due to the unrealistic assumption of identically and independently distributed returns.

In section 6 the portfolios which are considered during the analysis are described while in section 7 there is a first glance on the VaR estimation. By this first results we can already have an intuition on the dimension of the VaR for the different portfolios. Section 8 is devoted to test the accuracy of the VaR estimate during the last two years.

Comments and conclusions are in section 9. To make the reading “smooth” the most tedious subjects are placed in the appendixes.

2 A general definition of portfolio VaR

In a portfolio composed by n assets, let $X_t := [X_{1,t}, X_{2,t}, \dots, X_{n,t}]'$ be the vector of the asset prices such that $X_t \in \mathbb{R}^n$. Moreover let us define $q \in \mathbb{Z}^n$ the vector of the number of the assets in the portfolio between t and $t + \Delta t$ and V_t the price of the portfolio at time t such that $V_t = q'X_t$. We define the vector $a \in \mathbb{R}^n$ as the vector of the weights of each asset. For the i -th asset the weight a_i is equal to $\frac{q_i X_{it}}{V_t}$.

If α is the accepted loss probability the VaR measure between t and $t + \Delta t$ will be defined in the following way:

$$\mathbb{P}_t[V_{t+\Delta t} - V_t < -VaR_t] = \alpha ,$$

or with another notation

$$\mathbb{P}_t[V_t R_{t+\Delta t} < -VaR_t] = \alpha , \quad (1)$$

or again

$$\mathbb{P}_t\left[R_{t+\Delta t} < -\frac{VaR_t}{V_t}\right] = \alpha ,$$

where $R_{t+\Delta t}$ is the portfolio return i.e. the weighted average of the single asset returns $a' \bar{R}_{t+\Delta t}$ where

$$\bar{R}_{t+\Delta t} := [R_{1,t+\Delta t}, R_{2,t+\Delta t}, \dots, R_{n,t+\Delta t}]' ,$$

and

$$R_{i,t+\Delta t} := \frac{X_{i,t+\Delta t} - X_{i,t}}{X_{i,t}} \quad i = 1, 2, \dots, n .$$

By the above definition, it is clear that VaR depends on the available information in t , the horizon Δt , the portfolio allocation a (that is assumed constant during all the horizon) and the accepted loss probability α .

In particular the horizon that we will consider is the one day horizon ($\Delta t = 1$). Indeed, one of the problem to use longer time interval is that the assumption of a constant portfolio allocation becomes unrealistic. Moreover, as we will see in section 8, a longer horizon reduces drastically the power of the tests for the accuracy of the VaR estimation.

Note that the vector \bar{R}_{t+1} is the vector of the portfolio assets returns. Many times, instead to consider directly all the assets returns, it is convenient to regard only the variations of few risk factors. A risk factor can be defined as an index, a rate, a stock or a commodity that strongly affects the fluctuation of the assets. Typical risk factors are the interest rate for a bond or the variations of the market index for a stock return.

In the situation described above, it is necessary to define a relation between each asset return and the corresponding risk factor (or risk factors) return. Let $\tilde{R}_{t+1} \in \mathbb{R}^m$ be the m -vector of risk factors returns where $m \leq n$. A generic function $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$ can be defined such that

$$\bar{R}_{t+1} = g(\tilde{R}_{t+1}) .$$

For a stock belonging to a well diversified portfolio the function $g(\cdot)$ can be approximated with the Ross APT relation. In such a way, the stock returns are

$$R_{i,t+1} \approx r_f + \beta_i'(\tilde{R}_{t+1} - s_m r_f) \quad i = 1, 2, \dots, n^* ,$$

where n^* is the number of positions belonging to a well diversified portfolio, r_f is the risk free rate, $s_m = [1, 1, \dots, 1]'$ is a m -dimension unit vector and $\beta_i \in \mathbb{R}^m$ is the vector of the stock sensitivity to each risk factor.

For all the n^* assets the above equation becomes

$$\bar{R}_{t+1} \approx s_n r_f + \beta'(\tilde{R}_{t+1} - s_m r_f) ,$$

or equivalently

$$\begin{aligned} \bar{R}_{t+1} &\approx (s_n - \beta' s_m) r_f + \beta' \tilde{R}_{t+1} \\ &:= K + \beta' \tilde{R}_{t+1} , \end{aligned} \tag{2}$$

where the columns of β are the vectors of the stock sensitivity to each risk factor and s_n is a n^* -dimension unit vector.

Example 2.1 Let us assume that the asset returns are jointly normally distributed

$$\bar{R}_{t+1} \sim N(\mu_t, \Sigma_t) ,$$

where $\mu_t \in \mathbb{R}^n$ is the vector of asset mean returns and $\Sigma_t \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix². Then, the distribution of the portfolio return will be

$$R_{t+1} = a' \bar{R}_{t+1} \sim N(a' \mu_t, \sigma_t^2) ,$$

where

$$\sigma_t^2 = \text{var}_t(R_{t+1}) = \text{var}_t(a' \bar{R}_{t+1}) = a' \Sigma_t a ,$$

such that the daily VaR is

$$\begin{aligned} VaR_t &= -a' \mu_t - V_t \Phi^{-1}(\alpha) \sigma_t \\ &= -a' \mu_t + V_t \Phi^{-1}(1 - \alpha) \sigma_t , \end{aligned} \tag{3}$$

²Note that, due to the choice of one day as time horizon, means, variances and covariances are expressed on a daily basis.

where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal distribution function.

By using equation (2) and assuming that the risk factor returns is jointly normally distributed

$$\tilde{R}_{t+1} \sim N(\tilde{\mu}_t, \tilde{\Sigma}_t) ,$$

where $\tilde{\mu}_t \in \mathbb{R}^m$ is the vector of risk factor mean returns and $\tilde{\Sigma}_t \in \mathbb{R}^{m \times m}$ is a positive definite symmetric matrix.

In such a situation the portfolio return has the following distribution:

$$R_{t+1} \sim N(\nu_t, \tilde{\sigma}_t^2) ,$$

where

$$\begin{aligned} \nu_t &= E_t[R_{t+1}] = a'K + a'\beta'\tilde{\mu}_t \\ \tilde{\sigma}_t^2 &= \text{var}_t(R_{t+1}) = \text{var}_t(a'\beta'\tilde{R}_{t+1}) = a'\beta'\tilde{\Sigma}_t\beta a , \end{aligned}$$

such that the daily VaR is

$$\begin{aligned} VaR_t &= -\nu_t + V_t\Phi^{-1}(1 - \alpha)\tilde{\sigma}_t \\ &= -a'K - a'\beta'\tilde{\mu}_t + V_t\Phi^{-1}(1 - \alpha)\tilde{\sigma}_t . \end{aligned} \tag{4}$$

□

Once we accept the APT assumptions, the above relation gives no problem. Indeed, it is linear in the risk factors. If we consider portfolios containing options the function $g(\cdot)$ is no longer linear in the risk factor (the underlying).

For the estimate of the VaR in a portfolio containing a relevant position in non-linear assets, we have to face two kind problems:

- the first one is the non normality of the returns formerly noted by Mandelbrot [17] and Fama [11]. Figure 1 shows how the standardized returns of the Swiss Market Index and the Standard & Poor 500 cannot be considered normally distributed. The evidence accrues mostly by the QQ-plot where the plot diverges from the dashed line especially on the tails. This phenomenon is sometimes called heavy-tails or fat-tails problem and induces on a VaR underestimate also in portfolios with linear positions only.
- The second one due to the non-linearity of some positions. This makes difficult to understand what the distribution of the portfolio returns is and it increases the difficulty in the application of any analytical approach to estimate VaR.

To solve the last problem some methods can be used. Among them we can distinguish those based on approximation of the non-linear function $g(\cdot)$ from those based on the simulations and a full valuation of the non-linear positions.

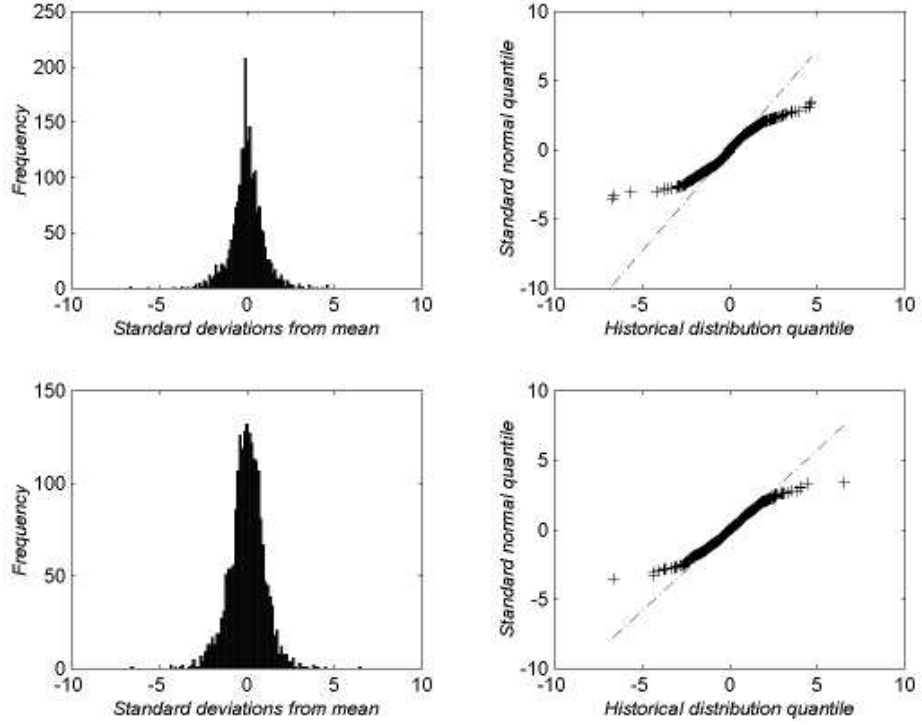


Figure 1: Frequency plot compared with the standard normal density and QQ-plot of the same historical distribution with respect to the standard normal distribution. The sample is composed by the standardized log-return of the Standard & Poor 500 (above) and of the Swiss Market Index (below) from May 1993 to May 2001. Source: Datastream.

