FRÉCHET AND ROBUST STATISTICS

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1. Introduction

It is a pleasure to discuss Fréchet's paper and I would like to thank the Editor for giving me this opportunity. The paper originally published in 1940, contains several interesting points which are still valuable today. In my discussion I focus in particular on the aspects related to robust statistics. More specifically, in Section 2 I comment on Fréchet's view of data analysis and on his quest for alternatives to the mean and the standard deviation. Section 3 will be devoted to discuss Fréchet differentiability in statistics, a topic which is not covered in the present paper but which plays an important role in the development of the theory of robust statistics and other fields.

2. Data Analysis and Robust Statistics

In Section A Fréchet makes some comments on the status of mathematical statistics. The points he raises are interesting since they come from a mathematician who had been working in the past decade mostly in the area of analysis. Fréchet complains about some results where the assumptions are not clearly stated, about the abuse of indicators like the correlation coefficient (which incidentally reminds J.W. Tukey statement "Does anyone know when a correlation coefficient is useful, as opposed to when it is used ? "), and about the almost exclusive place of the mean and the standard deviation in statistical analysis.

In particular he puts into perspective the optimality of these two estimators by reminding the reader that this result is obtained under an important condition, namely the normality assumption on the distribution of the observations. Even today this point is sometimes not well understood when the Gauss-Markov Theorem is invoked to justify the optimality of the least squares estimator (LS). It is true that LS is optimal even without the normality assumption, but in this case *only among linear* unbiased estimators of the parameter. Therefore, either LS is optimal in a general class of estimators but *under a very restrictive assumption (normality)*, or it is optimal under unrestricted distributional assumptions but *only in a very restricted class of estimators (linear)*. Clearly, in both cases the situation is not satisfactory. From an historical point of view, it is interesting to notice that Gauss in 1809 characterized the normal distribution by looking at the optimality of the mean

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from the opposite direction, i.e. *assuming* that the mean is the best (location) estimator, then the underlying distribution of the observations is normal.

More comments on the advantages and disadvantages of different types of estimators are provided in the Discussion, when Fréchet answers the questions raised by the audience. Incidentally the comparison between different measures of variability reminds the dispute between Eddington and Fisher around 1920 about the relative merits of the mean absolute deviation and the standard deviation; cf. Huber (1981), Chapter 1.1.

In spite of these criticisms (some of them shared by Gini), Fréchet is optimistic about the future of statistics and ends Section A by saying that these criticisms will open up new directions in the development of statistics. Indeed he was right : these ideas contributed among others to the development of robust statistics and modern data analysis; cf. the seminal paper by Tukey (1962) on the future of data analysis and the fundamental papers by Tukey (1960), Huber (1964), and Hampel (1968) on robust statistics.

Section B is devoted to the quantitative investigation of the variability of the median as measured by its variance but also by other measures of dispersion. Fréchet derives some bounds for these quantities and check the quality of these bounds by a simulation study(!). This is a modern approach to methodological research in statistics : the theoretical properties of some statistical procedure are first studied by means of asymptotic approximations and then the results are checked by simulation.

3. Fréchet Differentiability

In this section I briefly mention another (indirect) contribution of Fréchet to robust statistics. Hampel (1968,1974) recast the theory of robust statistics as the study of the stability properties of statistical functionals (such as an estimator, its variance, the level of a test etc.) and this opened the way to the use of different concepts of derivatives in functional analysis; cf. the books Huber (1981) and Hampel, Ronchetti, Rousseeuw, Stahel (1986). Three types of derivative are important for robust statistics (from the weakest to the strongest), the Gâteaux derivative (or influence function), the compact (or Hadamard) derivative, and the Fréchet derivative. A mathematical treatment of these concepts from a statistical perspective is provided by Fernholz (1983).

The Gâteaux derivative (or influence function) is a key tool in robust statistics. It can be computed for most of the existing estimators and test statistics and can be used to construct new robust statistical procedures. Fréchet differentiability is a stronger concept which implies the existence of the influence function and additionally guarantees the asymptotic normality of the corresponding statistic in a shrinking neighborhood around the model; cf. Clarke (1986) and Bednarski (1993). This property is important to insure the stability (robustness) of the corresponding functional in the presence of small deviations from the assumed model. Clarke (1986) gives general conditions which imply Fréchet differentiability for a large class of M-estimators. A key condition is the boundedness of the score function defining the M-estimator.

Finally, notice that the stability properties of a functional implied by Fréchet differentiability play an important role in other fields in statistics, including for instance the bootstrap.

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