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--INTEREST RATE BARRIER OPTIONS

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ABSTRACT

Less expensive than standard options, barrier options have become very popular in recent years as useful hedging instruments for risk management strategies. Thus far valuation approaches have largely focused on equity barrier options, where in certain instances analytical expressions may be available. In this paper we use Monte Carlo procedure to value barrier options based on the Chan, Karolyi, Longstaff and Sanders interest rate process. By performing simulations with and without including the recently suggested Sharp Large Deviations, we show that standard Monte Carlo procedure substantially misprices barrier options.

I. INTRODUCTION

Barrier options have become increasingly popular in the over-the-counter market as hedging instruments for risk management strategies. The study of barrier options goes back to Merton [1973] who provided a closed form solution for down-and-out options. Since then closed form solutions for a variety of European barrier options have been proposed. Rubinstein and Reiner [1991] developed analytical expressions for standard European barrier options. Heynan and Kat [1994] developed expressions for exotic barrier such as rainbow barriers. Kunitomo and Ikeda [1992] and Geman and Yor [1996] developed expressions for double barrier options incorporating curved boundaries.

By in large published research has focused on valuing equity barrier options where it is almost always assumed that the underlying asset price follows geometric Brownian motion. Empirical studies have indicated that stock prices are unlikely to be lognormally distributed. As a result several researchers have proposed numerical schemes for the pricing of barrier options. They have considered mainly two separate approaches – lattice approach and Monte Carlo simulation.

Boyle and Lau[1994] investigate the suitability of the binomial lattice to price barrier options. Their main findings indicated that convergence was poor unless the number of time steps is chosen in such a way as to ensure that a barrier lies on a layer of horizontal nodes in the tree. Ritchken [1995] used trinomial lattice to value a range of barrier options such as double barriers, curved barriers and rainbow barriers. He used the extra flexibility offered by trinomial lattices to ensure that tree nodes lined up with barriers. However, like Boyle and Lau's binomial method, Ritchken's method still required a large number of time steps if the initial stock price was close to a barrier. Cheuk and Vorst

[1996] further developed Ritchken's method by introducing a time dependent shift in the trinomial lattice. Although Cheuk and Vorst's method offered considerable improvement over Ritchken's method, it nevertheless still required a large number of time steps. Boyle and Tian [1999] and Tian [1999] use the trinomial lattice to value barrier options where the underlying asset follows the constant elasticity variance (CEV) process and general diffusion process respectively. Their particular contribution is to align grid points with barriers by constructing a grid which lies right on the barrier by adjusting a stretch parameter. Figlewski and Gao [1999] and Ahn, Figlewski and Gao [1999] use the trinomial lattice with an adaptive mesh. Their approach is to use a fine mesh in regions where it is required such as close to a barrier and then to graft the computed results from this onto a coarser mesh, which is used in other regions. Ahn et-al [1999] use Gao's [1997] Analytic High Order Trinomial (A-HOT) model in which the probabilities are positive constants. In the A-HOT model constant positive probabilities are achieved by detrending the drift. The detrending does not lead to constant positive probabilities in the case of interest rate processes exhibiting mean reversion. The probabilities vary from node to node and indeed may become negative under certain circumstances. In short the A-HOT model successfully tackles the difficulties of valuing equity barrier options using trinomial lattices; but cannot be adapted to value interest rate barrier options without introducing difficulties associated with some of the earlier schemes.

