

# Implied Volatility Surfaces for Inverse Gamma Models

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## Abstract

We study implied volatility surfaces when the squared volatility is driven by an inverse gamma process. We derive the first two conditional moments of the integrated volatility over the time to maturity to study theoretical term structure volatility patterns. We find that these patterns are in accordance with the empirical ones. Finally, we discuss some probabilistic properties of the volatility process.

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# 1 Introduction

We know since Mandelbrot (1963) and Fama (1965) that speculative log-returns are uncorrelated, not independent with leptokurtic distribution and conditional variance changing randomly over time. A good model for option pricing should capture the empirical evidence retaining the simplicity of the Black and Scholes (1973) model. Keeping the linearity of the drift and the diffusion components of the underlying asset, stochastic volatility models can generate log-returns with such characteristics.

In the sequel we assume the Hull and White (1987) framework. The extended model, defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ , is given by two dimensional stochastic differential equations (SDE),

$$dS_t = \alpha S_t dt + \sqrt{V_t} S_t dB_t \quad (1)$$

$$dV_t = \beta(V_t, \theta) dt + \gamma(V_t, \theta) dW_t, \quad (2)$$

where  $S_t$  is the underlying process,  $V_t$  is the latent process for the instantaneous squared volatility of  $dS_t/S_t$  and  $B_t$  and  $W_t$  are standard Brownian motions. Different SDE's have been proposed for the  $V_t$  process: the geometric Brownian motion (Hull and White, 1987), the Cox-Ingersoll-Ross process (Hull and White (1988), Heston 1993), the exponential of the Ornstein-Uhlenbeck process (Scott (1987), Stein and Stein (1991), Chesney and Scott 1989).

Unfortunately, it is difficult to derive analytical results for these models. Here we focus on models for foreign exchange options. These important financial markets are characterised by quite symmetric volatility smiles<sup>1</sup>, with convexity increasing when time to maturities decrease. Stochastic volatility models imply symmetric volatility smiles when underlying and volatility processes are uncorrelated.

Using the Hull and White (1987) option pricing formula, Taylor and Xu (1994) derive an approximation for the theoretical implied volatility of foreign exchange options which allows to compare theoretical and empirical volatility surfaces. The approximation involves the conditional mean and variance of the integrated volatility over the time to maturity. These moments have been derived only for few processes.

In this paper we assume that the squared volatility of the underlying asset is driven by an inverse gamma process and we derive the conditional mean and variance of the integrated volatility over the time to maturity to study implied volatility surfaces.

Sircar and Papanicolaou (1999), Fouque, Sircar and Papanicolaou (2000) and Zhu and Avellaneda (1998) derive similar results for equity and index

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<sup>1</sup>See for instance, Bollerslev and Zhou (2001), Taylor and Xu (1994) and references therein.

options<sup>2</sup>. Sircar and Papanicolaou results apply also to foreign exchange options. However, they assume that the volatility process is almost constant with infinitesimal stochastic variations.

The remaining of the paper is structured as follows: in section 2 we introduce the model, in section 3 we derive the conditional mean and variance of the integrated volatility over the time to maturity and in section 4 we study the implied volatility surface. Finally, section 5 concludes.

## 2 The model

We assume that the squared volatility of the underlying asset is driven by an inverse gamma process, i.e. a mean reverting geometric Brownian motion,

$$dV_t = (c_1 - c_2 V_t) dt + c_3 V_t dW_t, \quad (3)$$

where  $c_1, c_2 > 0$  and  $2c_2/c_3^2 > 1$  to ensure that the  $V_t$  process is mean reverting and with finite second moment. We assume that the underlying and the volatility process are uncorrelated. The initial time<sup>3</sup> is  $t = 0$ . When the  $V_t$  process starts according to its stationary distribution,

- the stationary distribution of  $V_t$  is the inverse gamma distribution with parameters  $1 + 2c_2/c_3^2$  and  $c_3^2/2c_1$ , i.e.  $1/V_1 \rightsquigarrow \Gamma(1 + 2c_2/c_3^2, c_3^2/2c_1)$ . The inverse gamma distribution has finite moments up to order  $r$  if and only if  $r < 1 + 2c_2/c_3^2$ .
- Given  $V_0$ , the unique strong solution of the SDE (2) is

$$\begin{aligned} V_t = & V_0 e^{-(c_2 + \frac{1}{2} c_3^2)t + c_3 W_t} \\ & + c_1 \int_0^t e^{(c_2 + \frac{1}{2} c_3^2)(s-t) + c_3(W_t - W_s)} ds. \end{aligned} \quad (4)$$

If  $V_0 > 0$  then  $V_t > 0$ .

- For all  $t, t \geq 0$ ,

$$\mathbb{E}[V_t | V_0] = \frac{c_1}{c_2} + \left( V_0 - \frac{c_1}{c_2} \right) e^{-c_2 t}. \quad (5)$$

Equation (5) and the stationarity of the  $V_t$  process imply

$$\mu_V := \mathbb{E}[V_1] = \frac{c_1}{c_2} \quad (6)$$

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<sup>2</sup>Usually, implied volatilities for equity and index options are asymmetric with respect to strike prices. To model this behaviour underlying and volatility processes are assumed correlated.