3 The quadratic approximation

As in example 2.1 the parametric approach is based on some assumptions on the return distribution. The most used hypothesis is to assume that the returns are jointly normally distributed. In spite of such an assumption, when there are option positions in the portfolio it is not clear where the portfolio return distribution is and then how to estimate VaR.

A widely used way to face the problem is to substitute the non-linear relation between the option returns and the underlying returns by a quadratic approximation of the function $g(\cdot)$.

3.1 The derivation of the approach

To make the notation easy, we will consider during all this section a portfolio composed by an option written on an underlying asset only. The multidimensional case will be considered in the following sections.

Let assume to have in the market an asset S_t , a risk free zero coupon bond B_t and an option X_t written on S_t . The asset and the bond have the

following dynamics:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dw_t , \\ dB_t &= r_f B_t dt , \end{aligned}$$

where r_f , μ and σ are constant parameters and w_t is a standard Brownian motion.

In such a simplified world the underlying asset is the only risk factor of the option.

Let us consider a small and discrete time interval Δt . By using a second order Taylor series expansion around S_t we can write the return on the derivative as a quadratic function of the return of the underlying asset:

$$X_{t+1} = X_t + \frac{\partial X}{\partial S}(S_{t+1} - S_t) + \frac{1}{2} \frac{\partial^2 X}{\partial S^2}(S_{t+1} - S_t)^2 + o_S(2) ,$$

or in the same way

$$\begin{aligned} \frac{X_{t+1} - X_t}{X_t} &= \frac{S_t}{X_t} \frac{\partial X}{\partial S} \left(\frac{S_{t+1} - S_t}{S_t} \right) + \\ &+ \frac{1}{2} \frac{S_t^2}{X_t} \frac{\partial^2 X}{\partial S^2} \left(\frac{S_{t+1} - S_t}{S_t} \right)^2 + o_S(2) . \end{aligned} \tag{5}$$

The ratios $\frac{\partial X}{\partial S}$ and $\frac{\partial^2 X}{\partial S^2}$ are called respectively *delta* and *gamma*, while the last term is the error made by the approximation.

In the following the asset return and the risk factor return will be noted respectively with R_{t+1} and \tilde{R}_{t+1} .

Remark 3.1 In the delta approach the Taylor series expansion is arrested at the first order. The advantage is that if we assume that the underlying return are normally distributed, also the distribution of the derivative return will be normal.

In spite of the analytical tractability, the assumptions in the delta-normal approach are very unrealistic. Indeed, both the normality of the return and the linearity of X_t with respect to S_t contribute to reduce the accuracy of the results.

More exactly for a long option position, we can say that the linear function with which we try to approximate the convex option function, always lies below the latter such to get a strongly overestimated VaR. On the other hand, for a short position the option function is concave and the linear approximation always lies above it such to get a strongly underestimated VaR.

For such a reasons we will not consider the delta-normal methodology in our analysis. \square

Remark 3.2 In the delta-gamma approach we are replacing the option function with a quadratic one. For such a substitution we expect to have an underestimated VaR for long option positions and an overestimated VaR for short option positions. \square

Another way to obtain the same result is by applying the Ito's lemma to the option price X_t such that

$$dX = \left(\frac{\partial X}{\partial t} + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial X}{\partial S} dS \quad (6)$$

$$\frac{dX}{X} = \left(\frac{\partial X}{\partial t} + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \sigma^2 S^2 \right) \frac{1}{X} dt + \frac{\partial X}{\partial S} \frac{S}{X} \frac{dS}{S} \quad (7)$$

where $\frac{\partial X}{\partial t}$ is called the *theta* of the call.

Note that in the delta approximation we assume that

$$\left(\frac{\partial X}{\partial t} + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \sigma^2 S^2 \right) = 0$$

while in the delta-gamma approximation we assume theta equal to zero such that

$$\frac{dX}{X} = \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \frac{1}{X} \sigma^2 S^2 dt + \frac{\partial X}{\partial S} \frac{S}{X} \frac{dS}{S} \quad (8)$$

$$\frac{dX}{X} = \frac{\partial X}{\partial S} \frac{S}{X} \frac{dS}{S} + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \frac{S^2}{X} \left(\frac{dS}{S} \right)^2 \quad (9)$$

where equation (9) is equivalent to equation (5) if we consider an infinitesimal variation of the underlying price.

Remark 3.3 Relaxing the assumption that theta is equal to zero, equation (9) becomes

$$\frac{dX}{X} = \frac{\partial X}{\partial S} \frac{S}{X} \frac{dS}{S} + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} \frac{S^2}{X} \left(\frac{dS}{S} \right)^2 + \frac{\partial X}{\partial t} dt ,$$

and by rearranging the term we obtain again equation (7).

The above equation is the starting point of the so called delta-gamma-theta approach. Note, however, that the equality is right only instantaneously i.e. for infinitesimal variations of the underlying. For large variation the equality is no longer true. \square

All the above equations are true (or approximately true) in a neighborhood of S_t . When we face large variations they are no longer true and moreover they are not in general a good approximation. Intuitively the problem is that by estimating VaR we are interested in large variations on the risk factor while the approximation is near to be true for small ones.

3.2 The distribution of the option returns

Also by assuming the conditional normality of the underlying returns, from equation (5) we cannot easily say anything about the distribution of the derivative returns. The most common ways to take into account the non normality of the derivatives returns are three:

Cornish-Fisher Expansion The expansion corrects the normal critical value of the α -percentile to deal with the kurtosis and the skewness of the option return distribution. The use of this approach is explained in Zangari [23].

Johnson Transformations The approach is based on the matching of the option return first four moments with a distribution belonging to the Johnson distributions family. The use of this approach is summarized in Zangari [24].

Fourier Transform The last approach is based on the inversion of the characteristic function of the approximated derivative returns. By the inversion of the Fourier transform, we can get the distribution of derivative return and obtain the required quantile.

In what follows the used approach is the last one. The advantage is that by Fourier inversion we get the exact distribution of approximated option returns. Moreover, some recent papers show the superiority of this methodology among the others³.

3.3 The multivariate framework

Let us define the risk factor return vector as

$$\tilde{R}_{t+1} = \left[\tilde{R}_{1,t+1}, \tilde{R}_{2,t+1}, \dots, \tilde{R}_{m,t+1} \right] .$$

By equation (5) the portfolio return can be approximated in the following way:

$$\begin{aligned} R_{t+1} &= a' \tilde{R}_{t+1} \\ &\approx a' C \tilde{R}_{t+1} + \tilde{R}_{t+1}' \tilde{a}' A \tilde{R}_{t+1} , \end{aligned}$$

where $C \in \mathbb{R}^{n \times m}$ is the matrix of the first term approximation coefficients, $A \in \mathbb{R}^{n \times m}$ is the matrix of the second term approximation coefficients and $\tilde{a} \in \mathbb{R}^{n \times m}$ is a matrix formed by the portfolio weights. The construction of the above matrixes will be clear in section 3.3.1.

³See among others Mina and Ulmer [19].

To be more general let us introduce a constant term in the above approximation such that⁴

$$R_{t+1} = a'K + a'C\tilde{R}_{t+1} + \tilde{R}_{t+1}'\tilde{a}'A\tilde{R}_{t+1} , \quad (10)$$

where $K \in \mathbb{R}^n$ is a vector of constants.

Remark 3.4 Note that when the portfolio return does not depend on the squared risk factor return, i.e. the portfolio is linear with respect to the risk factor, equation (10) becomes similar to equation (2):

$$R_{t+1} = a'K + a'C\tilde{R}_{t+1} .$$

Hence we can say that equation (2) is a special case of the more general equation (10). \square

With a more compact notation we can write the above equation in the following way:

$$R_t = \kappa + c'\tilde{R}_t + \tilde{R}_t'B\tilde{R}_t , \quad (11)$$

where $\kappa := a'K$, $c := C'a$ and $B := \tilde{a}'A$. We assume that B is symmetric.

Remark 3.5 Note that if $m = n$, the portfolio is composed only by plain vanilla options and the risk factors are the corresponding underlying assets, the matrixes C , A and \tilde{a} will be diagonal. More exactly

$$\begin{aligned} C &= \text{diag} \left(\frac{S_{1,t}}{X_{1,t}} \frac{\partial X_{1,t}}{\partial S_{1,t}}, \frac{S_{2,t}}{X_{2,t}} \frac{\partial X_{2,t}}{\partial S_{2,t}}, \dots, \frac{S_{n,t}}{X_{n,t}} \frac{\partial X_{n,t}}{\partial S_{n,t}} \right) , \\ A &= \frac{1}{2} \text{diag} \left(\frac{S_{1,t}^2}{X_{1,t}} \frac{\partial^2 X_{1,t}}{\partial S_{1,t}^2}, \frac{S_{2,t}^2}{X_{2,t}} \frac{\partial^2 X_{2,t}}{\partial S_{2,t}^2}, \dots, \frac{S_{n,t}^2}{X_{n,t}} \frac{\partial^2 X_{n,t}}{\partial S_{n,t}^2} \right) , \end{aligned}$$

while $\tilde{a} = \text{diag}(a_1, a_2, \dots, a_n)$ and $K = 0_n$. This is no longer true when $m < n$. Moreover, when the portfolio is composed also by equities and we decide to use an APT approximation, we can see from equation (2) that $K \neq 0_n$. \square

Let us assume as usual that the returns of the risk factors follow a multivariate conditional normal distribution

$$\tilde{R}_{t+1} \sim N(0, \Sigma) ,$$

where $\Sigma \in \mathbb{R}^{m \times m}$ is a positive definite symmetric matrix. The assumption of a zero mean is not so strong for a one day horizon. Moreover, for some authors such an approximation performs better than an estimate based on historical data⁵.

⁴Note that to simplify the reading, in all the section we will use the equality by neglecting the error term.

⁵See among others Figlewski [13].

Definition 3.1 *Let Y be a random variable and u a real number, we call characteristic function of Y the expected value of e^{iuY} . \square*

If we assume that R_{t+1} is a continuous random variable with probability density function $f(r)$, the characteristic function can be written as

$$\varphi_R(u) = E[e^{iuR}] = \int_{-\infty}^{+\infty} e^{iur} f(r) dr ,$$

where the last integral is called the Fourier transform.