Monte Carlo simulation is known for its high flexibility. However, in the case of barrier options it produces biased results for options, which depend on the continuously monitored sample path of some stochastic variable. In a Monte Carlo simulation, where stochastic variable values can be sampled at discrete times, information is lost about the parts of the continuous time path that lie between the sampling dates. Thus the discretely observed minimum will be too high and the discretely observed maximum too low compared with the real extremes of the continuous time process. As an example in the case of a knock-out option, this will mean the underestimation of the likelihood of the option being knocked out and thus overestimate the options value and vice versa for knock-in options. This bias in Monte Carlo simulation has been considered by Andersen and Brotherton-Ratcliffe [1996] and Beaglehole, Dybvig and Zhou [1997]. Their approach is to use the law of the maximum of the Brownian bridge in order to evaluate the probability that the underlying asset price process hits the barrier during each step of the simulation. Unfortunately the above mentioned researchers are restricted to single constant barriers. Baldi, Caramellino and Iovino [1999] use Sharp Large Deviation techniques to derive expressions for the exit probability in the context of single, double and time dependent barrier options where the underlying asset price follows a general diffusion process.

Numerical research into the pricing of barrier options using the lattice approach with the exception of researchers such as Tian [1999] have focused on equity options. No Monte-Carlo simulation scheme has been proposed to value interest rate barrier options.

The fixed income market is one of the largest sectors of the financial markets where billions of dollars worth of assets are traded daily. Over the years a variety of interest rate models, both single-factor and multi-factors have been proposed which have formed the basis for the valuation of fixed income instruments. The most general of the single-factor interest rate models is that proposed by Chan, Karolyi, Longstaff and Sanders (CKLS), [1992]. The CKLS model encloses many of the earlier single-factor models such as that proposed by Vasicek [1977] and Cox, Ingersoll and Ross [1985]. The main advantage of one-factor models is their simplicity, as the entire yield curve is a function of a single state variable.

In this paper we put forward a general Monte Carlo simulation to value barrier options where the underlying stochastic process follows the CKLS process. Our approach involves incorporating the results of Baldi et-al [1999] to demonstrate that the standard Monte Carlo simulation scheme can be successfully used to value a wide range of interest rate barrier options, once the bias has been corrected.

In Section II, we provide a description of the general problem. In Section III , we define the CKLS process and develop the algorithm in depth to value interest rate barrier option. In the final Section we summarise our results.

II. INTEREST RATE BARRIER OPTIONS

Barrier options differ from the conventional options due to the introduction of one or two boundaries affecting the options prices. These boundaries may be deterministic and time dependent. Furthermore the boundaries are contractually specified, and may nullify the value of the option or pay a pre-agreed rebate if the boundaries are breached by the underlying interest rate process. For example, a knock-and-out double barrier interest rate call option is equivalent to the corresponding standard call, provided that the underlying interest rate process does not hit either barrier, otherwise payoff is set to zero or a rebate rate. The pricing formula in a risk neutral world for a knock-and-out and knock-and-in barrier options are respectively:

$$H(t) = \hat{E}_t\left[H(T)\exp\left(-\int_t^T r(\tau)d\tau\right)\cdot 1_{\Sigma \geq T}\right] \quad (1)$$

$$H(t) = \hat{E}_t\left[H(T)\exp\left(-\int_t^T r(\tau)d\tau\right)\cdot 1_{\Sigma < T}\right] \quad (2)$$

where $H(T)$ denotes the payoff of the option at its expiry date. Σ denotes the first time the interest rate process hits the boundaries. As shown above for a knock-and-out option $\Sigma \geq T$ and for a knock-and-in option $\Sigma < T$. Using the results of Baldi et-al [1999] we set up Monte Carlo scheme to value interest rate barrier options which take into account the possibility of breaching the barrier between successive intervals of time. Our Monte Carlo scheme works in the following way. First we partition the life of the derivative security into N steps such that $[t = t_0 < t_1 < t_2 < \cdots < t_N]$, where $t_n = t + n\Delta t$ and the length of each time step is given by:

$$\Delta t = \frac{T - t}{N} \quad (3)$$

At each step n, the underlying interest rate process is simulated at time t_n by means of a suitable approximation scheme, giving the value r_n (see Section III for details). Since r_n and r_{n+1} may not have breached the barriers while r_t had during the time interval (t_n, t_{n+1}), $\Sigma^{\Delta t}$ provides a rough and strongly biased estimate of the hitting time Σ, as it has been pointed out by several authors (see e.g, Geman and Yor [1996] or also Baldi et-al [1999]). In particular, Σ turns out to be over estimated by $\Sigma^{\Delta t}$ whenever the interest rate process can be exactly simulated, giving an over estimate for knock-and-out options and an under estimate for knock-and-in ones. To account for this we calculate a sharp approximation $p_n^{\Delta t}$ of the probability that r_t hits the barriers during the time interval