<sup>3</sup>The reference scales of the parameters are  $[c_1] = 1/T_0^2$ ,  $[c_2] = 1/T_0$ ,  $[c_3] = 1/\sqrt{T_0}$  where  $T_0$  is the unit time interval, one year.

and

$$\mathbb{E}[V_0 V_t] = \mathbb{E}[V_0 \mathbb{E}[V_t | V_0]] = e^{-c_2 t} \text{Var} V_0 + (c_1/c_2)^2. \quad (7)$$

- The conditional and unconditional variance of the  $V_t$  process,  $\forall t \geq 0$ , are

$$\begin{aligned} \text{Var}[V_t | V_0] &= \frac{\mu_V^2}{2c_2/c_3^2 - 1} \\ &+ e^{-c_2 t} \frac{2\mu_V(V_0 - \mu_V)}{c_2/c_3^2 - 1} \\ &- e^{-2c_2 t} (V_0 - \mu_V)^2 \\ &+ e^{(c_3^2 - 2c_2)t} \left[ V_0^2 - \frac{2V_0\mu_V}{1 - c_3^2/c_2} \right. \\ &\left. + \frac{\mu_V^2}{(1 - c_3^2/2c_2)(1 - c_3^2/c_2)} \right] \end{aligned} \quad (8)$$

and

$$\sigma_V^2 := \text{Var} V_t = \frac{\mu_V^2}{2c_2/c_3^2 - 1}. \quad (9)$$

The  $V_t$  process is mean reverting,  $c_1/c_2$  is the run mean value and  $c_2$  is the reversion rate. For ‘small’  $c_2$  the mean reversion is weak and  $V_t$  tends to stay above (or below) the run mean value for long periods, i.e. to volatility cluster. For estimation purposes we will need the dynamic and the moments of the  $\ln S_t$  process. The model (1)-(3) implies

$$d \ln S_t = \left( \alpha - \frac{V_t}{2} \right) dt + \sqrt{V_t} dB_t \quad (10)$$

$$dV_t = (c_1 - c_2 V_t) dt + c_3 V_t dW_t. \quad (11)$$

The two dimensional diffusion process  $(\ln S_t, V_t)$  is a Markov process (see Genon-Catalot, Jeantheau and Laredo, 1998). For a given time interval<sup>4</sup>  $\Delta > 0$  we define

$$Z_i := \int_{(i-1)\Delta}^{i\Delta} d \ln S_s = \ln S_{i\Delta} - \ln S_{(i-1)\Delta},$$

$$M_i := \int_{(i-1)\Delta}^{i\Delta} \left( \alpha - \frac{V_s}{2} \right) ds,$$

$$\Sigma_i := \int_{(i-1)\Delta}^{i\Delta} V_s ds,$$

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<sup>4</sup>For daily log-returns  $\Delta = 1/250$  when the unit time interval is one year.

where  $i = 1, \dots, N$ ,  $N$  is the sample size and  $Z_1, \dots, Z_N$  the sequence of log-returns. The unconditional distribution of the  $Z_i$  process is not known but, given  $\sigma\{V_s, s \in [0, i]\}$ ,  $Z_i \rightsquigarrow \mathcal{N}(M_i, \Sigma_i)$ . For simplicity, we assume that the drift of the log returns is zero, i.e.  $M_i = 0, \forall i$ . This implies the following moment conditions,  $\forall i \neq j$

$$\mathbb{E}[Z_i] = 0,$$

$$\mathbb{E}[Z_i^p Z_j^p] = 0, \quad \forall p \text{ odd},$$

$$\text{Cov}(Z_i, Z_j) = 0, \tag{12}$$

$$\text{Var} Z_1 = \mathbb{E}[\Sigma_1] = (c_1/c_2) \Delta, \tag{13}$$

$$\begin{aligned} \text{Var} Z_1^2 &= \mathbb{E}[Z_1^4] - \mathbb{E}[Z_1^2]^2 = 3 \mathbb{E}[\Sigma^2] - (c_1/c_2)^2 \Delta^2 \\ &= 2 (c_1/c_2)^2 \Delta^2 + \frac{6(c_2 \Delta - 1 + e^{-c_2 \Delta})}{c_2^2} \cdot \frac{(c_1/c_2)^2}{(2c_2/c_3^2) - 1} \end{aligned} \tag{14}$$

$$\text{Cov}(Z_1^2, Z_2^2) = \frac{(1 - e^{-c_2 \Delta})^2}{c_2^2} \cdot \frac{(c_1/c_2)^2}{(2c_2/c_3^2) - 1} \tag{15}$$

and

$$\text{Corr}(Z_1^2, Z_2^2) = \frac{(1 - e^{-c_2 \Delta})^2 \frac{(c_1/c_2)^2}{(2c_2/c_3^2) - 1}}{2 c_1^2 \Delta^2 + 6(c_2 \Delta - 1 + e^{-c_2 \Delta}) \frac{(c_1/c_2)^2}{(2c_2/c_3^2) - 1}}.$$

**Observation 2.1** Equations (12) and (15) imply that  $Z_i$ 's are uncorrelated but not independent.

The excess-kurtosis of the stationary distribution of the  $Z_i$  process is given by

$$\begin{aligned} \mathbb{K}(Z_1) &:= \frac{\mathbb{E}Z_1^4}{(\mathbb{E}Z_1^2)^2} - 3 \\ &= \frac{6(c_2 \Delta - 1 + e^{-c_2 \Delta})}{c_2^2 \Delta^2} \frac{1}{(2c_2/c_3^2) - 1} > 0. \end{aligned} \tag{16}$$

Moreover,

$$\text{Corr}(Z_1^2, Z_2^2) < \frac{\frac{c_3^2}{2c_2 - c_3^2}}{2 + 3 \frac{c_3^2}{2c_2 - c_3^2}} = \frac{1}{\frac{2c_3^2}{2c_2 - c_3^2} + 3}.$$

When  $2c_2/c_3^2 \rightarrow 1^+$ , the kurtosis tends to infinity and  $\text{Corr}(Z_1^2, Z_2^2)$  approaches the upper limit  $1/3$ .