Proposition 3.1 *If we assume that \tilde{R}_{t+1} has a multivariate normal distribution with mean zero and covariance matrix Σ , then the random variable R_{t+1} will have the following characteristic function*

$$\varphi_R(u) = |D|^{-1/2} \exp \left[iu\kappa - \frac{1}{2} u^2 c' D^{-1} \Sigma c \right] , \quad (12)$$

where

$$D := I - 2iu\Sigma B .$$

\square

See appendix A for a proof.

Remark 3.6 When $B = O_m$ i.e. when the assets are linear with respect to the risk factors, the above characteristic function becomes

$$\varphi_R(u) = \exp \left[iu\kappa - \frac{1}{2} u^2 c' \Sigma c \right] ,$$

where $c' \Sigma c$ is the portfolio variance. Note that, as expected, in this framework we obtain the characteristic function of a multivariate normal distribution. \square

The probability density function $f(r)$ of the random variable R_{t+1} can be get by the result of the Fourier inversion theorem

$$f(r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iur} \varphi_R(u) du .$$

Moreover, we can also obtain the distribution function $F(r)$ of the same variable by the following theorem.

Theorem 3.1 *Let $f(r)$ and $\varphi_R(u)$ be Lebesgue-integrable, if the mean and variance of the random variable R_{t+1} exist, then its distribution function $F(r)$ will be*

$$F(r) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-iur} \varphi_R(u)}{iu} \right] du , \quad (13)$$

where $\operatorname{Re}[g(u)]$ means the real part of $g(u)$. \square

For more details see appendix while for a proof see Shephart [22].

Recalling that $F(r) = \mathbb{P}_t[R_{t+1} < r]$, we can guess that if $F(r) = \alpha$ then r will be equal to $-\frac{VaR}{V_t}$. From what above, we can obtain the portfolio VaR by numerically solving the following equation:

$$\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{1}{iu} \exp \left(iu \frac{VaR}{V_t} \right) \varphi_R(u) \right] du = \alpha . \quad (14)$$

3.3.1 How to construct K , C , A and \tilde{a}

For a stock, the elements of the vector K are obtained by the first part of the equation (2), while for an option they are zero.

The construction of the matrixes C , A and \tilde{a} is a bit more complicated. Indeed, with the exception of the framework in Remark 3.5, the three matrixes are not square and the coefficient cannot be placed on the diagonal.

A general rule is to put the coefficients of the i -th asset which depends on the j -th risk factor on the element i, j of each matrix. Hence, the coefficients of the assets written on the same risk factor will be on the same column.

Moreover, the linear part of the portfolio has to be a zero coefficient in both the matrix A and \tilde{a} . In the following example there is an application of such a rule.

Example 3.1 Let consider a portfolio composed by three options and a stock. Two of the options (the first and the third) are written on the S&P 500 Index, while the other on the NASDAQ 100 Index. Both indexes can be taken as risk factors for the stock such that $n = 4$ and $m = 2$.

The portfolio return can be approximated as in equation (10) where

$$\begin{aligned} C &= \begin{bmatrix} \frac{S_1}{X_1} \frac{\partial X_1}{\partial S_1} & 0 \\ 0 & \frac{S_2}{X_2} \frac{\partial X_2}{\partial S_2} \\ \frac{S_1}{X_3} \frac{\partial X_3}{\partial S_1} & 0 \\ \beta_1 & \beta_2 \end{bmatrix} & A &= \frac{1}{2} \begin{bmatrix} \frac{S_1^2}{X_1} \frac{\partial^2 X_1}{\partial S_1^2} & 0 \\ 0 & \frac{S_2^2}{X_2} \frac{\partial^2 X_2}{\partial S_2^2} \\ \frac{S_1^2}{X_3} \frac{\partial^2 X_3}{\partial S_1^2} & 0 \\ 0 & 0 \end{bmatrix} \\ \tilde{a} &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \\ a_3 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where S_1 and S_2 are the prices at time t of the S&P 500 Index and of the NASDAQ 100 Index, X_1 , X_2 and X_3 are the prices of the options while β_1 and β_2 are respectively the sensitivity of the stock to the first and second index.

By substituting the above matrixes in equation (10) and multiplying we have the following approximation for the portfolio return:

$$\begin{aligned}
R_{t+1} = & a_1 \frac{S_1}{X_1} \frac{\partial X_1}{\partial S_1} \tilde{R}_{1,t+1} + a_1 \frac{1}{2} \frac{S_1^2}{X_1} \frac{\partial^2 X_1}{\partial S_1^2} \tilde{R}_{1,t+1}^2 + \\
& + a_2 \frac{S_2}{X_2} \frac{\partial X_2}{\partial S_2} \tilde{R}_{2,t+1} + a_2 \frac{1}{2} \frac{S_2^2}{X_2} \frac{\partial^2 X_2}{\partial S_2^2} \tilde{R}_{2,t+1}^2 + \\
& + a_3 \frac{S_1}{X_3} \frac{\partial X_3}{\partial S_1} \tilde{R}_{1,t+1} + a_3 \frac{1}{2} \frac{S_1^2}{X_3} \frac{\partial^2 X_3}{\partial S_1^2} \tilde{R}_{1,t+1}^2 + \\
& + a_4 K + a_4 (\beta_1 \tilde{R}_{1,t+1} + \beta_2 \tilde{R}_{2,t+1}) ,
\end{aligned}$$

that is what we expect to have. \square

With the data of the above example the matrix B is symmetric. This is always true when each option price depends on a risk factor only. When this is not true, the symmetry is no longer ensured. In such a situation we can approximate the non symmetric matrix B with the symmetric matrix $\frac{1}{2}(B + B')$.

3.3.2 Some other application issues

As explained above, to find the VaR we have to numerically solve equation (14). To perform this task we can use an algorithm such as the Gauss-Newton one.

The problem is that on the left part of the equation there is an integral that we have to solve numerically. Hence, we need a method able to give quite accurate results in a short time⁶. For such a reason all the simulation based methods cannot be considered.

For our analysis we employed the Gauss-Laguerre approximation formula that is easy to apply for integrals defined in the interval $(0, \infty)$ and gives an error of order $2p - 1$ where p is the degree of the used Laguerre polynomial. In our application the Laguerre polynomial of degree 14 is considered.

The portfolios of section 6 contain short positions which can be considered by taking negative weights for them. Moreover, our portfolios are investment portfolios i.e. they have long total position or in an other way, the sum of all the weights has to be equal to one.

With a short total position i.e. where the sum of all the weights is equal to minus one the same approach cannot in general be applied. Indeed, in such a situation we have to consider the right tail instead of the left one. Intuitively, we can say that the VaR estimation in both situations would be equal only if the return distribution was symmetric. To make the issue clear let us show the following trivial example.

⁶Remember that, once solved, the integral has to take place in an other numerical procedure.

Example 3.2 Let us consider two portfolios. The first one is composed by a long position on a stock while the second by a short position on the same stock. The price of the stock at time t is X_t such as the values of the portfolios are respectively $V_{1,t} = X_t$ and $V_{2,t} = -X_t$.

Let $F(r)$ be the stock return distribution function. The VaR for the first portfolio is

$$\begin{aligned}\mathbb{P}_t [V_{t+1} - V_t < -VaR_1] &= \alpha \\ \mathbb{P}_t [R_{t+1} < -VaR_1/X_t] &= \alpha\end{aligned}$$

such that

$$VaR_1 = -X_t F^{-1}(\alpha) ,$$

where $F^{-1}(\alpha)$ is the α -quantile of the distribution function $F(r)$.

For the second portfolio the VaR is

$$\begin{aligned}\mathbb{P}_t [V_{t+1} - V_t < -VaR_2] &= \alpha \\ \mathbb{P}_t [-(X_{t+1} - X_t) < -VaR_2] &= \alpha \\ \mathbb{P}_t [X_{t+1} - X_t < VaR_2] &= 1 - \alpha \\ \mathbb{P}_t [R_{t+1} < VaR_2/X_t] &= 1 - \alpha\end{aligned}$$

such that

$$VaR_2 = X_t F^{-1}(1 - \alpha) ,$$

that is equal to VaR_1 only if $F^{-1}(1 - \alpha) = -F^{-1}(\alpha)$ i.e. if the probability density function of the stock returns is symmetric. \square

4 Non parametric method based on historical simulations

By taking a time series of data of dimension t as a sample of the whole population it is possible to get an empirical distribution of the portfolio. Indeed, at every date of the sample the past risk factor returns are used to revalue the portfolio and to get the empirical distribution of its return.

To avoid the misspecification problem of the function $g(\cdot)$, we will use directly the prices of the assets instead of the risk factors. While it is not so difficult for the stocks it becomes quite infeasible for options. Indeed a time series of the price of the options with the same strike price and the same maturity is not in general available. To face the problem the most common approach is to use the Black-Scholes formula to evaluate the option price at each simulated underlying value. Some problems due to this approach will be considered later on.

However, once we have the time series of the portfolio returns, we can get an empirical distribution. The VaR estimate is taken as the α -quantile of such a distribution. The idea is to replace in equation (1) the theoretical probability by an empirical frequency:

$$\frac{1}{t} \sum_{k=1}^t \mathbf{1}_{\{V_t R_{t-k+1} < -VaR\}} = \alpha ,$$

such that VaR can be get by the following minimization problem:

$$\widehat{VaR}_t = \arg \min_{VaR} \left[\left(\frac{1}{t} \sum_{k=1}^t \mathbf{1}_{\{-(V_t R_{t-k+1} + VaR) > 0\}} \right) - \alpha \right]^2 , \quad (15)$$

that has, in general, no closed form solution.

We can see equation (15) as a non-linear last square regression

$$\alpha = \mathbf{1}_{\{-(V_t R_{t-k+1} + VaR) > 0\}} + \varepsilon_{t-k} \quad k = 1, 2, \dots, t$$

where VaR is now the true VaR and $\varepsilon_{t-k} \sim \text{iid}(0, \sigma_\varepsilon)$. Summing and dividing by the sample dimension we get

$$\alpha = \frac{1}{t} \sum_{k=1}^t \mathbf{1}_{\{-(V_t R_{t-k+1} + VaR) > 0\}} + \frac{1}{t} \sum_{k=1}^t \varepsilon_{t-k} .$$

Note that the last term on the above equation go to zero if and only if all the error terms are independently and identically distributed with mean zero. Moreover, the distribution of ε_{t-k} depends on the distribution from what the sample is drawn. The consequence is that if we want to get a consistent estimator we have to assume that the returns are iid.