$[t_n, t_{n+1})$ given we know r_n and r_{n+1}. Thus we can stop the simulation with a probability $p_n^{\Delta t}$ and set $\Sigma^{\Delta t}$ equal to t_n. This procedure then provides an almost unbiased Monte Carlo estimator, being really unbiased if one can exactly simulate the underlying interest rate process and $p_n^{\Delta t}$ is the right exit probability as it does in some special case.

III. MONTECARLO SIMULATION OF THE CKLS DIFFUSION PROCESS

Consider the following CKLS [1992] model in a risk neutral world where the instantaneous short rat is pulled towards a long term mean of θ at a speed of adjustment κ

$$dr = \kappa(\theta - r)dt + \sigma r^{\gamma} dW \qquad (4)$$

In equation (4) substituting specific values of γ yields specific interest rate models. For example $\gamma = 0$ yields the Vasicek [1977] model, $\gamma = \frac{1}{2}$ yields the Cox, Ingersoll and Ross [1985] model and $\gamma = 1$ yields the Brennan and Schwartz [1979] model.

The usual discretized version of equation (4) is:

$$r_{n+1} = r_n + \kappa(\theta - r_n) + \sigma r_n^{\gamma} \varepsilon \sqrt{\Delta t} \qquad (5)$$

In equation (5) the Wiener process is usually approximated as $dW \approx \varepsilon\sqrt{\Delta t}$, where ε is a normally *N(0;1)* distributed random variable, however; this is only a first order approximation. It is possible to use a more accurate approximation of the Wiener process,which exploits the information in the drift and the volatility further. Observing equation (5) we note that the drift term is linearly dependent on *r*, and the volatility term is dependent on an arbitrary power of *r*. Thus both of these terms can be easily differentiated with respect to *r*. For our analysis we choose the strong Taylor approximation of equation (5) due to Platen and Wagner (see Kloeden and Platen [1999], pp. 351). By incorporating the derivatives of the drift and the volatility terms of equation (5) this scheme ensures any extreme fluctuations in the interest rate paths are minimal. Taking a as the drift function and b as the volatility function, Kloeden and Platen [1999] scheme states:

$$\begin{aligned} r_{n+1} = r_n &+ a\Delta t + b\Delta W_n + \frac{1}{2}bb'\left\{(\Delta W_n)^2 - \Delta t\right\} \\ &+ ba'\Delta Z_n + \frac{1}{2}\left\{aa' + \frac{1}{2}b^2a''\right\}\Delta t^2 \\ &+ \left\{ab' + \frac{1}{2}b^2b''\right\}\left\{\Delta W_n\Delta t - \Delta Z_n\right\} \\ &+ \frac{1}{2}b\left\{bb'' + (b')^2\right\}\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta t\right\}\Delta W_n \end{aligned} \tag{6}$$

$\Delta W_n = \varepsilon_{n,1}\Delta t^{\frac{1}{2}}$ and $\Delta Z_n = \frac{1}{2}\left(\varepsilon_{n,1} + \frac{1}{\sqrt{3}}\varepsilon_{n,2}\right)\Delta t^{\frac{3}{2}}$

where $\varepsilon_{n,1}$ and $\varepsilon_{n,2}$ are independent normally *N(0;1)* distributed random variables.