The moment conditions (13), (15) and (16) can be used for the estimation of the parameters  $c_1$ ,  $c_2$  and  $c_3$ .

**Remark 2.1** If the  $V_t$  process follows a Cox-Ingersoll-Ross process its stationary distribution is a gamma distribution. Hence, the  $Z_i$  process has moments of any order, the excess-kurtosis is at most 3 and the correlation between  $Z_1^2$  and  $Z_2^2$  is at most 1/5. The gamma distribution of  $Z_i$  can be only moderately heavy-tailed in contrast with the empirical evidence. The same objection does not apply to the inverse gamma model where heavy tails are obtained by choosing  $c_3^2$  closer to  $2c_2$ .

### 3 Financial applications

#### 3.1 The Taylor and Xu approximation

Empirical studies<sup>5</sup> show that implied volatilities are higher for in-the-money and out-of-the-money options than for at-the-money options. Traders are aware of shortcomings of the Black and Scholes model<sup>6</sup> but adverse to renounce to its simple formula. Assigning different implied volatilities to options with the same underlying and time to maturity but different strike prices, they answer to the imperfections of the model using the model itself. The different implied volatilities express their market views.

The dependence of implied volatilities on strike prices is referred to as ‘the volatility smile’. The shape depends on underlying assets and in foreign exchange options it is quite symmetric<sup>7</sup>.

Stochastic volatility models have been proposed to account for volatility smiles.

Assume the model (1)-(2) where the market price of risk is zero and the volatility process is uncorrelated with the underlying process. Let  $C$  be the option price for a European call option with time to maturity  $T$  and strike price  $K$ . Under these assumptions, Hull and White derive the option price  $C$ ,

$$\begin{aligned} C &= \mathbb{E}_{\bar{V}|V_0} [C_{BS}(K, \bar{V})] \\ &= \int_0^\infty C_{BS}(K, \bar{V}) f(\bar{V} | V_0) d\bar{V}, \end{aligned} \tag{17}$$

where  $C_{BS}$  is the Black and Scholes option price,  $\bar{V}$  is the integrated volatility over the time to maturity  $T$ , i.e.

$$\bar{V} = \frac{1}{T} \int_0^T V_t dt \tag{18}$$

and  $f(\bar{V} | V_0)$  is the conditional density function of  $\bar{V}$ . The model implies symmetric volatility smiles with respect to strike prices. This result is in

<sup>5</sup>See Hull and White 1987 and references therein.

<sup>6</sup>For instance, rare events are more likely than that allowed by the model assumptions.

<sup>7</sup>The term ‘smirk’ characterizes asymmetric implied volatilities decreasing monotonically with strike prices.

qualitative agreement with volatility smiles observed in foreign exchange option markets.

In this framework Taylor and Xu (1994) derive an approximation for the theoretical implied volatility ( $\sigma_{imp}$ ) induced by stochastic volatility models. The approximation allows to compare theoretical and empirical volatility smiles. It is based on the Hull and White option pricing formula and on the implied volatility equation

$$C = C_{BS}(K; \sigma_{imp}^2). \quad (19)$$

Let us to denote

$$\mu_{\bar{V}|V_0} := \mathbb{E}[\bar{V} | V_0] \quad \text{and} \quad \sigma_{\bar{V}|V_0}^2 := \text{Var}[\bar{V} | V_0].$$

Using a first order Taylor expansion of  $C_{BS}$  around  $\mu_{\bar{V}|V_0}$  in equation (19) and a second order Taylor expansion of  $C_{BS}$  in equation (17), Taylor and Xu derive

$$\sigma_{imp}^2(K) \approx \mu_{\bar{V}|V_0} + \frac{\sigma_{\bar{V}|V_0}^2}{4\mu_{\bar{V}|V_0}} \left[ \frac{\ln^2(F/K) - \mu_{\bar{V}|V_0}T - \frac{1}{4}\mu_{\bar{V}|V_0}^2T^2}{\mu_{\bar{V}|V_0}T} \right] \quad (20)$$

where  $F$  is the forward price of the underlying asset<sup>8</sup>. When  $T$  tends to infinity, the approximation improves because  $\bar{V} \rightarrow \mu_{\bar{V}|V_0}$  for the ergodicity property of the  $V_t$  process. Notice that the minimum of  $\sigma_{imp}$  is for  $K = F$ .

**Observation 3.1** The quadratic term  $\ln^2(F/K)$  predicts a symmetric smile whatever is the specification for the  $V_t$  process. This result is due to the assumption of uncorrelated Brownian motions in the model (1) - (2).

Let us to define the relative height of the smile  $h := \sigma_{imp}(K)/\sigma_{imp}(F)$ . Using equation (20) we have the following approximation for  $h$

$$\tilde{h} := \sqrt{1 + \ln^2(F/K) \cdot \frac{\sigma_{\bar{V}|V_0}^2}{\mu_{\bar{V}|V_0}T} \left[ \frac{1}{4\mu_{\bar{V}|V_0}^2 - \sigma_{\bar{V}|V_0}^2(1 + \mu_{\bar{V}|V_0}T/4)} \right]}. \quad (21)$$

If the second term in the squared root is “small”,  $\tilde{h}$  simplifies to

$$\tilde{h} := 1 + \ln^2(F/K) \frac{\sigma_{\bar{V}|V_0}^2}{8T\mu_{\bar{V}|V_0}^3}. \quad (22)$$

We will use equations (21) and (22) to study volatility smiles implied by the model (1)-(3).