Remark 4.1 Sometimes to smooth the returns distribution the indicator function is replaced by a Gaussian kernel such that equation (15) becomes

$$\widehat{VaR}_t = \arg \min_{VaR} \left\{ \frac{1}{t} \sum_{k=1}^t \Phi \left[-\frac{1}{h} (V_t R_{t-k+1} + VaR) \right] - \alpha \right\}^2 ,$$

where $\Phi(\cdot)$ is the normal distribution function and h is the bandwidth. The latter is a measure of the accuracy of the approximation such that the smaller is the bandwidth the higher will be the precision in using $\Phi(\cdot)$ to approximate the indicator.

Another function to approximate the indicator is the following:

$$\mathbf{1}_{x>0} \approx \frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{x}{h} \right) ,$$

such that equation (15) becomes

$$\widehat{VaR}_t = \arg \min_{VaR} \left\{ \frac{1}{2} + \frac{1}{t\pi} \sum_{k=1}^t \arctan \left[-\frac{1}{h} (V_t R_{t-k+1} + VaR) \right] - \alpha \right\}^2 , \quad (16)$$

where h has the same meanings than before. \square

4.1 Some hidden drawbacks

The easiness of the implementation and the absence of explicit model assumption on the risk factors returns are the main benefits of the approach. There are some drawback, though. The most relevant ones are the following:

- How to evaluate the options? It is necessary to introduce some assumptions in the underlying returns distribution that partially reduce one of the two main benefits.
- A lot of data have to be used (one year of daily data or more), otherwise the empirical distribution is not properly defined on the tails. This drawback is common to all the methodology based on simulations.
- An implicit assumption of the method is that in sample returns are iid. This assumption is violated by the evidence that the volatility changes over time. This causes an inconsistent estimation of VaR⁷.
- Moreover, the iid assumption allows us to say that every realized return has the same probability to occur again. By giving the same probability to occur for each return we are reducing the sensibility of VaR to the changes of risk due to market crashes.

To face the last two problems some methods can be used. In the next section a short description of the Boudoukh, Richardson and Whitelaw [7] method is given. In section 5 a filtered historical simulation method is described.

4.2 Generalized historical simulation method

As previously said, one of the drawbacks of the standard historical simulation method is the assumption of iid returns. This enables us to say that every realized return has the same probability to occur again. As noted by Pritsker [21], this reduces the sensibility of the VaR measure to the changes of risk due to market crashes.

To face the problem Boudoukh, Richardson and Whitelaw [7] suggest to give different weights to different realized returns such that equation (15) becomes

$$\widehat{VaR}_t = \arg \min_{VaR} \left[\left(\sum_{k=1}^t p_k \mathbf{1}_{\{-(V_t R_{t-k+1} + VaR) > 0\}} \right) - \alpha \right]^2, \quad (17)$$

where p_k is the weight and $\sum_k^t p_k = 1$.

⁷The same idea is discussed by Hendricks [14] and by McNeil and Frey [18].

In particular they suggest to give a higher weight to the most recent realizations by using a weight that decays with time

$$\begin{aligned} p_k &= \left(\sum_{i=1}^t \lambda^{i-1} \right)^{-1} \lambda^{k-1} \\ &= \frac{1-\lambda}{1-\lambda^t} \lambda^{k-1} \quad k = 1, 2, \dots, t \end{aligned}$$

where $\lambda \in (0, 1)$ is called the decay factor. Note that for λ equal to one we have equation (15).

Moreover, we can say that $\lambda p_k = p_{k-1}$. This shows that the lower λ is the higher the decay effect on the weights associated with far returns will be. This should increase the VaR sensibility to market crashes or in general to risk raises.

5 Filtered historical simulation

In previous sections we described the historical simulation approach. The assumption was of iid return. Such an assumption is violated by volatility changing over time. As noted above this leads to an inconsistent estimate of VaR.

Barone-Adesi, Bourgoin and Giannopoulos [3] introduced a method to face the problem. The approach is based on historical data. Indeed, the aim is to have a sort of independence from the assumptions on the risk factors distribution. The main idea is to obtain iid returns by standardizing them i.e. by dividing them by their volatility. Such a volatility can be estimated by assuming a model for the return with, say, some GARCH(1,1) errors.

Let us describe the idea of the method in the one dimension framework. For the j -th asset we can assume the following model:

$$R_{j,t} = \varepsilon_{j,t} \quad j = 1, 2, \dots, m ,$$

where

$$\varepsilon_{j,t} \sim N(0, h_{j,t}) ,$$

and

$$h_{j,t} = \alpha_0 + \alpha_1 R_{j,t-1}^2 + \alpha_2 h_{j,t-1} , \quad (18)$$

To implement the methodology one has to consider the following steps:

1. collect a set of observed daily returns $R_{j,t}$ for $t = 1, 2, \dots, T$;
2. estimate equation (18) to have the estimated variance $\hat{h}_{j,t}$ for each time t ;

3. define the standardized residuals in the following way:

$$e_{j,t} := \frac{R_{j,t}}{\sqrt{\hat{h}_{j,t}}} \quad t = 1, 2, \dots, T$$

4. pick randomly (with replacement) one of the T standardized residuals (let us define it e^*);
5. forecast the variance of the period $T + 1$ by equation (18);
6. define the simulated innovation forecast for time $T + 1$ as

$$z_{j,T+1}^* := e^* \sqrt{\hat{h}_{j,T+1}} ,$$

7. define the $T + 1$ simulated risk factor price as

$$S_{j,T+1}^* := S_{j,T}(1 + z_{j,T+1}^*)$$

8. by using $S_{j,T+1}^*$ and $\hat{h}_{j,T+1}$ calculate the simulated portfolio price and then the simulated portfolio returns.

By repeating the procedure from step 4 to step 8 we can obtain a simulated probability density function for the one day returns that may be used to calculate the VaR of the portfolio.

5.1 The model misspecification

In the step 4 the bootstrap from the sample can be done by assuming that all the standard residuals are equally probable. Therefore, the random date can be picked from a uniform distribution. Indeed, the use of a uniform distribution is the best thing to do since we are sure that the standardized residuals are iid, or, in an other way, that the model (18) is the true model for the market volatility.

If we consider the possibility that the model could be misspecified, we will not be sure about the iid of the standardized residuals. In such a situation, to use a uniform distribution could not be the best way to sample the random date. As in sections 4.2, to increase the sensibility of the VaR measure to risk changes, we can impute an higher probability of occurrence to the standard residuals obtained by the most recent returns. This can be done by extracting the random date from an exponential instead of a uniform distribution.

5.2 The multivariate framework

In the multivariate extension of the above method no variance-covariance matrix is used. Indeed, the bootstrap is done not directly on the residual returns, but on the past states of the world. A state of the world is represented by the m risk factor returns observed at a certain date.

In particular, we construct an $t \times m$ matrix where in the column there are the time series of the each risk factor return. The returns have to be standardized with the procedure described above.

The bootstrap is done by picking randomly a row vector from the standardized return matrix. Now \bar{e} is an m -vector and it is used to generate the simulated innovation as in step 6.

The portfolio is revalued for each randomly picked state of the world such to obtain an empirical distribution. From the empirical distribution the VaR estimate is obtained by the same techniques which are used for the historical simulation.

6 The considered portfolios

We will consider four portfolios which consist of the same kind of assets (equities and options on index) but with different composition rates. A certain percentage of equities are diversified inside the American biotechnological industry. The choice of this industry is motivated by the will to consider an high volatility portfolio whose stocks are quoted in a market that suffered by some strong crashes in the recent past. It enable us to observe the sensibility of the VaR estimate to an increase of risk due to a market crash.

Moreover, in some portfolios there is also a group of less volatile stocks traded in the NYSE.

In two of the four portfolios there are option positions. In particular, the options are four months at the money European calls and puts written on the S&P 500 Index and on the NASDAQ 100.

Later on, the different compositions of the four portfolios are shortly described. The percentages are expressed with respect to the market price of the assets.

Portfolio A All the portfolio is represented by the biotech equities such that only linear positions are considered.

The idea is to show what the error made by neglecting the firm specific risk is. Indeed, two measures of VaR for the parametric method will be calculated. The first one is got by using directly the asset returns while in the second one an APT approximation as in equation (2) will be used.

The difference between the two estimate should give a measure of the error made by considering only the market risk.

Portfolio B The 63% of the portfolio is represented by the biotech equities while the residual 37% by other stocks.

We will make an analysis equivalent to what above. The aim is to show the effect of the introduction of new equities with a reduced correlation with the former portfolio.

Portfolio C The biotech stocks are the 56% of the total portfolio while the 12% is represented by long positions in the S&P 500 Index European put options and in the NASDAQ 100 Index European put options. In the residual part there are other stocks.

Both the options are in the money. Their function is to reduce the market risk of the equity portfolio. Indeed, the number of put options and the strike price are chosen to reduce, in the next four month, the probability of a loss for more than the 95% of the equity portfolio. The aim is to show how such a hedging strategy can change the VaR estimate.

Portfolio D The biotech stocks are the 56% of the total portfolio while the 7% is represented by short positions in the S&P 500 Index European call options and in the NASDAQ 100 Index European call options. In the residual part there are other stocks.

Both the options are out of the money and are not used for hedging purposes but for a sort of speculative aim such to increase the portfolio profitability. This kind of behavior strongly increases the portfolio risk and the aim is to show how the different VaR methods are able to detect such a rise.

The exact composition of the portfolios is in appendix E.

7 A first glance on the empirical results

The different VaR estimates at 30.05.2001 are summarized in table 1 and 2 respectively for the accepted loss probability of 1% and 5%.

In the application of the parametric approach for the linear portfolios A and B the assumption is only that the returns are normally distributed,⁸ while for portfolio C and D a delta-gamma approximation is done.