Further a, a', a'', b, b', b'' are evaluated at r_n. From equation (4) we have:

$$a = \kappa(\theta - r)$$

$$a' = -\kappa$$

$$a'' = 0$$

$$b = \sigma r^{\gamma}$$

$$b' = \gamma\sigma r^{\gamma-1}$$

$$b'' = \gamma(\gamma-1)\sigma r^{\gamma-2}$$

Substituting equation (6) into equation (5) yields:

$$
\begin{aligned}
r_{n+1} &= r_n + \kappa(\theta - r_n)\Delta t + \sigma r_n^{\gamma}\Delta W_n + \frac{1}{2}\gamma\sigma^2 r_n^{2\gamma-1}\left\{(\Delta W_n)^2 - \Delta t\right\} \\
&- \kappa\sigma r_n^{\gamma}\Delta Z_n - \frac{1}{2}\kappa^2(\theta - r_n)\Delta t^2 \\
&+ \left\{\gamma\kappa\sigma(\theta - r_n)r_n^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)\sigma^3 r_n^{3\gamma-2}\right\}\left\{\Delta W_n \Delta t - \Delta Z_n\right\} \\
&+ \frac{1}{2}\sigma r_n^{\gamma}\left\{\gamma(\gamma-1)\sigma^2 r_n^{2\gamma-2} + \gamma^2\sigma^2 r_n^{2\gamma-2}\right\}\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta t\right\}\Delta W_n
\end{aligned} \tag{7}
$$

For European bond options the maturity value is given by:

$$
H(T) = \begin{cases} \max[0, P(T,s,r) - K] & \text{for a call option} \\ \max[0, K - P(T,s,r)] & \text{for a put option} \end{cases}
$$

where $P(T,s,r)$ is the price of a bond maturing at time s with q cash flows with payment c_i at each cashflow. The bond price is evaluated using the formula:

$$
P(T,s,r) = \sum_{i=1}^{q} c_i \left[\exp\left(-\int_t^{s_i} r(\tau)d\tau\right)\right] \tag{8}
$$

Except for specific cases such as $\gamma = 0$ and $\gamma = \frac{1}{2}$ no analytical function is available for the discount function. In such circumstances we can evaluate the discount function in equation (8) using the trapezium rule.

$$
\exp\left(-\int_t^{T} r(\tau)d\tau\right) = \exp\left(-\sum_{i=1}^{N-1}\frac{r_n + r_{n+1}}{2}\right) \tag{9}
$$

Assuming M simulations with Φ_j as the discounted payoff the option from the j-th simulation, we have the mean discounted payoff as:

$$
\Phi(t) = \frac{1}{M}\sum_{j=1}^{M}\Phi_j \tag{10}
$$

Baldi et-al [1999] show that the only random element in the estimation of exit probabilities enter through the volatility term and further this volatility is a denominator term. Thus a suitable transformation which converts the volatility term into a constant will eliminate any singularities which may arise due to very low interest rates, in the calculations of exit probabilities. Hence in order to calculate the exit probabilities, we use the transformation of Barone-Adesi, Dinenis and Sorwar [1997], who show that the CKLS process can be transformed in such a way that the volatility is independent of r. In particular we use:

$$\frac{\partial \phi}{\partial r}\sigma r^{\gamma} = \nu \qquad (11)$$

for some positive constant ν. This is equivalent to:

$$\frac{\partial \phi}{\partial r} = \frac{\nu}{\sigma} r^{-\gamma} \qquad (12)$$

Thus the transformation is given by:

$$\begin{aligned} \phi &= \frac{\nu}{\sigma(1-\gamma)} r^{1-\gamma} \text{ for } \gamma \neq 1 \\ \phi &= \frac{\nu}{\sigma} \ln r \text{ for } \gamma = 1 \end{aligned} \qquad (13)$$

Noting that the value of ν has no impact on the accuracy of the model, we choose ν = σ for convenience.