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<sup>8</sup>To ensure  $\sigma_{imp}^2 > 0$   $\mu_{\bar{V}}$  and  $\sigma_{\bar{V}}^2$  have to satisfy  $\sigma_{\bar{V}}^2 \leq \frac{4\mu_{\bar{V}}}{1+(1/4)T}$ . Usually, the inequality is verified for financial data.

### 3.2 The conditional mean and variance of $\bar{V}$

The Taylor and Xu approximation involves the first two conditional moments of the integrated volatility over the time to maturity  $\bar{V}$ . So far, these moments have been calculated analytically only for few processes:

1. the Cox-Ingersoll-Ross process by Bollerslev and Zhou (2002),
2. the mean reverting Ornstein-Uhlenbeck process by Cox and Miller (1972, Sec. 5.8),
3. the geometric Brownian motion by Hull and White (1987).

Unfortunately these models have some drawbacks when applied to real financial data. In the Cox-Ingersoll-Ross process the log-returns have finite moments of any order and the distribution can be only moderately heavy-tailed (see remark 2.1, p. 5). In the mean reverting Ornstein-Uhlenbeck process  $V_t$  has Gaussian density and the model can not be used to drive positive volatility processes. In the geometric Brownian motion the  $V_t$  process is not mean reverting and implies ‘term structure patterns’ of volatility smiles which do not match the empirical evidence. Indeed, volatility smiles are more convex for short time to maturity than for long ones while the model predicts the opposite (see appendix A).

Taylor and Xu (1994) assume that  $\ln \sqrt{V_t}$  follows a mean reverting Ornstein-Uhlenbeck process and use approximations (21) and (22) to study implied volatility surfaces. They have no analytical results for  $\ln \sqrt{V_t}$  and they use its discrete version to compute conditional moments of  $\bar{V}$ . They show that theoretical volatility smiles follow empirical term structure patterns but the theoretical convexity is approximately a half of the empirical one.

In the following, we give the conditional mean and variance of  $\bar{V}$  when  $V_t$  follows the inverse gamma process in equation (3). We recall that the integrated volatility is

$$\begin{aligned} \bar{V} &= \frac{1}{T} \int_0^T V_0 e^{-(c_2 + \frac{1}{2} c_3^2)t + c_3 W_t} \\ &\quad + \frac{c_1}{T} \int_0^T e^{(c_2 + \frac{1}{2} c_3^2)(s-t) + c_3 (W_t - W_s)} ds. \end{aligned}$$

**Proposition 3.1** Assume the  $V_t$  process follows the SDE in (3). Given  $V_0$ , the conditional mean of  $\bar{V}$  is

$$\mu_{\bar{V}|V_0} := \mathbb{E}[\bar{V} | V_0] = \mu_V + [V_0 - \mu_V] \frac{1 - e^{-c_2 T}}{c_2 T}. \quad (23)$$

The conditional variance of  $\bar{V}$  is

$$\begin{aligned}
\sigma_{\bar{V}|V_0}^2 := \text{Var}[\bar{V} | V_0] = & \\
& \frac{1}{c_2 T} \frac{2\mu_V^2}{2c_2/c_3^2 - 1} \\
& + \frac{1}{(c_2 T)^2} \left[ \frac{V_0^2}{2c_2/c_3^2 - 1} + \frac{\mu_V V_0}{2c_2/c_3^2 - 1} \right. \\
& \left. + \frac{\mu_V^2 (5 + 3(2c_2/c_3)^2 - 17/2(2c_2/c_3))}{(1 - c_2/c_3)(2c_2/c_3 - 1)^2} \right] \\
& + \frac{e^{-c_2 T}}{c_2 T} \left[ \frac{4\mu_V V_0 - 4\mu_V^2}{1 - c_2/c_3^2} \right] \\
& + \frac{e^{-c_2 T}}{(c_2 T)^2} \left[ \frac{2V_0^2}{1 - c_2/c_3^2} + \frac{4\mu_V V_0(c_2/c_3^2)}{(1 - c_2/c_3^2)^2} - \frac{4\mu_V^2}{(1 - c_2/c_3^2)^2} \right] \\
& - \frac{e^{-2c_2 T}}{(c_2 T)^2} [V_0 - \mu_V]^2 + \\
& - \frac{e^{-2c_2 T}(1 - c_3^2/2c_2)}{(c_2 T)^2} \left[ \frac{V_0^2}{(c_3^2/c_2 - 1)(c_3^2/2c_2 - 1)} \right. \\
& \left. + \frac{2V_0}{(c_3^2/c_2 - 1)^2(c_3^2/2c_2 - 1)} + \frac{\mu_V^2}{(c_3^2/c_2 - 1)^2(c_3^2/2c_2 - 1)^2} \right]. \tag{24}
\end{aligned}$$

The proof of the results are available from the authors on request.

**Remark 3.1** If  $c_1 = 0$  the process (3) reduces to the log normal process and equations (23) and (24) give the mean and variance of  $\bar{V}$  derived by Hull and White (1987) (see appendix A).

Equation (23) shows some properties of the integrated volatility implied by the inverse Gamma process:

- $\mu_{\bar{V}|V_0}$  does not depend on the parameter  $c_3$ ,
- when the time to maturity  $T$  goes to zero  $\mu_{\bar{V}|V_0} \rightarrow V_0$ ,
- when  $T$  is large  $\mu_{\bar{V}|V_0} \approx \mu_V$ ,
- $\mathbb{E}[\bar{V}] = \mathbb{E}[\mu_{\bar{V}|V_0}] = \mu_V$  as expected from the ergodicity of the  $V_t$  process.