For the portfolio A the only risk factor considered for the APT approximation is the NASDAQ Biotechnology Index. Moreover, the used betas are not exactly the estimated ones. Indeed, estimated betas are corrected when their R^2 ratio is lower than 0.4. By satisfying a prudential aim, in such a

⁸For both portfolios equations (3) and (4) are used respectively for the method 1 and 2.

Table 1: VaR estimated by the methods described in the previous sections for the four considered portfolios. The accepted loss probability is 1% ($\alpha = 0.01$) and the horizon is one day. The VaR is expressed in percentage with respect to the portfolio price at time t .

	Portfolios			
	A	B	C	D
1) Parametric VaR	7.50%	6.06%	3.92%	19.76%
2) Parametric VaR (APT)	9.15%	6.84%	2.11%	20.92%
3) Standard Historical Simulation	10.11%	7.41%	5.41%	10.10%
4) Weighted H. S. ($\lambda = 0.99$)	9.24%	7.40%	4.95%	10.77%
5) Weighted H. S. ($\lambda = 0.97$)	7.16%	6.24%	4.17%	10.09%
6) Filtered H. S. (Uniform)	9.32%	6.90%	8.57%	16.58%
7) Filtered H. S. (Exponential)	7.75%	5.93%	8.79%	16.53%

Table 2: VaR estimated by the methods described in the previous sections for the four considered portfolios. The accepted loss probability is 5% ($\alpha = 0.05$) and the horizon is one day. The VaR is expressed in percentage with respect to the portfolio price at time t .

	Portfolios			
	A	B	C	D
1) Parametric VaR	5.30%	4.28%	2.62%	12.55%
2) Parametric VaR (APT)	6.47%	4.83%	1.38%	16.47%
3) Standard Historical Simulation	5.88%	4.44%	3.39%	6.79%
4) Weighted H. S. ($\lambda = 0.99$)	6.35%	4.69%	3.51%	6.89%
5) Weighted H. S. ($\lambda = 0.97$)	4.73%	4.53%	2.23%	5.52%
6) Filtered H. S. (Uniform)	5.22%	4.34%	5.72%	10.55%
7) Filtered H. S. (Exponential)	4.34%	4.11%	4.88%	9.73%

situation the corrected beta is set equal to 1 when the estimated beta is lower than 1.1 and equal to 2 when it is higher than 1.1.

For the portfolios B, C and D four indexes are used as risk factors in the APT approximation: the NASDAQ Biotechnology Index, the NASDAQ 100 Index, the S&P 500 Index and the NYSE Financials Index. The choice of the above indexes is justified by the portfolio composition. Here the betas are not corrected. Indeed, the R^2 ratio is higher than in previous situation.

The variance-covariance matrixes are estimated by the so called Orthogonal GARCH(1,1). A short description of the methodology and some references are in the appendix C. The parameters of the GARCH model are estimated by using 300 daily observations.

Figure 2 shows the conditional probability density function of the return of the portfolios C and D compared with a normal distribution. Such a probability density function is obtained by assuming the normality of the risk factors return and by using a quadratic approximation for the options

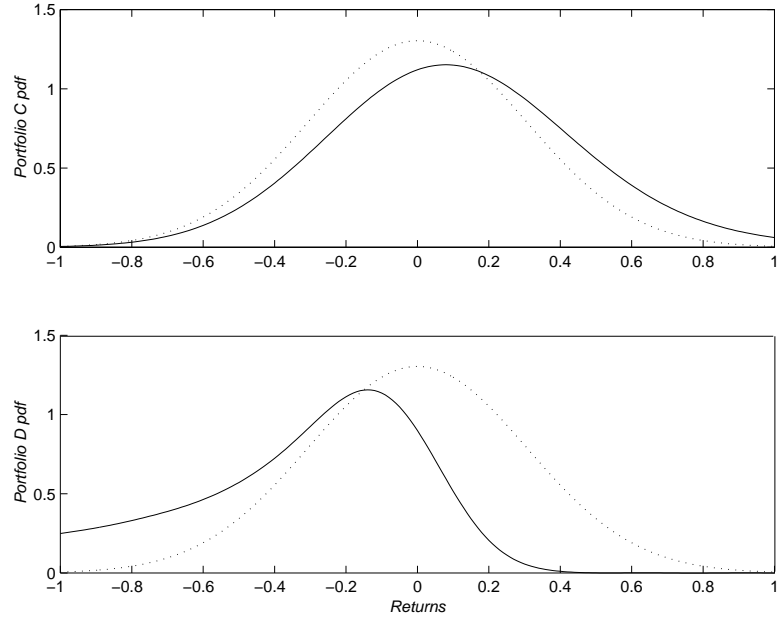


Figure 2: Conditional probability density function of the return of the portfolios C (above) and D (below) compared with a normal distribution (shaded line).

return. The asymmetric shape of the probability function due to the non linear positions inside the portfolio.

The historical simulations are done by using equation (16) with 500 past daily returns and a bandwidth equal to 0.01. We decided to use an approximation instead of the equation (15), because it works better in the minimization problem.

In portfolio C and D the options are revalued by using the Black-Scholes formula with a constant volatility equal to the implied volatility at date t .

The weighted historical simulation approach is performed by using two different decay factors. The first one (0.99) gives a lower decay effect than the second one (0.97). The latter yields a VaR estimate more sensitive to the changes in market risk.

In the filtered historical simulation method, the sample used is of 400 past observations. To reduce the impact on the arbitrary choice of the starting point on the volatility path, once estimated the GARCH(1,1) we let it run from 100 days before the furthest date.

Within the sample the standardized residuals are picked randomly 5000 times. The bootstrap is done either by picking randomly a date from a uniform distribution or by using an exponential distribution with parameter λ equal to 0.99.

Table 3: Number of failures x and their proportion \hat{p} with respect to the whole sample of 500 observations (from 07.06.1999 to 30.05.2001). The accepted loss probability is 1% ($\alpha = 0.01$) and the horizon is one day.

	Portfolios							
	A		B		C		D	
	x	\hat{p}	x	\hat{p}	x	\hat{p}	x	\hat{p}
1) Param. VaR	6	1.2%	5	1.0%	27	5.4%	5	1.0%
2) Param. VaR (APT)	6	1.2%	10	2.0%	42	8.4%	3	0.6%
3) S.H.S.	8	1.6%	9	1.8%	13	2.6%	12	2.4%
4) W.H.S. ($\lambda = 0.99$)	9	1.8%	10	2.0%	9	1.8%	6	1.2%
5) W.H.S. ($\lambda = 0.97$)	9	1.8%	10	2.0%	12	2.4%	10	2.0%
6) F.H.S. (Uniform)	7	1.4%	7	1.4%	8	1.6%	7	1.4%
7) F.H.S. (Exp.)	8	1.6%	8	1.6%	7	1.4%	8	1.6%

8 Testing the VaR estimations

To verify the accuracy in VaR estimate one of the most used methods is the so called reality check test based on the observation of the VaR performances during a period of time.

The observed variable is the number of VaR failures. We have a VaR failure when we observe a loss higher than the estimated VaR. The percentage of failure with respect to the total of the considered observations, say \hat{p} , should be as near as possible to the defined loss probability α .

Table 3 and 4 show the number of failures and their proportion with respect to the whole sample respectively for the accepted loss probability of 1% and 5%. They are obtained by estimating VaR for the different portfolios during a period of 500 days from June 1999 to May 2001. The obtained VaR is compared with the loss of the day after. If the latter is greater than VaR, we will have a failure.

By using a likelihood ratio statistic, a test can be done to decide whether to reject the null hypothesis that the probability of failure p is equal to α .

The starting point is to assume that the sample is drawn from a Bernoulli population where we can have two possible events: the VaR can cover the loss or the VaR is not sufficient to cover the loss. If the Bernoulli random variables are independent the probability to have x failure in a sample of size n will be given by a binomial distribution:

$$\mathbb{P}(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, \dots, n \\ 0 & \text{elsewhere} \end{cases}$$

For the null hypothesis $p = \alpha$ and assuming that x is the observed number

Table 4: Number of failures and their proportion (in brackets) with respect to the whole sample of 500 observations (from 07.06.1999 to 30.05.2001). The accepted loss probability is 5% ($\alpha = 0.05$) and the horizon is one day.

	Portfolios							
	A		B		C		D	
	x	\hat{p}	x	\hat{p}	x	\hat{p}	x	\hat{p}
1) Param. VaR	24	4.8%	23	4.6%	41	8.2%	10	2.0%
2) Param. VaR (APT)	15	3.0%	32	6.4%	70	14.0%	8	1.6%
3) S.H.S.	52	10.4%	50	10.0%	45	9.0%	33	6.6%
4) W.H.S. ($\lambda = 0.99$)	32	6.4%	32	6.4%	30	6.0%	29	5.8%
5) W.H.S. ($\lambda = 0.97$)	28	5.6%	28	5.6%	27	5.4%	28	5.6%
6) F.H.S. (Uniform)	40	8.0%	39	7.8%	30	6.0%	27	5.4%
7) F.H.S. (Exp.)	25	5.2%	24	4.8%	25	5.0%	26	5.2%

of failures, the likelihood ratio can be defined in the following way:

$$W(x, \alpha) = -2 \ln [\alpha^x (1 - \alpha)^{n-x}] + 2 \ln [(\hat{p})^x (1 - \hat{p})^{n-x}] ,$$

where \hat{p} is equal to $\frac{x}{n}$.

Under the null hypothesis, the test statistic $W(x, \alpha)$ is asymptotically distributed as a χ_1^2 such that if we accept a first kind error equal to 5% the rejection region will be

$$\{x : W(x, \alpha) \geq c_{0.05}\} ,$$

where $c_{0.05}$ is the 0.05 quantile of the χ_1^2 distribution that is equal to 3.841.

By solving the inequality inside the brackets with sample size of 500 observations, we have that the non rejection region is approximately equal to

$$\begin{aligned} 1 \leq x \leq 10 & \quad \text{if } \alpha = 0.01 \\ 16 \leq x \leq 35 & \quad \text{if } \alpha = 0.05 . \end{aligned}$$

According to the Neyman-Pearson theorem, the likelihood ratio test is the uniformly most powerful test against simple alternative hypothesis. Table 5 shows the power of the test for different alternative hypothesis. To increase the power of such a test a wider sample is necessary. This is one of the reasons why we chose a one day horizon.