The exit probability assuming a single upper barrier at U can be approximated as in Baldi et-al [1999]:

$$p_U^{\Delta t}(t, \phi_n, \phi_{n+1}) = \exp\left\{ -\frac{2}{\sigma^2 \Delta t}(\phi(U) - \phi_n)(\phi(U) - \phi_{n+1}) \right\} \qquad (14)$$

The exit probability assuming a single barrier at L can be approximated as in Baldi et-al [1999]:

$$p_L^{\Delta t}(t, \phi_n, \phi_{n+1}) = \exp\left\{-\frac{2}{\sigma^2 \Delta t}(\phi_n - \phi(L))(\phi_{n+1} - \phi(L))\right\} \tag{15}$$

The exit probability assuming a lower barrier at L and an upper barrier at U can be approximated as in Baldi et-al [1999]:

$$p_{U,L}^{\Delta t}(t, \phi_n, \phi_{n+1}) = \exp\left\{-\frac{2}{\sigma^2 \Delta t}(\phi(U) - \phi_n)(\phi(U) - \phi_{n+1})\right\} \text{if } \phi_n + \phi_{n+1} > \phi(U) + \phi(L) \tag{16}$$

$$p_{U,L}^{\Delta t}(t, \phi_n, \phi_{n+1}) = \exp\left\{-\frac{2}{\sigma^2 \Delta t}(\phi_n - \phi(L))(\phi_{n+1} - \phi(L))\right\} \text{if } \phi_{\mathrm{n}} + \phi_{n+1} < \phi(U) + \phi(L) \tag{17}$$

The correct Monte Carlo procedure works as follows: with probability equal to $p_n^{\Delta t}$ we stop the simulation and set t_n as the hitting time $\Sigma^{\Delta t}$; with probability $1 - p_n^{\Delta t}$ we carry on the simulation.

In Table1 – Table 3, we compare option values calculated both using standard Monte Carlo simulation and corrected Monte Carlo simulation. In each instance option prices are obtained by 20,000 paths of the underlying interest rate process. The time step size is set equal to 1/365[1]. The standard error is displayed in the brackets. We focus solely on $\gamma = \frac{1}{2}$, i.e. the CIR process, where analytical option prices are available in the case of no barriers. Our analysis holds for other values of γ. All the options have one year to expiry and are written on zero coupon bonds with five years to maturity. The bond pays \$100 on maturity. We value both call and put options across a wide range of strike prices

[1] This is also the time step chosen by Baldi et-al [1999].

varying from 62 to 74. For simplicity we assume zero rebate, in the instance of the barrier being breached.

Table 1 contains up-and-out option values. We find that the up-and-out option values are lower than the corresponding option with no barriers. This is due to the upper barrier being close to the initial interest rate. Further we also find that standard Monte Carlo simulation overprices. For example at a strike price of 62, standard Monte Carlo yields a call price of 4.4706, whereas corrected Monte Carlo yields 4.1231, this is a reduction of 7.77% compared to the standard Monte Carlo price.

We observe the same trends in Table 2 as in Table 1. In this case the down-and-out options are closer to the option prices without any barriers; due to a larger difference between the initial interest rate and the lower barrier.

In Table 3, we observe the most interesting results. Corrected Monte Carlo prices are significantly lower than standard Monte Carlo prices in the case of double knock-out options. For example at a strike price of 62, the standard Monte Carlo call option price is 0.5271, whereas corrected Monte Carlo price is 0.3511; this is a reduction of 33% compared to the standard Monte Carlo price.

IV. SUMMARY

We have used Monte Carlo simulation scheme to value barrier options based on single factor interest rate models. Further, we have incorporated the corrections terms of Baldi et-al [1999] into our scheme. Our findings reinforces the existing results found in options price literature, that standard Monte Carlo simulation produces biased barrier option values. In particular we find that that the standard Monte Carlo scheme overprices knock-out options and this bias becomes significant in the case of double barrier options. For example at a strike price of 62, standard Monte Carlo overprices up-and-out call option by 7.77% whereas it overprices a double knock-out call option by 33%.