Moreover, equation (23) shows that  $c_2$  controls ‘the reversion rate’ of  $\mu_{\bar{V}|V_0}$  to  $\mathbb{E}[\bar{V}]$ . We recall that  $c_2$  controls also the reversion rate of the  $V_t$  process to  $\mu_V$ .

## 4 The implied volatility surface

The analytical derivation of the conditional mean and variance of  $\bar{V}$  derived in proposition 3.1 allows us to use the Taylor and Xu approximation to study volatility surfaces implied by the model (1) - (3).

To verify if the theoretical volatility surface is in qualitative and quantitative agreement with the empirical evidence, we consider the relative height of volatility smiles of Mark-Dollar call options estimated by Taylor and Xu (1994). The option prices cover the period 1985–1994. Figure 1 shows the relative height of volatility smiles for time to maturities of 30, 40, 50 and 60

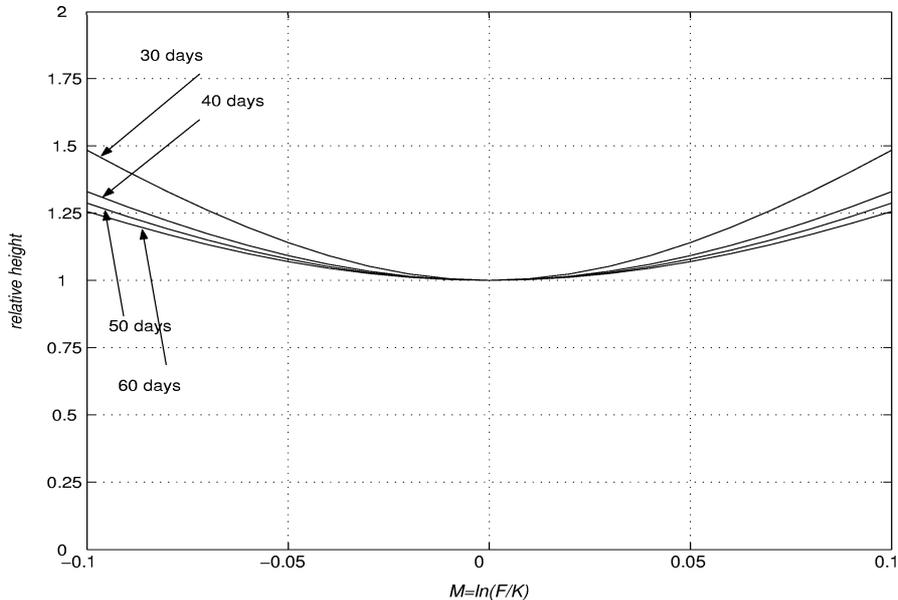


Figure 1: Relative height of volatility smiles of Mark-Dollar call options (1985-1992)

days<sup>9</sup>. Clearly, the volatility smiles are quite symmetric and more convex for short maturities.

To derive theoretical volatility surfaces we use the approximation of the height of the smile in equation (21). Unfortunately, we do not yet estimate the model. We choose reasonable parameters for the process  $V_t$  in accordance with the empirical evidence on log-returns. Specifically, we fix  $c_1 = 0.2$ ,  $c_2 = 23$ ,  $c_3 = 5$  and  $V_0 = E[V_1] = 0.01$ . This implies

1.  $\text{Var}[Z_1] = E[V_1] = c_1/c_2 = 0.01$ ,
2. half life<sup>10</sup> for the  $V_t$  process equals to 11 days;
3. kurtosis of  $Z_1$  equals to 7.

Figure 2 shows the volatility surface for these parameters and time to maturities between 0 and 150 days. Figure 3 shows the relative height of volatility smiles for time to maturities of 30, 40, 50 and 60 days.

In qualitative agreement with the empirical evidence, the volatility surface convexity increases when the time to maturity decreases.

Finally, we study how the volatility surface convexity depends on parameters under the following reasonable assumptions:

<sup>9</sup>Taylor and Xu find similar volatility smiles for others foreign exchange call options (Pound-Dollar, Yen-Dollar, Swiss-Dollar).

<sup>10</sup>The half life  $h$  is the time necessary to half the deviation of  $V_t$  from its run mean value given that there are no more random shocks. For the inverse gamma process  $h$  is equal  $\ln 2/c_2$