Let us consider the accepted loss probability of 1%. For the first method the results that we obtain for the two portfolios, which contain linear positions only, are quite the same. For both portfolios such a method seems to work quite well. The APT approximation works definitely better for the

Table 5: Power of the likelihood ratio statistic for a sample size of 500 observations.

Null Hypothesis $p=0.01$		Null Hypothesis $p=0.05$	
Alternative Hypothesis	Power	Alternative Hypothesis	Power
0.020	0.443	0.060	0.181
0.030	0.899	0.075	0.671
0.040	0.992	0.100	0.991

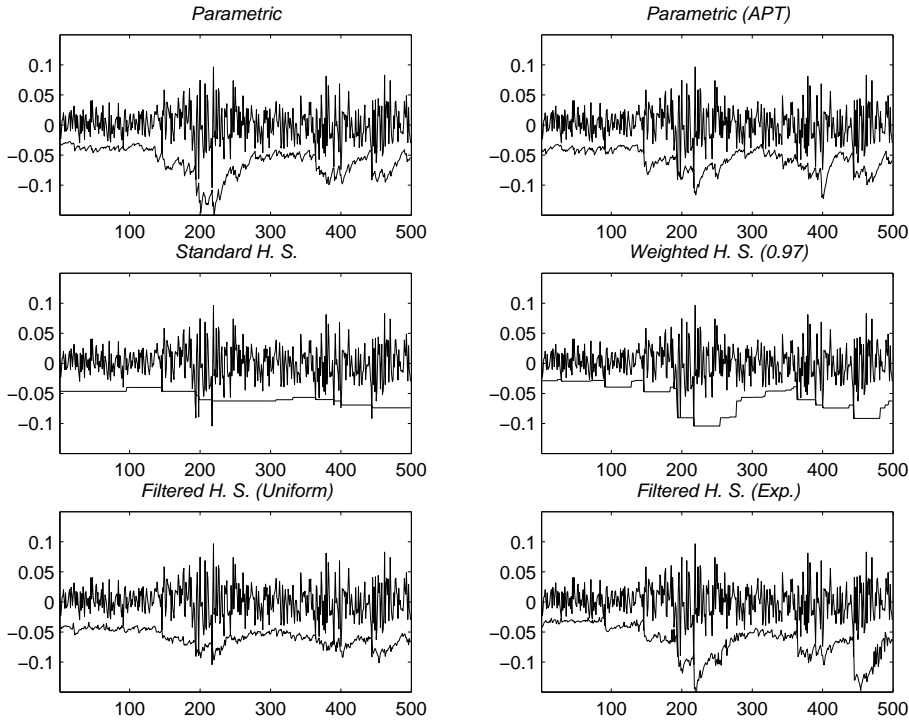


Figure 3: Portfolio B returns of the period from June 1999 to May 2001 compared with the 1% VaR.

portfolio A than for portfolio B. The reason is that for the first portfolio we made a prudential correction of the betas.

On the other hand, the delta-gamma approach, with or without APT approximation, seems to perform badly for both the portfolios containing options. As expected, the quadratic approximation underestimates the risk for a long position while overestimates the risk for a short position.

This is also evident from table 1 where the VaR estimated by the delta-gamma approach is two times the value obtained by the historical simulations. The conclusion is that for the portfolio C the value of the variable x is on the rejection region while the test suggests to accept the null hypothesis

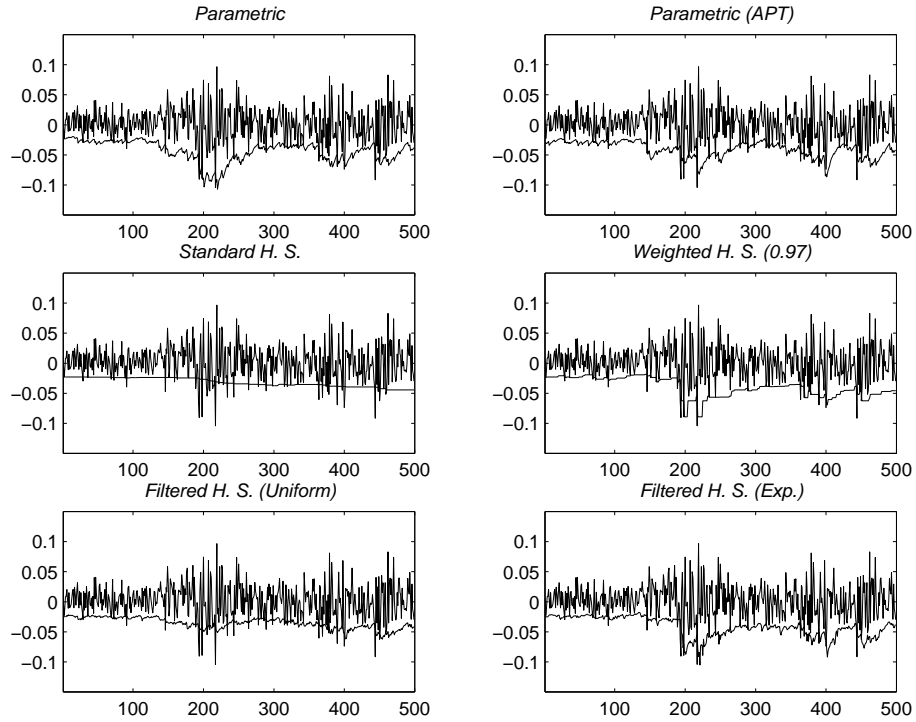


Figure 4: Portfolio B returns of the period from June 1999 to May 2001 compared with the 5% VaR.

for the portfolio D. The same results are confirmed for the accepted loss probability of 5%.

For the two equity portfolios, the number of failures obtained by using the historical simulation methods is inside the non rejection region. For 1% accepted loss probability the three non-parametric methods give quite the same results, while the filtered historical simulations seems to work slightly better.

The results are different for both the non linear portfolios. Indeed, in such a situation the standard historical simulation perform very poorly. This is due to the lack of sensibility of the method with respect to changes in market risk. Indeed, from figure 5 we can see the weak reaction of the standard historical simulation VaR to the recent NASDAQ crash. As expected, the weighted methods seems to work better during a market crash.

For the accepted loss probability of 5% the lack of sensibility of the standard historical simulation gives poor performances also for the equity portfolios. Figure 4 shows how the VaR estimated by using this method go through all the losses during the crash period.

The semi-parametric method of the filtered historical simulation gives some quite accurate results for each portfolio. Due to the GARCH forecast

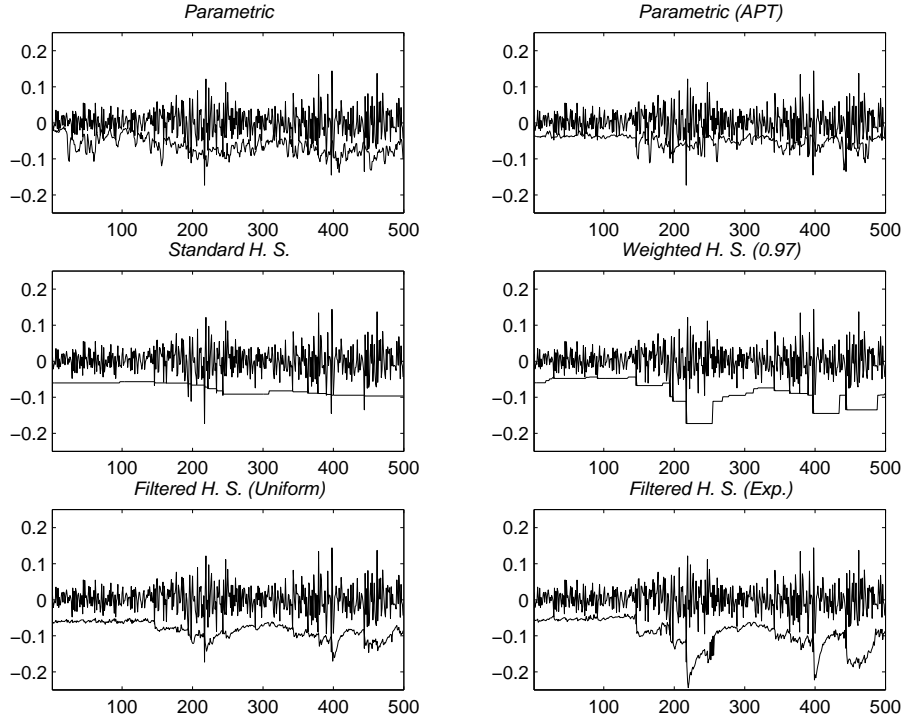


Figure 5: Portfolio C returns of the period from June 1999 to May 2001 compared with the 1% VaR.

of the variance, the semi-parametric methods are very sensitive to market crashes. This is evident in figure 5 and 6 where it is possible to note that the sensitivity is higher for the modified semi-parametric approach than for the standard one. For such a reason, the modified method performs slightly better for the accepted loss probability of 5%.

To have an intuition on the tails beyond VaR, one can observe the losses which exceed the VaR. The measure that can be used as an indicator of the exceeding losses is the mean of the ratio between the losses which exceed VaR and the VaR itself. Table 6 and 7 show such a ratio for the different methods considered. With the same aim, in the table there is also the maximum of such a ratio.

First of all, we can say that the mean ratio is always higher than what we expected to have under the normality of the portfolio returns. Indeed, the values of the ratio under the normality assumption⁹ are 1.146 for an accepted loss probability of 1% and 1.254 for an accepted loss probability of 5%.

⁹In appendix D is briefly described how to reach this result.