TABLE 1. VALUATION OF UP-AND-OUT OPTIONS

$\kappa = 0.1, \sigma = 0.2, \theta = 0.1, \gamma = 0.5, r_0 = 0.08$
$\Delta t = 1/365$

Price of 5 year bond = 68.4059

Simulation	Barrier	Option	Strike Price				
			62	65	68	71	74
	No barriers	Call	12.1565	9.8779	7.7781	5.8922	4.2521
Standard M.C.	Upper at 10%		4.4706	3.7047	2.9922	2.3407	1.7593
			(0.0455)	(0.0379)	(0.0311)	(0.0253)	(0.0205)
Corrected M.C.			4.1231	3.4224	2.7720	2.1788	1.6474
			(0.0439)	(0.0367)	(0.0302)	(0.0247)	(0.0200)
	No barriers	Put	0.9580	1.4475	2.1158	2.9980	4.1260
Standard M.C.	Upper at 10%		0.3690	0.5450	0.7744	1.0648	1.4253
			(0.0104)	(0.0129)	(0.0158)	(0.0189)	(0.0222)
Corrected M.C.			0.3560	0.5239	0.7421	1.0174	1.3545
			(0.0102)	(0.0127)	(0.0154)	(0.0185)	(0.0218)

TABLE 2. VALUATION OF DOWN-AND-OUT OPTIONS

$\kappa = 0.1, \sigma = 0.2, \theta = 0.1, \gamma = 0.5, r_0 = 0.08$
$\Delta t = 1/365$

Price of 5 year bond = 68.4059

Simulation	Barrier	Option	Strike Price 62	65	68	71	74
	No barriers	Call	12.1565	9.8779	7.7781	5.8922	4.2521
Standard M.C.	Lower at 4%		7.0523	5.7604	4.5441	3.4249	2.4452
			(0.0450)	(0.0367)	(0.0293)	(0.0230)	(0.0184)
Corrected M.C.			6.7898	5.5522	4.3891	3.3168	2.3763
			(0.0457)	(0.0373)	(0.0297)	(0.0235)	(0.0188)
	No barriers	Put	0.9580	1.4475	2.1158	2.9980	4.1260
Standard M.C.	Lower at 4%		0.3692	0.5654	0.8374	1.2064	1.7149
			(0.0090)	(0.013)	(0.0140)	(0.0169)	(0.0199)
Corrected M.C.			0.3682	0.5635	0.8333	1.1938	1.6862
			(0.0089)	(0.0113)	(0.0140)	(0.0169)	(0.0200)

TABLE 3. VALUATION OF DOUBLE KNOCK-OUT OPTIONS

$\kappa = 0.1, \sigma = 0.2, \theta = 0.1, \gamma = 0.5, r_0 = 0.08$
$\Delta t = 1/365$

Price of 5 year bond = 68.4059

			Strike Price				
Simulation	**Barrier**	**Option**	62	65	68	71	74
	No barriers	Call	12.1565	9.8779	7.7781	5.8922	4.2521
Standard M.C.	Upper at 10%		0.5271	0.4172	0.3074	0.2001	0.1069
	Lower at 4%		(0.0185)	(0.0146)	(0.0108)	(0.0070)	(0.0038)
Corrected M.C.			0.3511	0.2780	0.2048	0.1328	0.0697
			(0.0145)	(0.0115)	(0.0085)	(0.0055)	(0.0029)
	No barriers	Put	0.9580	1.4475	2.1158	2.9980	4.1260
Standard M.C.	Upper at 10%		0.0000	0.0000	0.0000	0.0025	0.0192
	Lower at 4%		(0.0000)	(0.0000)	(0.0000)	(0.0003)	(0.0012)
Corrected M.C.			0.0000	0.0000	0.0000	0.0010	0.0111
			(0.0000)	(0.0000)	(0.0000)	(0.0002)	(0.0008)

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