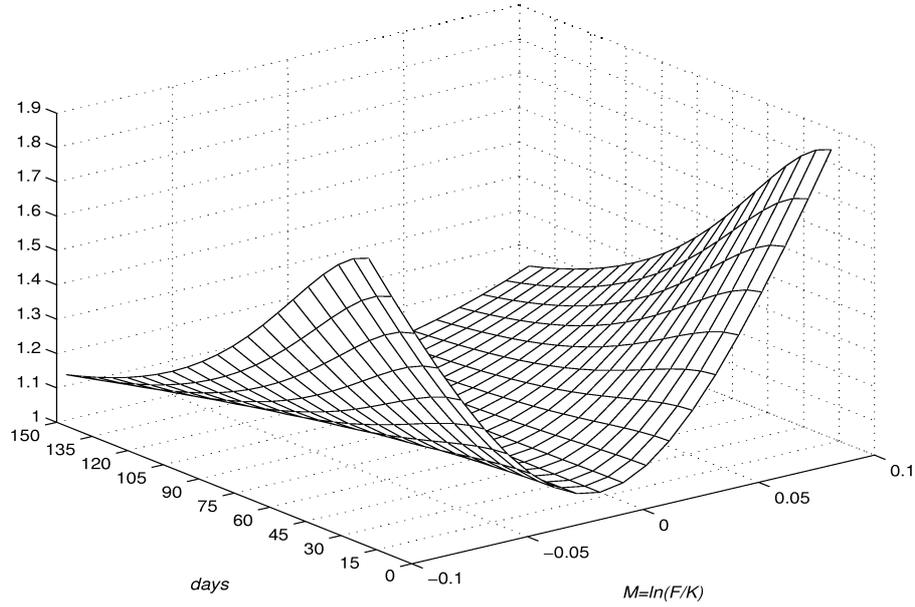


Figure 2: Relative height of smile.  $dV = 0.2 - 23V dt + 5V dW$ ;  $V_0 = 0.01$

- the initial value of the  $V_t$  process is equal to the run mean value

$$V_0 \approx \mathbb{E}[V_1] = c_1/c_2$$

and<sup>11</sup>

- $c_2T \gg 1$ .

Conditional moments (23) and (24) simplify to

$$\mu_{\bar{V}|V_0} \approx \mu_V, \quad (25)$$

$$\sigma_{\bar{V}|V_0}^2 \approx \frac{1}{c_2T} \frac{2\mu_V^2}{2c_2/c_3^2 - 1} \quad (26)$$

and

$$\tilde{h} \approx 1 + \frac{\ln^2 F/K}{T^2} \frac{1}{4 c_1(2c_2/c_3^2 - 1)}. \quad (27)$$

From equation (27) we deduce that

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<sup>11</sup>The second assumption is verified for fast mean reverting  $V_t$  process and/or long maturities.

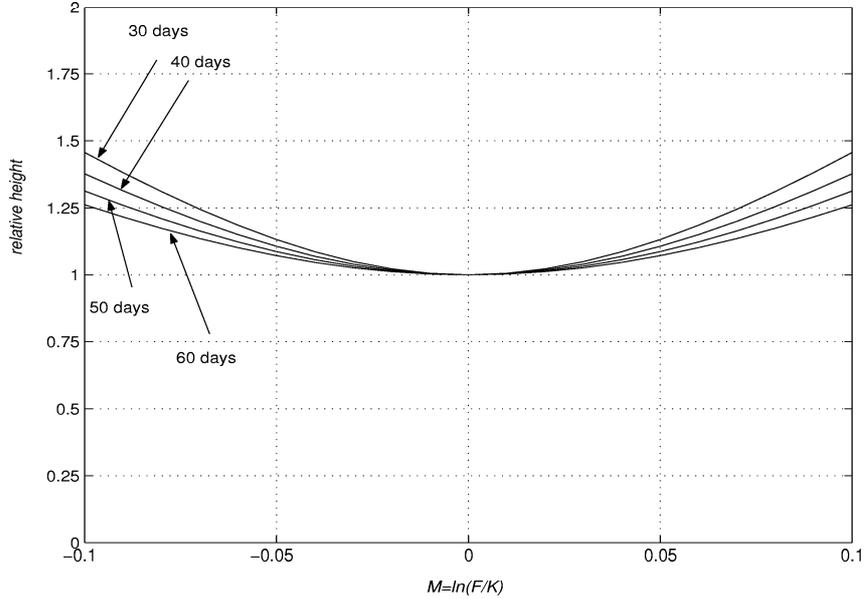


Figure 3: Relative height of smile (30-60 days).  $dV = 0.2 - 23V dt + 5V dW$ ;  $V_0 = 0.01$

- the relative height of smile is an increasing function of  $c_3^2$  and a decreasing function of  $c_1$  and  $c_2$ . Indeed, given  $\ln^2(F/K)/T^2$ ,

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial c_1} &= \frac{\ln^2(F/K)}{T^2} \frac{-1}{4 c_1^2 (2c_2/c_3^2 - 1)} < 0, \\ \frac{\partial \tilde{h}}{\partial c_2} &= \frac{\ln^2(F/K)}{T^2} \frac{-2/c_3^2}{4 c_1 (2c_2/c_3^2 - 1)^2} < 0, \\ \frac{\partial \tilde{h}}{\partial c_3^2} &= \frac{\ln^2(F/K)}{T^2} \frac{2c_2}{4 c_1^2 (c_3^2)^2 (2c_2/c_3^2 - 1)^2} > 0. \end{aligned}$$

- When  $2c_2/c_3^2 \rightarrow 1^+$  the convexity of  $\tilde{h}$  increases.

Notice that for fast mean reverting  $V_t$  process the excess kurtosis (16) can be approximated as

$$\mathbb{K}(Z_1) \approx \frac{1}{(2c_2/c_3^2) - 1}.$$

Hence, heavy tailed log-returns imply  $\tilde{h}$  sharply convex, as expected.

- Given  $\ln^2(F/K)$ , the volatility surface becomes more convex when the time to maturity goes to zero.

## 5 Concluding remarks

We derive the conditional mean and variance of the integrated volatility over the time to maturity when the squared volatility follows the inverse gamma process. The analytical results allow us to study implied volatility surfaces using the Taylor and Xu approximation. We find qualitative agreement between theoretical and empirical term structure patterns. Moreover, we show that for fast mean reverting volatility processes and/or long time to maturities, the convexity of implied volatilities increases when the kurtosis increases and/or the reversion rate decreases. The next step will be the model estimation, using Mark-Dollar exchange rates.