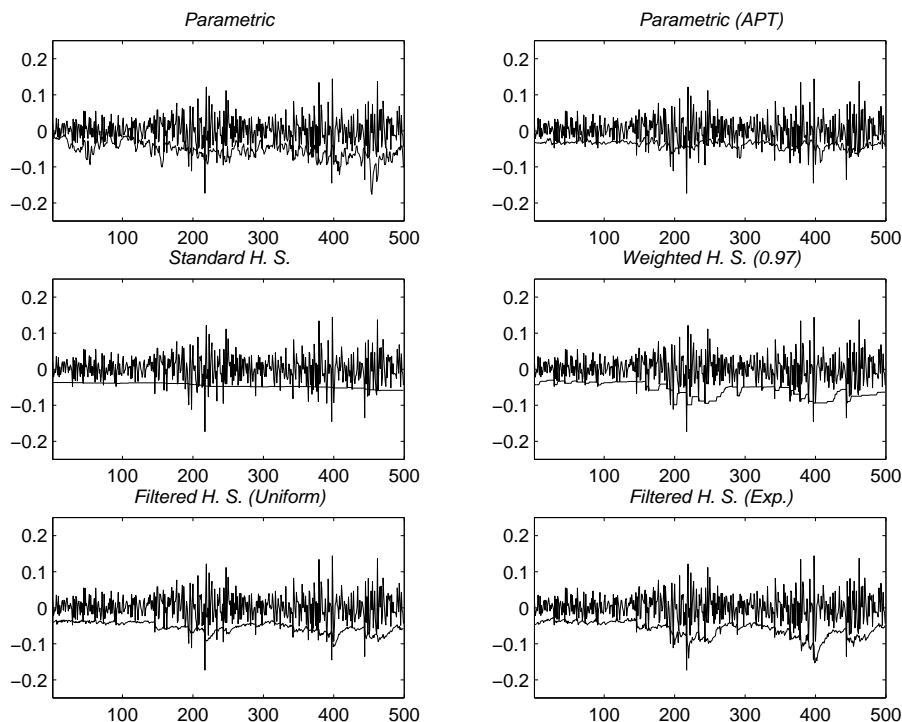


Figure 6: Portfolio C returns of the period from June 1999 to May 2001 compared with the 5% VaR.

Moreover, the conclusion that we can get is that, once again, while for equity portfolios the performances of the seven methods are quite the same, for the non linear portfolios the historical simulation approaches work definitely better than the delta-gamma approximation. Indeed, for the two parametric methods both the mean and the maximum of the ratio are higher than the same ratio obtained for the historical simulation methods.

9 Conclusion

In the first part of this work some of the most known VaR estimation methods are described. In such a description we try to focus mainly on the application problems of each approach, which are modified to take into account non linear positions.

The second part is devoted to measuring the performances of the different estimation methods, by testing them on the last two years. The observed variable is the number of times that the loss exceed VaR. To have an idea on what the returns behavior beyond the quantile is, we consider also the mean ratio between the losses which exceed VaR and the VaR itself. The sample

Table 6: Mean value and maximum value of the ratio between the losses which exceed VaR and the VaR itself (from 07.06.1999 to 30.05.2001). The accepted loss probability is 1% ($\alpha = 0.01$) and the horizon is one day.

	Portfolios							
	A		B		C		D	
	mean	max	mean	max	mean	max	mean	max
Param.	1.30	1.77	1.26	1.90	1.45	3.56	1.96	2.42
Param. (APT)	1.17	1.33	1.23	1.57	1.42	3.42	1.92	2.60
S.H.S.	1.47	1.81	1.40	1.92	1.36	2.61	1.35	2.50
W.H.S. (0.99)	1.37	1.60	1.26	1.49	1.36	1.75	1.31	1.93
W.H.S. (0.97)	1.32	1.67	1.34	1.66	1.27	1.62	1.33	1.88
F.H.S. (Unif.)	1.39	1.74	1.23	1.63	1.29	1.76	1.34	1.96
F.H.S. (Exp.)	1.31	1.87	1.31	1.50	1.31	1.57	1.35	1.90

period considered for the test includes the recent crash of the NASDAQ.

The conclusions are quite different for the different portfolios considered. Indeed, for linear portfolios all the methods give quite accurate results while the best performance are obtained by the parametric method with a covariance matrix estimated by the Orthogonal GARCH. Unfortunately, most of the financial institutions have portfolios with strong positions on options or other non linear instruments.

The results are visibly different for portfolios which contain non-linear positions. Indeed the increase of volatility due to the leverage effect of option positions, reduces the accuracy of some methods. Here the performances are very different among the approaches and, unlike to the former results, the worst performances are obtained by the parametric method. As expected the quadratic approximation underestimates the risk on long option positions while it overestimates the short option positions.

The generalization of the standard historical simulation strongly improves the accuracy on VaR estimate and, particularly, its sensitivity to crashes of the market. This is evident from figure 3 and figure 5 but also from the lower ratio of failure with both the accepted loss probability.

The best performance is got by the semi-parametric method of the filtered historical simulations. The standardization of the returns reduce the non iid problem while the bootstrap enable us to have a well defined simulated distribution. In the standard approach the assumption is that the volatility model is correctly specified. By accounting a possible model misspecification, the modified filtered historical simulation method seems to perform better for the accepted loss probability of 5%.

Note, however, that the filtered historical simulation method is computationally more time consuming than the other approaches, such that no

Table 7: Mean value and maximum value of the ratio between the losses which exceed VaR and the VaR itself (from 07.06.1999 to 30.05.2001). The accepted loss probability is 5% ($\alpha = 0.05$) and the horizon is one day.

	Portfolios							
	A		B		C		D	
	mean	max	mean	max	mean	max	mean	max
Param.	1.35	2.50	1.31	2.68	1.55	4.28	1.43	4.25
Param. (APT)	1.32	1.87	1.33	2.22	1.52	4.22	1.53	4.61
S.H.S.	1.56	2.22	1.50	3.56	1.51	3.74	1.59	3.74
W.H.S. (0.99)	1.51	3.81	1.51	3.38	1.53	3.15	1.50	3.24
W.H.S. (0.97)	1.42	2.75	1.44	2.55	1.47	2.67	1.47	3.00
F.H.S. (Unif.)	1.46	2.85	1.39	2.80	1.42	2.80	1.42	2.92
F.H.S. (Exp.)	1.46	2.62	1.40	2.16	1.41	2.37	1.40	2.27

one of the method can be considered the best from all the perspectives.

The concluding remark is that while the first three methods give strongly inaccurate results in the considered framework, by the last four methods we may have more accurate performances. However, among these it is not possible to define the best method but only to describe the trade-off between estimation accuracy and computational time.

A Proof of the Proposition 3.1

By the properties of the positive definite matrixes we can decompose Σ as

$$\Sigma := HH' ,$$

such that

$$\tilde{R}_{t+1} = HZ_{t+1} ,$$

where Z_{t+1} is an $m \times 1$ vector of independent standard normal variables.

By using the above notation, equation (11) can be written as

$$R_{t+1} = \kappa + c'HZ_{t+1} + Z'_{t+1}H'BHZ_{t+1} .$$

From the assumption that B is symmetric also $H'BH$ will be symmetric such that

$$H'BH = P\Lambda P' ,$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$ is the diagonal matrix formed from the eigenvalues of $H'BH$, while P is the matrix of the orthogonal and normalized eigenvectors of $H'BH$. Note that for the symmetry of $H'BH$ we can say that all the eigenvalues are real. The portfolio return takes the following form:

$$\begin{aligned} R_{t+1} &= \kappa + c'HPP'Z_{t+1} + Z'_{t+1}P\Lambda P'Z_{t+1} \\ &= \kappa + \eta'\tilde{Z}_{t+1} + \tilde{Z}'_{t+1}\Lambda\tilde{Z}_{t+1} , \end{aligned}$$

where

$$\begin{aligned} \eta &:= P'H'c , \\ \tilde{Z}_{t+1} &:= P'Z_{t+1} . \end{aligned}$$

The vector $\tilde{Z}_t = [\tilde{Z}_{1,t}, \tilde{Z}_{2,t}, \dots, \tilde{Z}_{m,t}]$ also consists of independent standard normal variables. The above equation can be written as

$$R_{t+1} = \kappa + \sum_{j=1}^m \left(\eta_j \tilde{Z}_{j,t+1} + \lambda_j \tilde{Z}_{j,t+1}^2 \right) .$$

Moreover, let us consider the random variable

$$X_j = h + \eta_j Z_j + \lambda_j Z_j^2 \quad j = 1, 2, \dots, m$$

where $h = \frac{\kappa}{m}$. To have a more clear notation, later on we will leave the subscript j out.

The characteristic function of X is

$$\begin{aligned}
\varphi_X(u) &= E[e^{iu(h+\eta Z+\lambda Z^2)}] \\
&= \frac{e^{iuh}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[iu(\eta z + \lambda z^2)] \exp\left[-\frac{z^2}{2}\right] dz \\
&= \frac{e^{iuh}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2(1-2iu\lambda) - 2iu\eta z)\right] dz \\
&= \exp\left[iuh - \frac{u^2\eta^2}{2(1-2iu\lambda)}\right] \times \\
&\quad \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(z\sqrt{1-2iu\lambda} - \frac{i u \eta}{\sqrt{1-2iu\lambda}}\right)^2\right] dz
\end{aligned}$$

by defining the variable y as

$$y = z\sqrt{1-2iu\lambda},$$

we can solve the above integral in the following way:

$$\begin{aligned}
\varphi_X(u) &= \frac{1}{\sqrt{1-2iu\lambda}} \exp\left[iuh - \frac{u^2\eta^2}{2(1-2iu\lambda)}\right] \times \\
&\quad \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(y - \frac{i u \eta}{\sqrt{1-2iu\lambda}}\right)^2\right] dy \\
&= \frac{1}{\sqrt{1-2iu\lambda}} \exp\left[iuh - \frac{u^2\eta^2}{2(1-2iu\lambda)}\right].
\end{aligned}$$

By the above result we can say that

$$\begin{aligned}
\varphi_R(u) &= E\left[\exp\left(iu \sum_{j=1}^m X_j\right)\right] \\
&= \exp(iu\kappa) \left(\prod_{j=1}^m \frac{1}{\sqrt{1-2iu\lambda_j}}\right) \exp\left(-\frac{1}{2} \sum_{j=1}^m \frac{u^2\eta_j^2}{1-2iu\lambda_j}\right) \\
&= \exp(iu\kappa) |I - 2iu\Lambda|^{-1/2} \exp\left(-\frac{1}{2} \sum_{j=1}^m \frac{u^2\eta_j^2}{1-2iu\lambda_j}\right)
\end{aligned}$$