## A Implied volatility surface with log normal process

We study implied volatility patterns when the squared volatility is driven by the log-normal process

$$dV_t = \tilde{c}_2 V_t dt + \tilde{c}_3 V_t dW_t. \quad (28)$$

We consider separately the case  $\tilde{c}_2 = 0$  (log normal process without drift) and the case  $\tilde{c}_2 \neq 0$  (log normal process with drift).

1. Case  $\tilde{c}_2 = 0$  and  $\tilde{c}_3 \neq 0$ . Hull and White derive

$$\mu_{\bar{V}|V_0} = V_0, \quad (29)$$

$$\sigma_{\bar{V}|V_0}^2 = 2V_0^2 \left[ \frac{e^{\tilde{c}_3^2 T} - \tilde{c}_3^2 T - 1}{\tilde{c}_3^4 T^2} - \frac{1}{2} \right]. \quad (30)$$

The asymptotic behaviour of  $\sigma_{\bar{V}|V_0}^2$  is the following:

$$\sigma_{\bar{V}|V_0}^2 \approx V_0^2 \left[ \frac{\tilde{c}_3^2 T}{3} + \frac{\tilde{c}_3^4 T^2}{12} \right] \rightarrow 0, \quad \tilde{c}_3^2 T \rightarrow 0; \quad (31)$$

$$\sigma_{\bar{V}|V_0}^2 \approx 2V_0^2 \left[ \frac{e^{\tilde{c}_3^2 T}}{\tilde{c}_3^4 T^2} \right] \rightarrow \infty, \quad \tilde{c}_3^2 T \gg 1. \quad (32)$$

In the first case ( $\tilde{c}_3^2 T \rightarrow 0$ ), substituting the approximation (31) in equation (22), we get

$$\tilde{h} \approx 1 + \frac{\ln^2(F/K)}{V_0} \left[ \frac{\tilde{c}_3^2}{24} + \frac{\tilde{c}_3^4 T}{96} \right]. \quad (33)$$

The approximation (33) shows that the smile convexity increases as the time to maturity increases, in contrast with the empirical evidence<sup>12</sup>. Figure 4 is obtained using equation (21) with exact moments (29) and (30) and confirms the result.

In the second case ( $\tilde{c}_3^2 T \gg 1$ ), the Taylor and Xu approximation does not apply because  $\sigma_{\bar{V}|V_0}^2 \rightarrow \infty$ . However, relevant time to maturities are less than or equal to one year.

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<sup>12</sup>Gesser and Poncet (1997) find empirical evidence of this inconsistency using Dollar/Mark call options traded in February 1996.

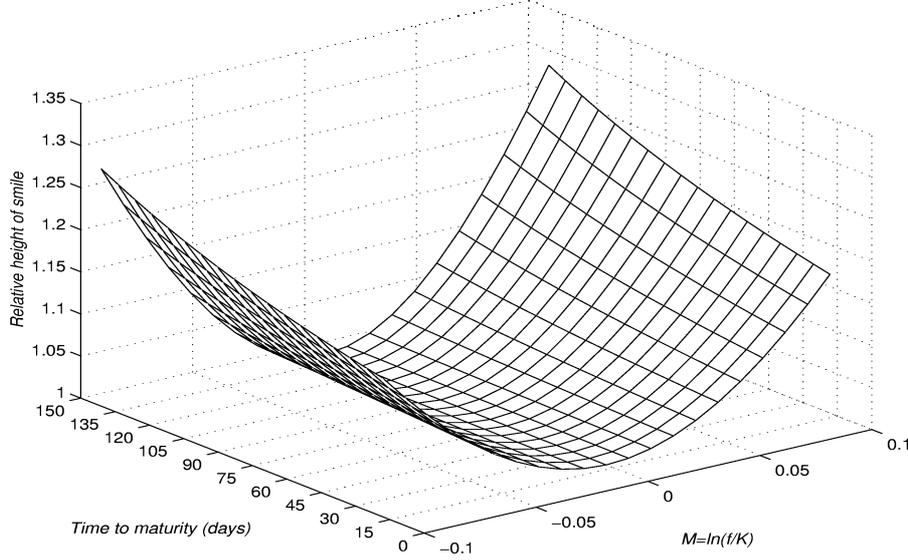


Figure 4: Relative height of volatility smiles.  $dV = 2VdW$  (same parameters for the  $V_t$  process used in Hull and White 1987).

2. Case  $\tilde{c}_2 \neq 0$ ,  $\tilde{c}_3 \in \mathbb{R}^+ \setminus \{\tilde{c}_2, -2\tilde{c}_2\}$ . Conditional moments are

$$\mu_{\bar{V}|V_0} = V_0 \left[ \frac{e^{\tilde{c}_2 T} - 1}{\tilde{c}_2 T} \right], \quad (34)$$

$$\sigma_{\bar{V}|V_0}^2 = V_0^2 \left[ \frac{2e^{(2\tilde{c}_2 + \tilde{c}_3^2)T}}{(\tilde{c}_3^2 + \tilde{c}_2)(\tilde{c}_3^2 + 2\tilde{c}_2)T^2} - \frac{e^{2\tilde{c}_2 T}}{\tilde{c}_2^2 T^2} + \frac{e^{\tilde{c}_2 T}}{\tilde{c}_2^2 T^2} \left( \frac{2\tilde{c}_3^2}{\tilde{c}_3^2 + \tilde{c}_2} \right) - \frac{1}{\tilde{c}_2^2 T^2} \left( \frac{\tilde{c}_3^2}{\tilde{c}_3^2 + 2\tilde{c}_2} \right) \right]. \quad (35)$$

When  $T \rightarrow 0$

$$\begin{aligned} \mu_{\bar{V}} &\approx V_0, \\ \sigma_{\bar{V}}^2 &\approx V_0^2 [\tilde{c}_3^2 T + \tilde{c}_3^2 [5\tilde{c}_2 + \tilde{c}_3^2] T^2] / 3 \end{aligned} \quad (36)$$

and, using these approximations in equation (22),

$$\tilde{h} \approx 1 + \frac{\ln^2(F/K)}{24V_0} [\tilde{c}_3^2 + \tilde{c}_3^2 (5\tilde{c}_2 + \tilde{c}_3^2) T]. \quad (37)$$

As in the previous case, equation (33) shows that the smile convexity increases with the time to maturity. Figure 5, obtained using equation (21) and exacts moments (34) and (35), confirms the result.

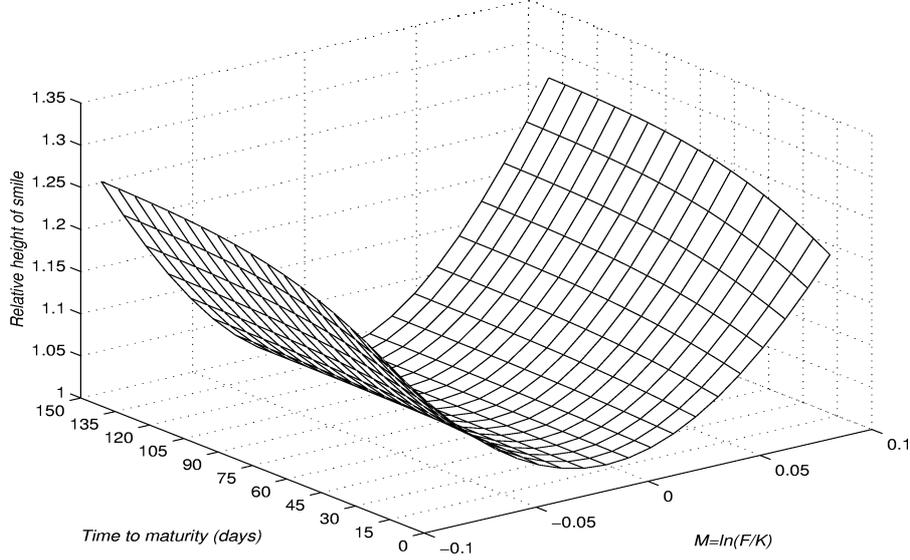


Figure 5: Relative height of volatility smiles.  $dV = -13Vdt + 3VdW$  .

When  $T \rightarrow \infty$  and  $2\tilde{c}_2 + \tilde{c}_3^2 < 0$ ,

$$\sigma_V^2 \approx \frac{-\tilde{c}_3^2}{\tilde{c}_2^2 T^2 (\tilde{c}_3^2 + 2\tilde{c}_2)} \rightarrow 0. \quad (38)$$

When  $T \rightarrow \infty$  and  $2\tilde{c}_2 + \tilde{c}_3^2 > 0$ ,

$$\sigma_V^2 \approx \frac{2e^{(2\tilde{c}_2 + \tilde{c}_3^2)T}}{(\tilde{c}_3^2 + \tilde{c}_2)(\tilde{c}_3^2 + 2\tilde{c}_2)T^2} \rightarrow \infty.$$

In the first case ( $2\tilde{c}_2 + \tilde{c}_3^2 < 0$ ) the Taylor and Xu approximation holds and the equation (22) implies

$$\tilde{h} \approx 1 - \frac{\ln^2(F/K)}{T^3} \left[ \frac{\tilde{c}_3^2}{8\tilde{c}_2^2 V_0 [\tilde{c}_3^2 + 2\tilde{c}_2]} \right]. \quad (39)$$

The equation (39) shows that the smile convexity decreases as the time to maturity increases. Unfortunately, there are no traded options with time maturity  $T \gg 1$  year.

In the second case ( $2\tilde{c}_2 + \tilde{c}_3^2 > 0$ ) the Taylor and Xu approximation does not apply because  $\sigma_V^2 \rightarrow \infty$ .

Finally, when  $t \rightarrow \infty$   $V_t \rightarrow 0$  or  $V_t \rightarrow \infty$  (exponentially). Indeed, given  $V_0$ , the solution of the equation (28) is given by

$$V_t = V_0 e^{(\tilde{c}_2 - \tilde{c}_3^2/2)t + \tilde{c}_3 W_t}.$$

For the iterated logarithm law and the large number law (Karatzas and Shreve 1988), we have

1. if  $\tilde{c}_2 < \tilde{c}_3^2/2$ ,

$$\lim_{t \rightarrow \infty} V_t = 0 \quad a.s.$$

2. if  $\tilde{c}_2 > \tilde{c}_3^2/2$ ,

$$\lim_{t \rightarrow \infty} V_t = \infty \quad a.s$$

3. if  $\tilde{c}_2 = \tilde{c}_3^2/2$ ,

$$\inf_{0 \leq t \leq \infty} V_t = 0 \quad a.s.,$$

$$\sup_{0 \leq t \leq \infty} V_t = \infty \quad a.s$$

This is in contrast with the mean reverting trend of the historical volatility<sup>13</sup>.

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<sup>13</sup>We recall that the historical volatility is used for estimation purposes assuming a constant risk premium.

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