Moreover, we can show that

$$\begin{aligned}
|I - 2iu\Lambda| &= |P(P'IP - 2iuP'\Lambda P)P'| \\
&= |(I - 2iuP'\Lambda P)| \\
&= |(I - 2iuH'BH)| \\
&= |H'[(H')^{-1}H^{-1} - 2iuB]H| \\
&= |\Sigma||\Sigma^{-1} - 2iuB| \\
&= |I - 2iu\Sigma B|,
\end{aligned}$$

while the last expression of the above equation can be written as

$$\begin{aligned}
\sum_{j=1}^m \frac{\eta_j^2}{1 - 2iu\lambda_j} &= \eta'(I - 2iu\Lambda)^{-1}\eta \\
&= \eta'P'P(I - 2iu\Lambda)^{-1}P'P\eta \\
&= \eta'P'[P(I - 2iu\Lambda)P']^{-1}P\eta \\
&= c'H(I - 2iuH'BH)^{-1}H'c \\
&= c'(\Sigma^{-1} - 2iuB)^{-1}c \\
&= c'(I - 2iu\Sigma B)^{-1}\Sigma c ,
\end{aligned}$$

such that

$$\varphi_R(u) = |(I - 2iu\Sigma B)|^{-1/2} \exp \left[iu\kappa - \frac{1}{2}u^2 c'(I - 2iu\Sigma B)^{-1}\Sigma c \right] .$$

□

B Some notes on Theorem 3.1

Theorem B.1 *Let $f(y)$ and $\varphi_Y(u)$ be Lebesgue-integrable, if the mean and variance of the random variable Y exist, then its cumulative density function $F(y)$ will be*

$$F(y) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \Delta_u \left[\frac{e^{-iuy}\varphi(u)}{iu} \right] du , \quad (19)$$

where $\Delta_u g(u) = g(u) + g(-u)$.

□

For a proof see Shephart [22].

Here we have to proof that equation (19) and equation (13) are equal or equivalently that

$$\int_0^\infty \operatorname{Re} \left[\frac{e^{-iuy}\varphi(u)}{iu} \right] du = \frac{1}{2} \int_0^\infty \left[\frac{e^{-iuy}\varphi(u) - e^{iuy}\varphi(-u)}{iu} \right] du .$$

Let us consider equation (13). By the definition of the characteristic function we can write

$$\begin{aligned}
&\int_0^\infty \operatorname{Re} \left[\frac{1}{iu} e^{-iuy} \int_{-\infty}^{+\infty} e^{iux} f(x) dx \right] du \\
&\int_0^\infty \operatorname{Re} \left[\frac{1}{iu} \int_{-\infty}^{+\infty} e^{i(u x - u y)} f(x) dx \right] du ,
\end{aligned}$$

where $f(x)$ is the probability density function. By applying Euler's rule we have

$$\begin{aligned} \int_0^\infty \operatorname{Re} \left[\int_{-\infty}^{+\infty} \frac{\cos(ux - uy)f(x)}{iu} dx + \right. \\ \left. + \int_{-\infty}^{+\infty} \frac{\sin(ux - uy)f(x)}{u} dx \right] du \\ \int_0^\infty \left[\int_{-\infty}^{+\infty} \frac{\sin(ux - uy)f(x)}{u} dx \right] du . \end{aligned}$$

On the other hand, equation (19) can be written as

$$\frac{1}{2} \int_0^\infty \frac{1}{iu} \left[\int_{-\infty}^{+\infty} \left(e^{i(ux-uy)} - e^{-i(ux-uy)} \right) f(x) dx \right] du .$$

By applying again Euler's rule

$$\begin{aligned} \frac{1}{2} \int_0^\infty \left[\int_{-\infty}^{+\infty} \frac{\cos(ux - uy) + i \sin(ux - uy)}{iu} f(x) - \right. \\ \left. - \frac{\cos(ux - uy) - i \sin(ux - uy)}{iu} f(x) dx \right] du \\ \int_0^\infty \left[\int_{-\infty}^{+\infty} \frac{\sin(ux - uy)}{u} f(x) dx \right] du . \end{aligned}$$

C The Orthogonal GARCH

One of the most important drawbacks of the GARCH method is the difficulty to have an estimate in a multivariate framework. Indeed, in such a situation, the number of GARCH parameters is very high. A possible solution is to restrict the dimension of the parameter space by the introduction of some simplifying assumptions. The problem of many of such simplifications is that they may get a non-positive definite covariance matrix.

Orthogonal GARCH is an estimation method based on the principal component analysis (PCA). By using the PCA we can extract the most important uncorrelated sources of information contained in the data and use them to construct a positive semi-definite covariance matrix.

Let us define X_t the $T \times n$ matrix whose columns are the time series of the n asset returns at time t . The principal components matrix at the same time, P_t , is defined as

$$P_t = X_t W_t , \tag{20}$$

where W_t is the orthogonal matrix of the normalized eigenvectors of $X_t' X_t$. The column inside the matrix W_t are ordered according to the size of the corresponding eigenvalue (the first eigenvector corresponds to the highest eigenvalue and so on).

From what above, we can note that:

1. each principal component is a linear combination of the columns on the matrix X_t ;
2. the weights of such a linear combination are the eigenvectors of the matrix $X_t'X_t$;
3. the explanation power of the principal components is structured so that the first one accounts for the maximum amount of the total variation in X_t , the second one accounts for the maximum amount of the remaining variation, and so on;
4. the principal components are uncorrelated with each other.

To show the last statement let us define Λ_t the diagonal matrix of the eigenvalues of $X_t'X_t$ such that

$$P_t'P_t = W_t'X_t'X_tW_t = W_t'W_t\Lambda_tW_t'W_t = \Lambda_t .$$

Since Λ_t is diagonal the columns of P_t are incorrelated.

By inverting equation (20) we have

$$X_t = P_tW_t' ,$$

such that

$$\text{var}(X_t) = \text{var}(P_tW_t') = W_tD_tW_t' ,$$

where D_t is the diagonal matrix of the principal component variances, that can be estimated by a GARCH model.

The advantage is that, due to the non-correlation of the principal components, we have to estimate only n variances (recall that the matrix D_t is diagonal). This can be done also by estimating the coefficients independently from each other with no risk to obtain a negative definite matrix.

For more details on the method the work of Alexander [1] can be seen. In Byström [8] there is an application to Nordic stock market during the Asian financial crisis.

D Ratio between losses and VaR for a normally distributed random variable

Let Z be a standard normally distributed random variable. The expected value of the ratio between losses which exceed VaR and the VaR itself is

$$X = \frac{1}{VaR} \int_{-\infty}^{+\infty} z\mathbb{P}(Z = z|Z < -VaR)dz . \quad (21)$$

The above conditional probability can be written as

$$\mathbb{P}(Z = z | Z < -VaR) = \frac{\mathbb{P}(Z = z, Z < -VaR)}{\mathbb{P}(Z < -VaR)},$$

where

$$\begin{aligned} \mathbb{P}(Z = z, Z < -VaR) &= \begin{cases} 0 & \text{se } z > -VaR \\ \mathbb{P}(Z = z) & \text{se } z < -VaR \end{cases} \\ \mathbb{P}(Z < -VaR) &= \alpha. \end{aligned}$$

Under the normality assumption, equation (21) can be written as

$$\begin{aligned} X &= \frac{1}{VaR} \frac{1}{\alpha} \int_{-\infty}^{-VaR} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{VaR} \frac{1}{\alpha} \int_{VaR}^{+\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \end{aligned}$$

Moreover, being Z a standard normal random variable, the VaR will be equal to the critical value to obtain the α quantile, say $-z_\alpha$, such that

$$X = -\frac{1}{\alpha z_\alpha \sqrt{2\pi}} \int_{z_\alpha}^{+\infty} z e^{-\frac{z^2}{2}} dz = -\frac{1}{\alpha z_\alpha \sqrt{2\pi}} e^{-\frac{z_\alpha^2}{2}},$$

that is equal to 1.146 for $\alpha = 0.01$ and to 1.254 for $\alpha = 0.05$.

E Portfolios composition

The composition of the portfolios is the following:

Assets	Portfolios			
	A	B	C	D
Advanced Tissue Sciences	3.15%	2.00%	1.76%	2.30%
Alexion Pharmaceuticals	4.99%	3.16%	2.79%	3.64%
Amgen	10.32%	6.54%	5.76%	7.53%
Applera Biosystems Group	2.82%	1.79%	1.58%	2.06%
Aradigm	2.27%	1.44%	1.27%	1.66%
Atrix Labs	3.05%	1.93%	1.70%	2.23%
Avigen	3.60%	2.28%	2.01%	2.62%
Aviron	9.99%	6.33%	5.58%	7.29%
Cell Genesys	11.17%	7.08%	6.24%	8.15%
Chiron	6.68%	4.23%	3.73%	4.87%
Cortex Pharmaceuticals	4.31%	2.73%	2.41%	3.14%
Diversa	3.63%	2.30%	2.03%	2.65%
Genome Therapeutics	2.68%	1.70%	1.50%	1.96%
Genzyme-GENL Division	6.75%	4.28%	3.77%	4.93%
Human Genome Sciences	8.52%	5.40%	4.76%	6.21%
Immunex	2.01%	1.27%	1.12%	1.46%
Medimmune	5.10%	3.23%	2.85%	3.72%
Myriad Genetics	4.06%	2.57%	2.26%	2.96%
Sciclone Pharmaceuticals	2.57%	1.63%	1.43%	1.87%
Valentis	2.34%	1.48%	1.30%	1.70%
Intel	-	8.80%	7.76%	10.14%
Cisco	-	7.86%	6.93%	9.05%
Lucent Technologies	-	0.67%	0.59%	0.78%
General Electric	-	4.06%	3.58%	4.67%
Wells Fargo & Co	-	11.59%	10.22%	13.35%
US Bancorp	-	3.66%	3.22%	4.21%
S&P 500 put option	-	-	3.87%	-
NASDAQ 100 put option	-	-	7.99%	-
S&P 500 call option	-	-	-	-9.43%
NASDAQ 100 call option	-	-	-	-5.72%

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